J. Aust. Math. Soc. **117** (2024), 85–104
doi:10.1017/S1446788723000174

BRATTELI–VERSHIKISABILITY OF POLYGONAL BILLIARDS ON THE HYPERBOLIC PLANE

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(Received 14 February 2023; accepted 19 October 2023; first published online 15 December 2023)

Communicated by Milena Radnović

Abstract

Bratteli–Vershik models of compact, invertible zero-dimensional systems have been well studied. We take up such a study for polygonal billiards on the hyperbolic plane, thus considering these models beyond zero-dimensions. We describe the associated Bratteli models and show that these billiard dynamics can be described by Vershik maps.

2020 *Mathematics subject classification*: primary 37B10; secondary 37D40, 37D50.

Keywords and phrases: polygonal billiards, pointed geodesics, subshifts of finite type, Bratteli diagrams, Vershik maps.

1. Introduction

The analysis of billiard trajectories forms a crucial component of quantum, classical and relativistic mechanics. The motion of a point particle bouncing at the sides of a compact domain in the Euclidean plane captures many important flavours of chaos. This dynamics is better understood using symbols leading to the deeper study of symbolic dynamics [10]. We refer to [14] for an elementary introduction to billiard dynamics. Such a study misses out on a subtle point, as when it comes to following the trajectories of billiard balls, the codings generated are not uniquely attached to a single trajectory. Considering billiards on hyperbolic tables compensates for this deficiency as every generated coding can be associated to a unique trajectory.

A hyperbolic plane is a two-dimensional manifold with a constant negative Gaussian curvature. We prefer the *Poincaré disc model* $\mathbb{D}$ for our work as it is Euclidean compact, and this lets us follow the trajectories to the boundary, which is the essence of traditional billiards. Billiards in polygons on hyperbolic tables was described in [11], which investigated billiards on a class of polygons that were subsets of $\mathbb{D}$ with their boundaries comprising finitely many geodesic segments that are piecewise smooth and intersect in vertices at angles of either 0 or those that divide π into integer parts. Therein, the symbolic codes for the billiard trajectories inside such ideal, rational and semi-ideal hyperbolic polygons were studied and in all the cases, it was observed that the billiard trajectories were conjugate to either a mixing subshift of finite type or a dense subset of a mixing subshift of finite type.

Bratteli diagrams are infinite graphs used to model various topological dynamical systems. They were first introduced by O. Bratteli to study approximately finite-dimensional (AF) algebras, which happen to be the direct limits of directed sequences of finite-dimensional C*-algebras. Later, some ideas of A. Vershik were found to be closely related to Bratteli diagrams. Thus, *Bratteli–Vershik models* were forged becoming important tools in the study of Cantor dynamical systems. A correspondence was set up between a minimal Cantor invertible system and its Bratteli diagram via the *Vershik map* by Herman *et al.* [7]. Subsequently, this approach was adopted to study various Cantor systems. Medynets [9] showed that aperiodic Cantor systems admit Bratteli–Vershik (BV) models. These models were also found useful to study substitution systems [3, 6, 13]. A general survey on this theory can be seen in [2].

Later, Downarowicz and Karpel [4, 5] extended this study to zero-dimensional systems with periodic points, by referring such systems to be *Bratteli–Vershikisable* if the Vershik map on the set of nonmaximal infinite paths on the attached ordered Bratteli diagram can be extended uniquely to a homeomorphism on the whole path space of the Bratteli diagram. In particular, they showed that an invertible zero-dimensional system is Bratteli–Vershikisable if and only if the set of aperiodic points is dense or its closure misses one periodic orbit. Shimomura [12] showed that all invertible zero-dimensional systems admit nontrivial Bratteli–Vershik models.

The study of polygonal billiards on the hyperbolic plane was initiated in [11] and here we further consider Bratteli–Vershik models for the same. These trajectories are curves in $\mathbb{D}$. Though these hyperbolic polygonal billiards can be conjugated to zero-dimensional systems, we observe that the known theory fails here since in the case of ideal or semi-ideal polygons, the billiard trajectories fail to form a compact space. However, if we strictly consider only the forward trajectories, we can also take into account all those trajectories that bounce their way out to infinity between two adjacent sides of a polygon to be valid, and then these billiard trajectories form a compact space (we avoid all the trajectories that jump to infinity in finite time and so our billiard trajectories are those that remain entirely on the billiard table for infinite time). However, then our system cannot be regarded as invertible. We show that despite such constraints, we do indeed have a Vershik map that describes the billiard dynamics here.

This opens up a path to study *K-theoretic properties* attached to billiards.

We discuss the general theory considered in Section 2 and our main results in Section 3.

2. Preliminaries

2.1. Some dynamical notions. A dynamical system is a pair (X, f), where X is a compact metric space and f a homeomorphism or a continuous self-map on X. We call (X, f) a cascade if f is a homeomorphism and when f is just a continuous map, we call (X, f) a semicascade. For any $x \in X$, the trajectory of x is $\{f^n(x) : n \in \mathbb{Z}\}$ (for a cascade (X, f)) and its forward trajectory is $\{f^n(x) : n \in \mathbb{N}\}$ (taken as trajectory if (X, f) is a semicascade). The point x is *periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. The system is called *topologically transitive* when for every pair of nonempty, open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Note that for transitive systems in a separable and complete metric space without isolated points, equivalently there exists a dense set of points each with a dense trajectory. In such a case, we call the system (X, x_0, f) a *pointed system*, where x_0 has a dense trajectory. Here, (X, f) is called *topologically mixing* if for every pair V, W of nonempty open sets in X, there is an $N > 0$ such that $f^n(V) \cap W$ is nonempty for all $n \geq N$.

A *conjugacy* $\pi : (X_1, f_1) \to (X_2, f_2)$ is a homeomorphism $\pi : X_1 \to X_2$ such that $f_2 \circ \pi = \pi \circ f_1$.

In particular, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_1 \\ \pi\downarrow & & \downarrow\pi \\ X_2 & \xrightarrow{f_2} & X_2 \end{array}$$

commutes, and we write $(X_1, f_1) \simeq (X_2, f_2)$. Conjugate systems are dynamically equivalent.

Shift spaces are built on a finite set $\mathcal{A}$ of symbols. We define the *full* $\mathcal{A}$ *shift* as the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. It is denoted by $\mathcal{A}^{\mathbb{Z}} = \{x = \cdots x_{-1} \cdot x_0 x_1 \cdots : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$.

The product topology on $\mathcal{A}^{\mathbb{Z}}$ is metrisable and a compatible metric defined on it can be given as

$$d(x, y) = \inf\left\{\frac{1}{2^m} : x_n = y_n \text{ for } |n| < m\right\}$$

for any two sequences $x = \cdots x_{-1} \cdot x_0 x_1 \cdots$ and $y = \cdots y_{-1} \cdot y_0 y_1 \cdots \in \mathcal{A}^{\mathbb{Z}}$. Observe that for $x, y \in X$, there exists $m > 0$ for which $d(x, y) < 2^{-m}$ which is equivalent to the requirement that there exists $k > 0$ for which $x_{[-k,k]} = y_{[-k,k]}$, that is, the centre blocks $x_{-k} \cdots x_{-1} \cdot x_0 x_1 \cdots x_k = y_{-k} \cdots y_{-1} \cdot y_0 y_1 \cdots y_k$.

The *shift map* σ maps a point x to the point $\sigma(x)$ whose ith coordinate is

$$(\sigma(x))_i = x_{i+1}.$$

A *shift space* is a closed set $X \subseteq \mathcal{A}^{\mathbb{Z}}$.

For the shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the system (X, σ) is called a *subshift* when $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is closed and invariant (that is, $\sigma(X) \subseteq X$).

The set $\mathcal{A}^{\mathbb{N}} = \{x = x_1x_2\cdots : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{N}\}$ of infinite sequences is a *one-sided shift space*.

For integers $i < j$, the block $x_i \cdots x_j$ is called a *word*. The words that arise as blocks of all $x \in X$ are called *admissible*, and those that do not are called *forbidden*. The set of admissible words of the shift space X is called its *language*, $\mathcal{L}(X)$. The subshift $(X_{\mathcal{F}}, \sigma)$ is called a *subshift of finite type (SFT)*. One can find a finite set of forbidden words from which the set $\mathcal{F}$ of forbidden words can be generated by means of concatenation. For the theory of shift dynamics, we refer to [8].

2.2. Billiards on hyperbolic plane. The hyperbolic plane is a two-dimensional Riemannian manifold with a constant negative curvature that is not embeddable in $\mathbb{R}^2$. We use the Poincaré disk model $\mathbb{D}$ to study it. We refer to [1] for the standard theory of hyperbolic geometry.

The Poincaré disk model is described by the space $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with metric given by $ds^2 = (4(dx^2 + dy^2))/((1 - (x^2 + y^2))^2)$. Its boundary is $\partial\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and the geodesics are the euclidean diameters of the disc and the euclidean semicircles hitting the boundary orthogonally.

As in [11], we work with billiard trajectories on tables that are finite sided convex polygons on the hyperbolic plane. We recall some basics studied there.

The *ideal polygons* are polygons formed by the geodesics with vertices placed on $\partial\mathbb{D}$, the *rational polygons* are polygons where all the vertices are placed inside $\mathbb{D}$ with interior angles dividing π by an integer, and their generalised versions *semi-ideal rational polygons* comprise the vertices of both kinds: ideal and rational. We parametrise the boundary of the Poincaré disc $\mathbb{D}$ using the azimuthal angle by considering it as a subset of $\mathbb{C}$. Thus, we can represent a directed geodesic by the pair (θ, ϕ), where θ and ϕ are the intercepts made by a directed geodesic on $\partial\mathbb{D}$ with the direction being from θ to ϕ. In this setting, we have a natural metric on $\partial\mathbb{D}$ given by

$$d_{\partial\mathbb{D}}(\phi_1, \phi_2) = |\phi_1 - \phi_2|.$$

A point moving along a geodesic inside the polygon Π gets reflected from the sides of the polygon under specular reflection rules. We label the reflection map as T. Thus, we obtain a billiard trajectory as a curve that is parametrised by the arc-length and comprises the geodesic arcs that are reflected by the sides of the polygon Π. Initially, as in [11], we are interested in only the geodesic arcs that hit the sides of the polygon and not any vertex. Therefore, a trajectory can be expressed as

$$\gamma = \{(\theta_n, \phi_n)_{n\in\mathbb{Z}}\},$$

where

$$(\theta_n, \phi_n) = T(\theta_{n-1}, \phi_{n-1}).$$

DEFINITION 2.1. Let $\gamma = (\theta_n, \phi_n)_{n\in\mathbb{Z}}$ be a billiard trajectory in a polygon Π in $\mathbb{D}$. We call (θ_0, ϕ_0) a *base arc* of the trajectory $\gamma = \{(\theta_n, \phi_n)_{n\in\mathbb{Z}}\}$.

A base arc uniquely determines the billiard trajectory under the restrictions imposed by the specular reflection rule. The base arcs are compact subsets of $\mathbb{D}$.

Recall that for a metric space (X, d), we denote as $\mathcal{K}(X)$ the space of all compact subsets of X, endowed with the Hausdorff topology, with a natural induced metric.

Given a point $p \in X$ and a closed set $A \subseteq X$, recall $d(p, A) = \inf_{a\in A} d(p, a)$.

For $A, B \in \mathcal{K}(X)$, we define the *Hausdorff metric* as

$$d_H(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big\}.$$

DEFINITION 2.2. For the base arc (θ, ϕ) defining γ, we call $(\gamma, (\theta, \phi))$ a pointed geodesic.

Thus, a pointed geodesic $(\gamma, (\theta, \phi))$ is identified with the element

$$\cdots (T^{-1}(\theta, \phi)) \cdot (\theta, \phi)(T(\theta, \phi)) \cdots \in \mathcal{K}(\mathbb{D})^{\mathbb{Z}},$$

by clearly establishing the position of the base arc (θ, ϕ). A natural way of encoding a pointed geodesic is to seize the order in which it hits the sides of Π, starting from the side hit by the base arc, and then reading the past and future hits of the trajectory and pointing out the symbol corresponding to the base arc. If we label the sides of Π anti-clockwise from 1 to k, then every pointed geodesic produces a bi-infinite sequence $\cdots a_{-1} \cdot a_0 a_1 \cdots$ with $a_j \in \{1, \ldots, k\}$. It is remarked that throughout we would assume mod k additivity on the symbols $1, 2, \ldots, k$, that is, $k + 1$ would be treated the same as 1, $k + 2$ as 2 and so on.

DEFINITION 2.3. Define

$$\mathbb{G} = \mathbb{G}_\Pi := \{(\gamma, (\theta, \phi)) : \gamma = (T^n(\theta, \phi))_{n\in\mathbb{Z}}\}$$

as the space of all pointed geodesics on Π.

Here, $\mathbb{G} \subseteq \mathcal{K}(\mathbb{D})$ and so $\mathbb{G}$ can be equipped with the natural Hausdorff metric d_H, and so is endowed with the Hausdorff topology.

We define a function $d_{\mathbb{G}} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ as

$$d_{\mathbb{G}}((\gamma, (\theta, \phi)), (\gamma', (\theta', \phi'))) = \max\{d_{\partial\mathbb{D}}(\theta, \theta'), d_{\partial\mathbb{D}}(\phi, \phi')\},$$

where $d_{\partial\mathbb{D}}$ is the arc distance on $\partial\mathbb{D}$. Then $d_{\mathbb{G}}$ defines a metric on $\mathbb{G}$.

Recall that the Hausdorff topology on $\mathbb{G}$ is given by d_H on $\mathbb{G}$, which can be given as

$$\begin{aligned} d_H((\gamma, (\theta, \phi)), (\gamma', (\theta', \phi'))) &:= d_H((\theta, \phi), (\theta', \phi')) \\ &= \max\Big\{\sup_{Q\in(\theta,\phi)} d(Q, (\theta', \phi')), \sup_{Q\in(\theta',\phi')} d(Q, (\theta, \phi))\Big\}. \end{aligned}$$

THEOREM 2.1 [11]. *Let $\mathbb{G}$ be the space of pointed geodesics on a polygon Π in $\mathbb{D}$, then $d_{\mathbb{G}}$ and d_H generate the same topology on $\mathbb{G}$.*

Again from [11], the map $\tau : \mathbb{G} \to \mathbb{G}$ with its action on $\mathbb{G}$ given as $\tau((\gamma, (\theta, \phi))) = (\gamma, T(\theta, \phi))$ is a homeomorphism. Here for $n \in \mathbb{Z}$, $\tau^n((\gamma, (\theta, \phi))) = (\gamma, T^n(\theta, \phi))$ where $T^0(\theta, \phi) = (\theta, \phi)$ and the specular reflection map T gives the geodesic $\gamma = \{T^n(\theta, \phi)\}_{n\in\mathbb{Z}}$.

We thus get a metric cascade $(\mathbb{G}, \tau)$, with the action of τ on $\mathbb{G}$ described as

$$\tau((\gamma, (\theta, \phi))) = (\gamma, T(\theta, \phi)).$$

Intuitively, a typical pointed geodesic behaves like a long rope divisible into sections of finite length, gripped at one of its sections, with each successive section being gripped on forward movement while the preceding section is gripped on backward movement. Thus, the map τ acts on a pointed geodesic by shifting the grip to its future consecutive section.

A natural way of encoding a pointed geodesic is to seize the order in which it hits the sides of Π, starting from the side hit by the base arc, and then reading the past and future hits of the trajectory and pointing out the symbol corresponding to the base arc. If we label the sides of Π anti-clockwise from 1 to k, then every pointed geodesic produces a bi-infinite sequence $\cdots a_{-1} \cdot a_0 a_1 \cdots$ with $a_j \in \{1, \ldots, k\}$.

THEOREM 2.2 [11]. *Let $\Pi \subset \mathbb{D}$ be an ideal polygon with anti-clockwise enumeration $1, \ldots, k$ and $\mathbb{G}$ be the space of pointed geodesics on Π. Suppose X is the space of all bi-infinite sequences $\cdots a_{-1} \cdot a_0 a_1 \cdots \in \{1, \ldots, k\}^{\mathbb{Z}}$ satisfying the rules:*

(1) *$a_j \neq a_{j+1}$ for all $j \in \mathbb{Z}$ and*
(2) *$\cdots a_{-1} \cdot a_0 a_1 \cdots$ does not contain an infinitely repeated sequence or bi-infinite sequence of labels of two adjacent sides.*

Then the cascade $(\mathbb{G}, \tau) \simeq (X, \sigma)$.

Here, X is not closed, as the limit points of X of type $\overline{i(i+1)}$, $w\overline{i(i+1)}$, $\overline{i(i+1)}w$ do not lie in X. In this context, w represents an arbitrary word in $\mathcal{L}(X)$ and the notation $\overline{w'}$ signifies the infinite repetition of the word w' in either feasible direction. Thus, we further look for the closure of X in $\{1, \ldots, k\}^{\mathbb{Z}}$ and define $\tilde{X} = X \cup X'$, where X' is the set of all limit points of X. Hence,

$$\tilde{X} = \{\cdots x_{-1} \cdot x_0 x_1 \cdots \in \{1, \ldots, k\}^{\mathbb{Z}} : x_i \neq x_{i+1} \text{ for all } i\},$$

and thereby is a mixing SFT with forbidden set $\{11, 22, \ldots, kk\}$. Thus, X is a dense subset of an SFT. We notice that $\tilde{X}$ is a completion of X, therefore it is also a compactification of X.

THEOREM 2.3 [11]. *Let $\Pi \subset \mathbb{D}$ be a compact rational polygon with anti-clockwise enumeration of sides labelled $1, 2, \ldots, k$. Label the vertices of Π as $v_1, \ldots, v_k$ with $\Omega_1, \ldots, \Omega_k$ being the respective interior angles such that the adjacent sides of v_i are i and $i+1$. Further, assume that $\lambda_i = \pi/\Omega_i \in \mathbb{N}$ for each $i \in \{1, \ldots, k\}$. Let $\mathbb{G}$ be the space of pointed geodesics on Π and X the space of all bi-infinite sequences $\cdots a_{-1} \cdot a_0 a_1 \cdots \in \{1, \ldots, k\}^{\mathbb{Z}}$ satisfying:*

(1) *$\cdots a_{-1} \cdot a_0 a_1 \cdots$ does not contain any immediate repetitions of symbols, that is, $a_j \neq a_{j+1}$ for all $j \in \mathbb{Z}$;*
(2) *$\cdots a_{-1} \cdot a_0 a_1 \cdots$ does not contain more than λ_i repetitions of two successive symbols i and $i+1$ for every $i \in \{1, \ldots, k\}$.*

Then the cascade $(\mathbb{G}, \tau) \simeq (X, \sigma)$.

Here X is compact and is a mixing SFT.

THEOREM 2.4 [11]. *Let $\Pi \subset \mathbb{D}$ be a semi-ideal rational polygon with anti-clockwise enumeration of sides labelled $1, 2, \ldots, k$. Label the vertices of Π as $v_1, \ldots, v_k$ with $\Omega_1, \ldots, \Omega_k$ being the respective interior angles such that the adjacent sides of v_i are i and $i+1$. Further, assume that $v_i \in \mathbb{D}$ for all $i \in \Lambda \subset \{1, \ldots, k\}$ and $v_i \in \partial\mathbb{D}$ for all $i \in \{1, \ldots, k\} - \Lambda$ with $\lambda_i = \pi/\Omega_i \in \mathbb{N}$ for all $i \in \Lambda$. Let $\mathbb{G}$ be the space of pointed geodesics on Π and X the corresponding space of all bi-infinite sequences $\cdots a_{-1} \cdot a_0 a_1 \cdots \in \{1, \ldots, k\}^{\mathbb{Z}}$ satisfying:*

(1) *$\cdots a_{-1} \cdot a_0 a_1 \cdots$ does not contain any immediate repetitions of symbols, that is, $a_j \neq a_{j+1}$ for all $j \in \mathbb{Z}$;*
(2) *$\cdots a_{-1} \cdot a_0 a_1 \cdots$ does not contain more than λ_i repetitions of two successive symbols i and $i+1$ for every $i \in \Lambda$;*
(3) *$\cdots a_{-1} \cdot a_0 a_1 \cdots$ does not contain an infinitely repeated sequence or bi-infinite sequence of labels of two adjacent sides i and $i+1$ for all $i \in \{1, \ldots, k\} - \Lambda$.*

Then the cascade $(\mathbb{G}, \tau) \simeq (X, \sigma)$.

Here again, X is not compact but is a dense subset of a compact $\tilde{X}$ which is a mixing SFT.

2.3. Bratteli diagrams and Vershik maps. Bratteli diagrams were introduced to encode finite-dimensional C^*-algebras, and got a dynamical interpretation by Vershik's approach, which associated ordered Bratteli diagrams with a lexicographic map called the Vershik map.

A *Bratteli diagram* is an infinite directed graph $B = (\Upsilon, E)$, where $\Upsilon = \Upsilon_0 \cup \Upsilon_1 \cup \Upsilon_2 \cup \cdots$ is the vertex set and $E = E_1 \cup E_2 \cup \cdots$ is the edge set comprising a union of disjoint finite sets with the following conditions:

(1) $\Upsilon_0 = \{\upsilon_0\}$;
(2) Υ_n and E_n are finite sets for all $n \in \mathbb{N}$ and
(3) there exists a range map $r : E \to \Upsilon$ with $r(E_n) \subset \Upsilon_n$ and a source map $s : E \to Y$ with $s(E_n) \subseteq \Upsilon_{n-1}$ for $n \in \mathbb{N}$. Furthermore, $s^{-1}(\upsilon) \neq \emptyset$ for all $\upsilon \in \Upsilon$ and $r^{-1}(\upsilon) \neq \emptyset$ for all $\upsilon \in \Upsilon \setminus \Upsilon_0$.

Thus, there are no *leaf* vertices in the traditional sense. Here, Υ_i represents the ith level of B and E_i is the collection of all edges from Υ_i to Υ_{i+1}. We draw Bratteli diagrams in a natural downward sense with Υ_0 at the top and further levels appearing as we move down. This gives a natural direction to the edges in the graph. If we follow any finite or infinite sequence of edges, say (e_i) (we would prefer a more coalesced notation $e_1 e_2 \cdots$ to represent infinite paths) with $r(e_i) = s(e_{i+1})$, it gives us a finite or an infinite path, respectively, on a Bratteli diagram.

Here, A_n is the $|\Upsilon_{n+1}| \times |\Upsilon_n|$th incidence matrix of (Υ, E), where each row of A_n gives the number of edges between each vertex in Υ_{n+1} and each vertex in Υ_n.

Given $e_k \in E_k, e_{k+1} \in E_{k+1}, \ldots, e_{k+m} \in E_{k+m}$ such that $r(e_i) = s(e_{i+1})$ for $i = k, k+1, \ldots, k+m-1$, we call the sequence $e_k, \ldots, e_{k+m}$ a path in (Υ, E) starting at $s(e_k) \in \Upsilon_{k-1}$ and terminating at $r(e_{k+m}) = \upsilon_{k+m}$. For $i < j$ with $\upsilon \in \Upsilon_i$ and $\mu \in \Upsilon_j$, we denote the set of all paths from υ to μ by $E(\upsilon, \mu)$.

Let $\mathbb{P}$ denote the associated infinite path space, that is,

$$\mathbb{P} = \{(e_1, e_2, \ldots) : e_i \in E_i, r(e_i) = s(e_{i+1});\ i = 1, 2, \ldots\}.$$

Topologize $\mathbb{P}$ by a basis of cylinder sets

$$[e_1, e_2, \ldots, e_k] = \{(f_1, f_2, \ldots) \in \mathbb{P} : f_i = e_i, 1 \leq i \leq k\}.$$

This gives $\mathbb{P}$ the product topology.

Each $[e_1, \ldots, e_k]$ is also closed, as is easily seen, and so $\mathbb{P}$ becomes a compact Hausdorff space with a countable basis of clopen sets, that is, a zero-dimensional space. The $\mathbb{P}$ with this topology is called the *Bratteli compactum* associated with $B = (\Upsilon, E)$.

An ordered Bratteli diagram $B = (\Upsilon, E, <)$ is a Bratteli diagram (Υ, E) together with a partial order '$<$' on E, so that edges $e, e' \in E$ are *comparable* if and only if $r(e) = r(e')$, that is, we have a linear order on each set $r^{-1}(\upsilon)$, where $\upsilon \in \Upsilon \setminus \Upsilon_0$. Two finite paths are *comparable* if they have a common source and the same length, that is, a path $p = (e_1, \ldots, e_l)$ precedes $p' = (e'_1, \ldots, e'_l)$ if there exists an index $1 \leq i \leq l$ such that $e_j = e'_j$ for all $j > i$ (then $s(e_i) = s(e'_i)$) and $e_i < e'_i$. This gives us a partial order on $\mathbb{P}$. Two paths $(e_i)_{i\in\mathbb{N}}$, $(f_i)_{i\in\mathbb{N}} \in \mathbb{P}$ are *comparable* if there exists j such that $e_i = f_i$ for all $i > j$, that is, they agree from some place downwards on B. If $(e_i)_{i\in\mathbb{N}}$ and $(f_i)_{i\in\mathbb{N}} \in \mathbb{P}$ are *comparable* with a j such that $e_i = f_i$ for all $i > j$ and $e_j > f_j$, then $(e_i)_{i\in\mathbb{N}} > (f_i)_{i\in\mathbb{N}}$. Consider $(e_1, e_2, \ldots) \in \mathbb{P}$, then its *successor* (if it exists) is defined as the path $(e^0_1, e^0_2, \ldots, e^0_{k-1}, \bar{e}_k, e_{k+1}, e_{k+2}, \ldots)$, where the index $k = \min\{n \geq 1 : e_n$ is not maximal$\}$. And let $\bar{e}_k$ be the successor of e_k in the set $r^{-1}(r(e_k))$ and $(e^0_1, e^0_2, \ldots, e^0_{k-1})$ be the minimal path in $E(\upsilon_0, s(\bar{e}_k))$. In turn, $(e_1, e_2, \ldots)$ is called the *predecessor* of $(e^0_1, e^0_2, \ldots, e^0_{k-1}, \bar{e}_k, e_{k+1}, e_{k+2}, \ldots)$.

A path is called *maximal (minimal)* if it has no successor (predecessor). We denote the set of all maximal (minimal) paths as $\mathbb{P}_{\max}$ ($\mathbb{P}_{\min}$).

With the ordered Bratteli diagram $B = (\Upsilon, E, <)$ and the associated path space $\mathbb{P}$, the lexicographically defined homeomorphism ϕ_V on $\mathbb{P} \setminus \mathbb{P}_{\max}$, mapping each nonmaximal path to its successor is called the *Vershik map*. Its range is the set of all nonminimal paths.

For an invertible zero-dimensional system (X, T), we say that an ordered Bratteli diagram B is a *decisive BV model* for (X, T) if the Vershik map ϕ_V allows a unique continuous extension $\phi_{\mathbb{P}}$ to $\mathbb{P}$ such that $(\mathbb{P}, \phi_{\mathbb{P}})$ and (X, T) are topologically conjugate.

A compact, invertible dynamical system (Y, g) is called *Bratteli–Vershikisable* if it admits a decisive BV model with an associated ordered Bratteli diagram $B = (\Upsilon, E, <)$ and path space $\mathbb{P}$, together with homeomorphism $\phi_{\mathbb{P}}$, and is conjugate to $(\mathbb{P}, \phi_{\mathbb{P}})$. In such a situation, $(\mathbb{P}, \phi_{\mathbb{P}})$ is called a *decisive BV model* of (Y, g).

THEOREM 2.5 [5]. *A compact, invertible zero-dimensional system (Y, g) is Bratteli–Vershikisable if and only if either the set of aperiodic points is dense or its closure misses one periodic orbit.*

3. Main results

We describe Bratteli models for billiards in hyperbolic polygons and show that their dynamics can be described by a Vershik map.

The case of compact rational polygons follows the known results. However, in the ideal and semi-ideal polygon cases, the system is not compact. Hence, we take forward iterates here, which can be realised as compact and for such a forward case, we generate the Vershik map.

3.1. The case of compact rational polygons. The coding rules for the case of compact rational polygons are described in Theorem 2.3.

The map τ on the space $\mathbb{G}$ of pointed geodesics is a homeomorphism. Additionally, we see that $(\mathbb{G}, \tau) \simeq (X, \sigma)$, where (X, σ) is a mixing subshift of finite type. Hence, by Theorem 2.5, (X, σ) and hence $(\mathbb{G}, \tau)$ is Bratteli–Vershikisable.

3.2. The case of ideal polygons. We note that for the ideal case, again the map τ on the space $\mathbb{G}$ of pointed geodesics is a homeomorphism. Additionally, we see that $(\mathbb{G}, \tau) \simeq (X, \sigma)$, where (X, σ) is a dense subset of a mixing subshift of finite type $(\tilde{X}, \sigma)$. Here, (X, σ) is not compact, and so the previous studies on Bratteli models cannot be directly applied here.

However, we can consider such models in this case by restricting $\mathbb{G}$ and hence X here.

3.2.1. Forward pointed geodesics. Since the dynamics for billiards follows the specular rule, the forward iterates can well define the backward iterates. Thus, there is no harm in considering only the forward orbits as the complete picture is well depicted by just considering the forward case; the backward iterates can be back traced uniquely. Hence, we consider only the forward pointed geodesics here.

So we are interested in the trajectories starting at a point, thereby we can truncate their pasts and consider only the future. Thus, the *forward billiard trajectories* can be represented as

$$\gamma = \{(\theta_n, \phi_n)\}_{n \in \mathbb{Z}_+},$$

where

$$(\theta_n, \phi_n) = T(\theta_{n-1}, \phi_{n-1}) \quad \text{for all } n \geq 1.$$

Let $\gamma = \{(\theta_n, \phi_n)\}_{n \in \mathbb{Z}_+}$ be a forward billiard trajectory in an ideal polygon Π in $\mathbb{D}$. We call (θ_0, ϕ_0) a *base arc*. Once a base arc is declared, all the segments of γ with $n < 0$ 'fade' away from it in the sense that we only take the future arcs starting at instant 0 into consideration. We note that the base arcs are compact subsets of $\mathbb{D}$. A base arc determines uniquely the forward billiard trajectory under the impositions of the reflection map T.

DEFINITION 3.1. For the base arc (θ, ϕ) defining $\gamma = \{(T^n(\theta, \phi))\}_{n \in \mathbb{Z}_+}$, we call $(\gamma, (\theta, \phi))$ a *forward pointed geodesic*.

Thus, a forward pointed geodesic $(\gamma, (\theta, \phi))$ is identified with the element

$$(\theta, \phi)(T(\theta, \phi)) \cdots \in \mathcal{K}(\mathbb{D})^{\mathbb{N}}.$$

DEFINITION 3.2. Define

$$\mathbb{G}^+ = \mathbb{G}^+_\Pi := \{(\gamma, (\theta, \phi)) : \gamma = \{(T^n(\theta, \phi))\}_{n\in\mathbb{Z}_+}\},$$

as the *space of all forward pointed geodesics* on Π.

Here, $\mathbb{G}^+ \subseteq \mathcal{K}(\mathbb{D})$ and so $\mathbb{G}^+$ can be equipped with the natural Hausdorff metric d_H, and so is endowed with the Hausdorff topology.

Define a map $\tau : \mathbb{G}^+ \to \mathbb{G}^+$ with its action on $\mathbb{G}^+$ described as

$$\tau((\gamma, (\theta, \phi))) = (\gamma, T(\theta, \phi)) \quad \text{for all } (\gamma, (\theta, \phi)) \in \mathbb{G}^+.$$

Under the action of the map τ, if we move to a new base arc (θ_m, ϕ_m), then to realise the new forward pointed geodesic, we drop from consideration all the arcs (θ_n, ϕ_n) from γ with $n < m$. Here, (θ_m, ϕ_m) acts as the base arc for the newly obtained forward pointed geodesic.

The metric on $\mathbb{G}^+$ is defined by $d_{\mathbb{G}^+} : \mathbb{G}^+ \times \mathbb{G}^+ \to \mathbb{R}$ as

$$d_{\mathbb{G}^+}((\gamma, (\theta, \phi)), (\gamma', (\theta', \phi'))) = \max\{d_{\partial\mathbb{D}}(\theta, \theta'), d_{\partial\mathbb{D}}(\phi, \phi')\},$$

where $d_{\partial\mathbb{D}}$ is defined as

$$d_{\partial\mathbb{D}}(\phi_1, \phi_2) = |\phi_1 - \phi_2|.$$

The Hausdorff topology on $\mathbb{G}^+$ is the same as the topology on $\mathbb{G}^+$ given by $d_{\mathbb{G}^+}$ (follows from [11]). Note that d_H on $\mathbb{G}^+$ can be expressed as

$$\begin{aligned} d_H((\gamma, (\theta, \phi)), (\gamma', (\theta', \phi'))) &:= d_H((\theta, \phi), (\theta', \phi')) \\ &= \max\Big\{ \sup_{Q\in(\theta,\phi)} d(Q, (\theta', \phi')), \sup_{Q\in(\theta',\phi')} d(Q, (\theta, \phi))\Big\}. \end{aligned}$$

Additionally, $\mathbb{G}^+$ is not compact in the Hausdorff topology.

We start with a finite set of symbols and deal with the collections of sequences of these symbols that are closed under the *shift map*. Here, we are interested in one-sided shift spaces. We consider a finite set $\mathcal{A}$, called the *alphabet*, and equip it with the discrete topology. Thus, $\mathcal{A}$ is compact and by Tychonoff's theorem, $\mathcal{A}^{\mathbb{N}}$ with the product topology is also compact. The *shift map* σ on the shift space $\mathcal{A}^{\mathbb{N}}$ maps a point $x = (x_i)_{i\in\mathbb{N}}$ to the point $\sigma(x)$ whose ith coordinate is

$$(\sigma(x))_i = x_{i+1}.$$

The shift map is continuous and the pair $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is called the *one-sided shift space*. A *subshift* is a closed, invariant (that is, $\sigma(\Sigma) \subseteq \Sigma$) subset $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$ together with the shift map as the restriction of σ on Σ. Additionally, Σ being closed is also compact.

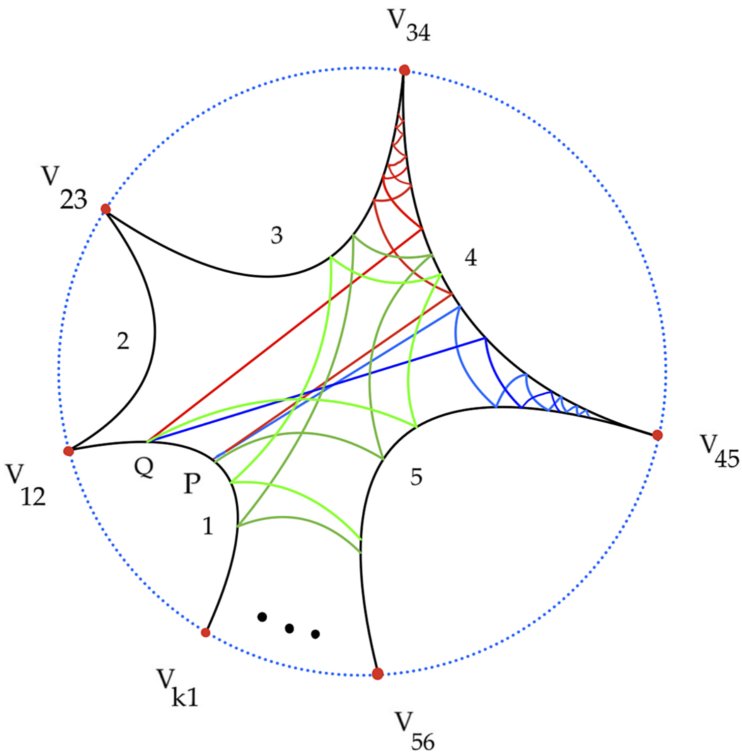

FIGURE 1. Forward trajectories in $\mathbb{D}$.

For a natural number k greater than 2, we fix k distinct vertices on the boundary of $\mathbb{D}$. This determines an ideal polygon Π for the given vertices. Label the vertices via an anti-clockwise ordering on the boundary by the symbols $V_{i,i+1}$ taking $(k+1) \equiv 1$. The side defined by the vertices $V_{i-1,i}$ and $V_{i,i+1}$ is labelled i. Fix a point P on any fixed side, say l. Given P, Π and a fixed trajectory γ_0 originating from P, an arbitrary trajectory in the neighbourhood of γ_0 can be described by displacement from P along l (see Figure 1).

On the ideal polygon Π on a fixed side, generating forward trajectories from a fixed point P is an overkill as the corresponding space of codes is preserved on varying the point P on the side. To start with, we deduce the coding rules for such forward trajectories. The characterisation of $\mathbb{G}^+$ can be pulled out using the characterisation of pointed geodesics that have been worked out in [11].

The map τ defined on the space $\mathbb{G}$ is a homeomorphism and Theorem 3.2 gives a conjugacy with a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ which is dense in the subshift $\tilde{X} \subseteq \mathcal{A}^{\mathbb{Z}}$. Thus, our space of forward pointed geodesics $\mathbb{G}^+$ should then be conjugate to $X^+ \subseteq \mathcal{A}^{\mathbb{N}}$ obtained by considering only the forward iterates in X.

By considering only forward pointed geodesics, we are also able to consider those trajectories that are asymptotic to infinity since they also take a path along $\mathcal{K}(\mathbb{D})^{\mathbb{N}}$, and so remain on the billiard table for the whole time. Such trajectories are valid billiard trajectories and so need to be also considered.

For the fixed ideal polygon Π, we denote the collection of all forward pointed geodesics asymptotic to its vertices by $\mathbb{G}^{+\infty}$. We can partition $\mathbb{G}^{+\infty}$ into the collections of the forward pointed geodesics associated with each vertex. We label the set of all forward pointed geodesics associated with a vertex $V_{i,i+1}$ as $\mathbb{G}^{+\infty}_{i,i+1}$. Thus,

$$\mathbb{G}^{+\infty} = \bigcup_i \mathbb{G}^{+\infty}_{i,i+1}.$$

We can also split $\mathbb{G}^+$ further into $\mathbb{G}^+_j$ terms, where each $\mathbb{G}^+_j$ represents the forward pointed geodesics that start from a side with label j, with $j \in \{1, 2, \dots, k\}$. The shift space conjugate to each $\mathbb{G}^+_j$ is labelled $X_{\mathbb{G}^+_j}$, where $X_{\mathbb{G}^+_j} = \{(x_i) : x_1 = j\}$ along with the restrictions imposed by Theorem 2.2.

As the counterparts to $\mathbb{G}^+_j$, we define $\mathbb{G}^{+\infty}_j$ as the space of those forward pointed geodesics that start from a side j and enter any vertex of Π. Further, $\mathbb{G}^{+\infty}$ can be split into $\mathbb{G}^{+\infty}_{i,i+1}$ terms, where $\mathbb{G}^{+\infty}_{i,i+1}$ represents the space of forward pointed geodesics ending in vertex $V_{i,i+1}$. We combine $\mathbb{G}^+$ and $\mathbb{G}^{+\infty}$ into what is called the *full space of forward pointed geodesics* attached to a given Π which is labelled as $\widehat{\mathbb{G}}^+$, that is, $\widehat{\mathbb{G}}^+ = \mathbb{G}^+ \cup \mathbb{G}^{+\infty}$. Same goes for those associated with a particular side, that is, $\widehat{\mathbb{G}}^+_j = \mathbb{G}^+_j \cup \mathbb{G}^{+\infty}_j$.

Once we have the rules for the codes of the pointed geodesics associated with the nonvertex trajectories of an ideal polygon, we can pull out from them specifically those codes that have a fixed symbol present. This collection comprises the pointed geodesics that hit the side with the corresponding label at least once. Since the codes are reversible, we can truncate their pasts (symbols associated with the portion before the base arc) to get the codes for the forward pointed geodesics associated with vertices. We denote the symbolic space attached to $\mathbb{G}^+$ and $\mathbb{G}^{+\infty}$ by X^+ and $X^{+\infty}$, respectively. The symbolic space corresponding to the full space of forward pointed geodesics is labelled $\widehat{X}^+$, that is, $\widehat{X}^+ = X^+ \cup X^{+\infty}$. Thus, we get the following result that gives us the complete description of the symbolic space for the space of forward pointed geodesics.

THEOREM 3.1. *Let $\Pi \subset \mathbb{D}$ be an ideal polygon with anti-clockwise enumeration $1, \dots, k$ and $\mathbb{G}^+$ be the space of forward pointed geodesics on Π. Suppose X^+ be the space of all forward sequences $a_1 a_2 \cdots \in \{1, \dots, k\}^{\mathbb{N}}$ satisfying the rules:*

(1) *$a_j \neq a_{j+1}$ for all $j \in \mathbb{N}$ and*
(2) *$a_0 a_1 \cdots$ does not contain an infinitely repeated sequence of labels of two adjacent sides.*

Then the semicascade $(\mathbb{G}^+, \tau) \simeq (X^+, \sigma)$.

The proof follows by restricting the conjugacy h from Theorem 2.2.

By imposing an additional constraint $a_1 = j$, we get a full characterisation of $\mathbb{G}_j^+$. Next, we characterise the symbolic space associated with $\mathbb{G}^{+\infty}$, thereby laying the groundwork for the study of the complete space $\widehat{\mathbb{G}}^+$.

THEOREM 3.2. *Let $\Pi \subset \mathbb{D}$ be an ideal polygon with anti-clockwise enumeration $1, \ldots, k$. An equivalence class $[a_1 a_2 \cdots]$ represented by (a_j) where $a_1 a_2 \cdots \in \{1, \ldots, k\}^{\mathbb{N}}$ is in $X^{+\infty}$ if and only if:*

(1) $a_j \neq a_{j+1}$ *for all* $j \in \mathbb{N}$ *and*
(2) (a_j) *eventually ends as an infinitely repeated sequence of labels of two adjacent sides.*

In particular, $X_{i,i+1}^{+\infty}$ is determined by conditions (1) and (2), constrained by the presence of the subsequence $(i\ i+1)^\infty$ at the tail. Moreover, the semicascade $(\mathbb{G}^{+\infty}, \tau) \simeq (X^{+\infty}, \sigma)$.

PROOF. The necessity of condition (1) follows from the geometry of the ambient space. Indeed, two geodesic arcs in $\mathbb{D}$ can either overlap completely or can have at most one intersection point. If condition (1) were not true, then it would account for exactly two intersections which would be a contradiction. Now, suppose condition (2) does not hold. This would mean that either the trajectories associated are of first category, in which case the corresponding forward pointed geodesics are not in $X^{+\infty}$ by definition, or they are the ones with no infinite repetition of symbols in the future, in which case they do not converge to any vertex (refer [11]). Thus, the necessity of condition (2) follows. Now, consider $a_1 a_2 \cdots$ to be a forward sequence under conditions (1) and (2). We produce a unique forward billiard trajectory associated with it. The infinite repetition of labels of the two adjacent sides of Π determines uniquely a limit point of the unfolded trajectory, which is the common vertex to the corresponding sides, call it $V_{i,i+1}$ for the repeated symbols i and $i+1$. We can represent $a_1 a_2 \cdots$ as $a_1 a_2 \cdots j(i\ i+1)^\infty$ with $j \neq i, i+1$.

Fixing a point P on the side j by using the geodesic arcs generated by the reverse of the finite subsequence $a_1 a_2 \cdots a_j$, which respects the specular reflection rule at P, gives us a unique forward trajectory associated with $a_1 a_2 \cdots$. Thus, we get an element in $\mathbb{G}_{i,i+1}^{+\infty}$ generated by the coding rules (1) and (2) generating the subsequence $(i\,i+1)^\infty$ at its tail.

Define

$$h : (\mathbb{G}^{+\infty}, \tau) \to (X^{+\infty}, \sigma)$$

by

$$h(\gamma, (\theta, \phi)) = a_{(\theta,\phi)} a_{T(\theta,\phi)} \cdots,$$

where $a_{(\theta,\phi)} \in \{1, 2, \ldots, k\}$ denotes the label of the side of Π that the pointed geodesic $(\gamma, (\theta, \phi))$ meets.

Now $h(\gamma, (\theta, \phi)) = h(\gamma', (\theta', \phi'))$ implies $a_{(\theta,\phi)} a_{T(\theta,\phi)} \cdots = a_{(\theta',\phi')} a_{T(\theta',\phi')} \cdots$, which further implies $(a_{T^n(\theta,\phi)})_{n\in\mathbb{N}} = (a_{T^n(\theta',\phi')})_{n\in\mathbb{N}}$.

Thus, $(T^n(\theta,\phi))_{n\in\mathbb{N}} = (T^n(\theta',\phi'))_{n\in\mathbb{N}}$ and $a_{(\theta,\phi)} = a_{(\theta',\phi')}$ implies $(\gamma,(\theta,\phi)) = (\gamma',(\theta',\phi'))$. Thus, h is injective.

The surjectivity of h is established from the fact that each $(a_j)_{j\in\mathbb{N}} \in X^{+\infty}$ defines a unique forward trajectory γ as seen above. The corresponding $a_1a_2\cdots$ generates a unique base symbol a_1, which gives us a base arc (θ,ϕ) on γ, giving a unique forward pointed geodesic in $\mathbb{G}^{+\infty}$, that is,

$$h(\gamma,(\theta,\phi)) = a_1a_2\cdots .$$

Recall that $(\sigma(x))_i = x_{i+1}$ and $\tau\colon \mathbb{G}^+ \to \mathbb{G}^+$ is defined as

$$\tau((\gamma,(\theta,\phi))) = (\gamma, T(\theta,\phi)) \quad \text{for all } (\gamma,(\theta,\phi)) \in \mathbb{G}^+.$$

Thus,

$$\begin{aligned} h\circ\tau((\gamma,(\theta,\phi))) &= h(\tau((\gamma,(\theta,\phi)))) = h((\gamma,T(\theta,\phi))) = h((\gamma,(\theta_1,\phi_1))) \\ &= a_{(\theta_1,\phi_1)}a_{T(\theta_1,\phi_1)}\cdots = a_{T(\theta,\phi)}a_{TT(\theta,\phi)}\cdots = a_{T(\theta,\phi)}a_{T^2(\theta,\phi)}\cdots \\ &= \sigma(h(\gamma,(\theta,\phi))) = \sigma\circ h(\gamma,(\theta,\phi)). \end{aligned}$$

Therefore, $h\circ\tau = \sigma\circ h$, implying that h is a bijective homomorphism.

Further, consider an open set $U = B_\epsilon(\gamma,(\theta,\phi))$ in $\mathbb{G}^{+\infty}$. So, $(\gamma',(\theta',\phi')) \in U$ if and only if $d_{\partial\mathbb{D}}(\theta,\theta') < \epsilon$ and $d_{\partial\mathbb{D}}(\phi,\phi') < \epsilon$. Now tessellating $\mathbb{D}$ with the reflected copies of Π about its sides generated by γ, we can label the vertices of the ith copy of Π by $A_1^i, A_2^i, \ldots, A_k^i$. Let $h(\gamma,(\theta,\phi))_{[1,k]} = x_1x_2\cdots x_k$. Take r as the largest positive integer with each of $A_1^i, A_2^i, \ldots, A_k^i \notin (\theta-\epsilon,\theta+\epsilon)\times(\phi-\epsilon,\phi+\epsilon)$ for all $i = 0,1,\ldots,r$. Then, $h^{-1}([x_1x_2\cdots x_r]) \supset U$. This implies the continuity of h^{-1}.

Now since $h^{-1} : (X^{+\infty},\sigma) \to (\mathbb{G}^{+\infty},\tau)$ is a continuous bijection and $X^{+\infty}$ is closed and hence compact, this implies that h is a homeomorphism, establishing that h gives a conjugacy. □

Since $(\mathbb{G}^+,\tau) \simeq (X^+,\sigma)$ and $(\mathbb{G}^{+\infty},\tau) \simeq (X^{+\infty},\sigma)$, we claim that the disjoint unions on the respective sides give us the conjugacy between the disjoint unions. The metric on $\widehat{\mathbb{G}}^+$ extends naturally as $d_{\widehat{\mathbb{G}}^+} : \widehat{\mathbb{G}}^+ \times \widehat{\mathbb{G}}^+ \to \mathbb{R}$ with

$$d_{\widehat{\mathbb{G}}^+}((\gamma,(\theta,\phi)),(\gamma',(\theta',\phi'))) = \max\{d_{\partial\mathbb{D}}(\theta,\theta'), d_{\partial\mathbb{D}}(\phi,\phi')\}.$$

THEOREM 3.3. *Let $\Pi \subset \mathbb{D}$ be an ideal polygon with anti-clockwise enumeration $1,\ldots,k$ and $\widehat{\mathbb{G}}^+$ be the space of all forward pointed geodesics on Π. Suppose $\widehat{X}^+$ is the space of all forward sequences $a_1a_2\cdots \in \{1,\ldots,k\}^{\mathbb{N}}$ satisfying the rule:*

(#) $a_j \neq a_{j+1}$ for all $j \in \mathbb{N}$.

Then the semicascade $(\widehat{\mathbb{G}}^+,\tau) \simeq (\widehat{X}^+,\sigma)$.

PROOF. Define $h : \widehat{\mathbb{G}}^+ \to \widehat{X}^+$ by

$$h(\gamma,(\theta,\phi)) = a_{(\theta,\phi)}a_{T(\theta,\phi)}\cdots ,$$

where $a_{(\theta,\phi)} \in \{1, 2, \ldots, k\}$ denotes the label of the side of Π that the pointed geodesic $(\gamma, (\theta, \phi))$ meets. Indeed, the bijection follows trivially from Theorems 3.1 and 3.2.

For continuity and openness, if we consider $x \in X^+$, the continuity and openness of h follows from the corresponding properties of h restricted to X^+ as we can choose a sufficiently small neighbourhood of x such that no element of $X^{+\infty}$ is in it. Thus, we only need to take care of the case when $x \in X^{+\infty}$. Take $x \in X^{+\infty}$. Without any loss of generality, suppose $x = x_1x_2 \cdots x_m(j, j+1)^\infty$ for some $j \in \{1, 2, \ldots, k\}$. Consider an open set $U = [x_1x_2 \cdots x_m(j, j+1)^l] \in \widehat{X}^+$ for any $l \in \mathbb{N}$ then $x \in U$. No other member of $X^{+\infty}$ lies in U, except x. Further, for any $(\gamma', (\theta', \phi')) \in h^{-1}(U - \{x\})$, $d(x, h((\gamma', (\theta', \phi')))) < 2^{-(l+m+1)}$. Thus, $h^{-1}(U)$ is open, implying the continuity of h. Conversely, consider $(\gamma, (\theta, \phi)) \in \mathbb{G}^{+\infty}$. Theorem 3.2 implies that $h(\gamma, (\theta, \phi)) = y_1y_2 \cdots y_n(j, j+1)^\infty = y$ (say) for some $n \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$. Pick $\epsilon > 0$ such that for any $(\theta', \phi') \in (\theta - \epsilon, \theta + \epsilon) \times (\phi - \epsilon, \phi + \epsilon)$ with $\gamma' = (T^n(\theta', \phi'))_{n\in\mathbb{N}}$, $(\gamma', (\theta', \phi')) \notin \mathbb{G}^{+\infty}$, and let $B((\gamma, (\theta, \phi)); \epsilon)$ be an ϵ−ball centred at $(\gamma, (\theta, \phi))$. Then there exists $s \in \mathbb{N}$ such that $h^{-1}([y_{[1,s]}] - y) \in \mathbb{G}^+ \subset \widehat{\mathbb{G}}^+$. Therefore, $h^{-1}([y_{[1,s]}]) \subset B((\gamma, (\theta, \phi)); \epsilon)$ implying that h is open. □

Note that $\widehat{X}^+$ is compact and $(\widehat{X}^+, \sigma)$ is a (one-sided) mixing subshift of finite type.

COROLLARY 3.4. *For forward pointed geodesics, $\widehat{\mathbb{G}}^+ \subseteq \mathcal{K}(\mathbb{D})$ is compact and the semicascade $(\widehat{\mathbb{G}}^+, \tau)$ is conjugate to a one-sided mixing subshift of finite type.*

We can say more about the dynamics of forward pointed geodesics.

THEOREM 3.5. *Let $\Pi \subset \mathbb{D}$ be a k-sided ideal polygon with k sides labelled $1, 2, \ldots, k$ in anti-clockwise ordering and with corresponding labels $V_{1,2}, V_{2,3}, \ldots, V_{k,1}$ for the vertices. Fix a side l. Let γ_0 be a fixed trajectory originating from side l from a point A and hitting a vertex $V_{m,m+1}$. Then:*

(i) *there exists no open set U in the neighbourhood of A such that all trajectories γ originating from U hit the vertex $V_{m,m+1}$;*

(ii) *for every ϵ-neighbourhood of A, there exists infinitely many trajectories starting from A hitting each vertex.*

PROOF. Consider a vertex $V_{m,m+1}$ and a side l of Π. Then by Theorem 3.2, an arbitrary forward sequence starting from l and ending in $V_{m,m+1}$ can be taken as $\gamma_0 = h^{-1}(a_0 \cdots a_t(m, m+1)^{+\infty})$ for any arbitrary index $t \in \mathbb{N}$. Suppose γ_0 starts from a point A_l on l and U_{A_l} be an open set about A_l such that all trajectories hit $V_{m,m+1}$. Then l can be expressed as a disjoint union of $l \cap U_{A_l}$ terms and $l \cap U_0$ (from which every forward geodesic is not associated with any vertex). This leads to more than $k+1$ boundary points on l that are not part of this union, giving a contradiction. Next, consider an ϵ-neighbourhood, U_ϵ of point A on l. Suppose there exists a vertex $V_{m,m+1}$ such that only finitely many forward trajectories enter $V_{m,m+1}$ after starting from U_ϵ. Then by symmetry, this is true for each vertex, which implies the collection of all forward trajectories starting from U_ϵ and ending in a vertex is finite. This gives a contradiction as this collection can be obtained from Theorem 3.2 as $\bigcup_t \bigcup_m h^{-1}(a_0 \cdots a_t(m, m+1)^{+\infty})$, which is infinite. □

3.2.2. *Bratteli–Vershikisability.*

DEFINITION 3.3. A semicascade (Y, g) is called *Bratteli–Vershikisable* if it can be realised as a canonical factor of a zero-dimensional compact, invertible (Z, T) that admits a decisive BV model.

We see that the Bratteli diagrams can be set up in the case of forward pointed geodesics $(\widehat{\mathbb{G}}^+, \tau)$ for the ideal polygons in $\mathbb{D}$ with respect to $(\widehat{\mathbb{X}}^+, \sigma)$. The correspondence between the two comes from the coding rules described in the previous section.

Consider an ideal polygon Π in the hyperbolic plane with its sides labelled by $1, 2, \ldots, k$. We establish a decisive BV model, whose quotient space corresponds to $\widehat{\mathbb{G}}^+$.

For the finite alphabets $\mathcal{A}_1, \mathcal{A}_2, \ldots$, an *array system* is defined as a closed, shift-invariant subset of $\Pi_k \mathcal{A}_k^{\mathbb{Z}}$. A typical element of an array system is represented as $x = [x_{k,n}]_{k\in\mathbb{N}, n\in\mathbb{Z}}$, where $x_{k,n} \in \mathcal{A}_k$. The action on this space is described by the map σ, where $\sigma(x) = [x_{k,n+1}]_{k\in\mathbb{N}, n\in\mathbb{Z}}$. Thus, every array system is a zero-dimensional system. The converse also holds true [5].

The utility of the array systems comes via additional symbols called *markers*. We place short vertical bars in a particular row of x to the left of the symbol at position n, if $\sigma^n(x) \in F$, where F is a clopen set. This provides us a conjugate *array representation with markers* for any zero-dimensional, compact, invertible system (Y, T). We can think of this representation of (Y, T) as an array system built upon enlarged alphabets $\mathcal{A}_k^* = \{\emptyset, |\} \times \mathcal{A}_k$. More details on array systems with markers can be found in [4, 5]. We follow their method here.

Next we give the terminology used in [5] that we need in the design of a decisive BV model. A *k-block* is a block of symbols between two adjacent markers appearing in the kth row of some array $x \in Y$. The embracing markers are also included in the block and while concatenating two $k-$ blocks, we 'glue' the markers meeting at the contacts. A *k-rectangle* is a rectangular block of symbols comprising stacked blocks from row 1 to k. A *k-trapezoid* is a pattern appearing in some array $x \in Y$, consisting of a k-rectangle enlarged in rows 1 through $k-1$ by two $(k-1)$-rectangles (placing one on each side), then, in rows 1 through $k-2$ by two $(k-2)$-rectangles (placing one on each side) and so on. Each $(k+1)$-rectangle R has a concatenation of k-rectangles, say $R_1, \ldots, R_q$ appearing in its top k rows, whereas a $(k+1)$-trapezoid S extending R has in its top k rows an 'overlapping concatenation' of $q+2$ k-trapezoids, say $S_0, \ldots, S_{q+1}$. The trapezoids $S_1, \ldots, S_q$ are called *internal* while S_0 and S_{q+1} are called *external*. Note that the internal k-trapezoids extend the k-rectangles included in R, whereas the external ones do not.

Now consider a zero-dimensional, compact, invertible system (Y, T). We recall the method in [5] to form a decisive BV model corresponding to (Y, T). The level 0 consists of vertex υ_0 and for $k \geq 1$, Υ_k consists of all possible k-trapezoids occurring in the array representation corresponding to Y. The edges to a target $(k+1)$-trapezoid S connect it to all of its internal k-trapezoids, ordered corresponding to the natural order as in the 'overlapping concatenation'. A typical infinite path on this diagram starting at υ_0 and developing along the nested sequence of congruent and growing k-trapezoids corresponds to an array representation of $x \in Y$, setting up a homeomorphism between the corresponding path space $\mathbb{P}$ and Y.

Recall that an equivalence relation $\frown$ on Z is a closed subset of $Z \times Z$. Let (Z, T) be a dynamical system. Then the equivalence relation $\frown$ on Z gives a *canonical factor* $Z/\frown$ with the factor map $f : Z \to Z/\frown$ given as $f(x) = f(y)$ if and only if $x \frown y$.

THEOREM 3.6. *Let* $\Pi \subset \mathbb{D}$ *be an ideal polygon with anti-clockwise enumeration* $1, \ldots, k$ *and* $\widehat{\mathbb{G}}^+$ *be the space of forward pointed geodesics on* Π. *Then the semicascade* $(\widehat{\mathbb{G}}^+, \tau)$ *is topologically conjugate to a canonical factor of a BV model.*

PROOF. We recall that $(\widehat{\mathbb{G}}^+, \tau) \simeq (\widehat{X}^+, \sigma)$ and note that $(\widehat{X}^+, \sigma)$ is an SFT corresponding to the matrix M. We take the full SFT corresponding to the matrix M and note that it is the same as the SFT for $(\tilde{X}, \sigma)$ as defined in §2. Now recalling the methods in Theorem 2.5, we have a Vershik system given by $(\mathbb{P}, \phi_{\mathbb{P}})$, corresponding to the associated ordered Bratteli diagram $(B, E, <)$. Now let $\psi : \mathbb{P} \to \tilde{X}$ be the corresponding homeomorphism.

Define an equivalence relation on $\tilde{X}$ given by $\cdots x_{-1} \cdot x_0 x_1 \cdots \frown \cdots y_{-1} \cdot y_0 y_1 \cdots$, if and only if $x_i = y_i$ for all $i \geq 0$. The map $\bar{\sigma} : \tilde{X}/\frown \to \tilde{X}/\frown$ between the equivalence classes, defined by $\bar{\sigma}([\cdots x_{-1} \cdot x_0 x_1 \cdots]) = [\cdots x_{-1} x_0 \cdot x_1 \cdots]$, is a homeomorphism. Then we have $\tilde{X}/\frown \simeq \hat{X}^+$ via the conjugacy map $\zeta : \tilde{X}/\frown \to \hat{X}^+$ defined by $\zeta([\cdots x_{-1} . x_0 x_1 \cdots]) = x_0 x_1 \cdots$.

Now using the above equivalence relation, we define an equivalence relation $\approx$ on $\mathbb{P}$, where $e_1 e_2 e_3 \cdots \approx f_1 f_2 f_3 \cdots$ if and only if $\psi(e_1 e_2 e_3 \cdots) \frown \psi(f_1 f_2 f_3 \cdots)$. Define $\psi^+ : \mathbb{P}/\approx \; \to \tilde{X}/\frown$ between the equivalence classes by $\psi^+([e_1 e_2 e_3 \cdots]) = [\psi(e_1 e_2 e_3 \cdots)]$.

Now

$$
\begin{aligned}
\psi^+([e_1 e_2 e_3 \cdots]) = \psi^+([f_1 f_2 f_3 \cdots]) &\implies [\psi(e_1 e_2 e_3 \cdots)] = [\psi(f_1 f_2 f_3 \cdots)] \\
&\implies \psi(e_1 e_2 e_3 \cdots) \frown \psi(f_1 f_2 f_3 \cdots) \\
&\implies e_1 e_2 e_3 \cdots \approx f_1 f_2 f_3 \cdots \\
&\implies [e_1 e_2 e_3 \cdots] = [f_1 f_2 f_3 \cdots],
\end{aligned}
$$

implying the injectivity of ψ^+. Also, for $[\cdots x_{-1} \cdot x_0 x_1 \cdots] \in \tilde{X}/\frown$, we have $[\psi^{-1}(\cdots x_{-1} \cdot x_0 x_1 \cdots)] \in \mathbb{P}/\approx$, such that $\psi^+([\psi^{-1}(\cdots x_{-1} \cdot x_0 x_1 \cdots)]) = [\cdots x_{-1} \cdot x_0 x_1 \cdots]$, implying the surjectivity of ψ^+. Thus, ψ^+ is a bijection.

Define $\phi_{\mathbb{P}}^+ : \mathbb{P}/\approx \; \to \; \mathbb{P}/\approx$ as $\phi_{\mathbb{P}}^+ = {\psi^+}^{-1} \circ \sigma \circ \psi^+$. Thus, ψ^+ commutes with $\phi_{\mathbb{P}}^+$ and σ. Consider V open in $\tilde{X}/\frown$. Label $\{x \in \tilde{X} : [x] \in V\}$ as W. Then W is open in $\tilde{X}$. This implies that there exists an open set U in $\mathbb{P}$, such that $\psi(U) \subset W$. Then the image $\psi^+(\{[u] : u \in U\}) \subset V$. This implies that ψ^+ is continuous. Thus, ψ^+ is a continuous bijection on a compact, Hausdorff space and thereby is a homeomorphism. □

The following diagram sums up all the commuting relationships between different spaces:

$$
\begin{array}{ccccccc}
\mathbb{P} & \xrightarrow{\psi} & \tilde{X} & & & & \\
\phi_{\mathbb{P}}\downarrow & & \downarrow\sigma & & & & \\
\mathbb{P} & \xrightarrow{\psi} & \tilde{X} & & & & \\
\approx\downarrow & & \downarrow\frown & & & & \\
\mathbb{P}/\approx & \xrightarrow{\psi^+} & \tilde{X}/\frown & \xrightarrow{\zeta} & \widehat{X}^+ & \xrightarrow{h^{-1}} & \widehat{\mathbb{G}}^+ \\
\phi^+_{\mathbb{P}}\downarrow & & \downarrow\bar{\sigma} & & \downarrow\sigma & & \downarrow\tau \\
\mathbb{P}/\approx & \xrightarrow{\psi^+} & \tilde{X}/\frown & \xrightarrow{\zeta} & \widehat{X}^+ & \xrightarrow{h^{-1}} & \widehat{\mathbb{G}}^+
\end{array}
$$

EXAMPLE 3.7. We describe an ordered decisive Bratteli diagram for our system $(\tilde{X}, \sigma)$ corresponding to an ideal polygon with three sides, which is a mixing SFT. For any $x \in \tilde{X}$, we generate the array representation by stacking x in every row. The markers are placed by endowing the set of all possible blocks of a given length with the lexicographical order. For the array corresponding to $x \in \tilde{X}$, we place a marker to the left of the symbol at position n in the kth row if and only if there exists $i \in [n-k+1, n]$, such that the block $x[n, n+k)$ dominates the blocks $x[i, i+k), x[i+1, i+k+1), \ldots, x[i+k-1, i+2k-1)$.

We next describe some initial levels of the Bratteli diagram for $\tilde{X}$.

In Υ_1, there are markers at all horizontal positions. Thus, Υ_1 has three vertices: $|1|$, $|2|$ *and* $|3|$ (see Figure 2).

In Υ_2, we have 13 vertices corresponding to the following trapezoids:

$$
\begin{array}{c|c|c} 1 & 2 & 3 \\ & 2 & \end{array},\quad
\begin{array}{c|c|c} 2 & 3 & 2 \\ & 3 & \end{array},\quad
\begin{array}{c|c|c} 1 & 3 & 2 \\ & 3 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 2 \mid & 1 & 2 \\ & 2 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 2 & 3 \mid & 2 & 3 \\ & 3 & 2 & \end{array},
$$

$$
\begin{array}{c|cc|c} 1 & 3 \mid & 1 & 3 \\ & 3 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 2 \mid & 1 & 3 \\ & 2 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 3 & 2 \mid & 1 & 2 \\ & 2 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 3 \mid & 2 & 3 \\ & 3 & 2 & \end{array},
$$

$$
\begin{array}{c|cc|c} 2 & 3 \mid & 1 & 2 \\ & 3 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 2 & 3 \mid & 1 & 3 \\ & 3 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 3 & 2 \mid & 1 & 3 \\ & 2 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 3 \mid & 1 & 2 \\ & 3 & 1 & \end{array}
$$

In Υ_3, we have 17 vertices corresponding to the following trapezoids:

$$
\begin{array}{c|c|c} 2 & 3 & 2 \\ & 3 & \\ & 3 & \end{array},\quad
\begin{array}{c|c|c} 1 & 2 & 3 \\ & 2 & \\ & 2 & \end{array},\quad
\begin{array}{c|c|c} 1 & 3 & 2 \\ & 3 & \\ & 3 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 2 \mid & 1 & 2 \\ & 2 & 1 & \\ & 2 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 3 \mid & 1 & 3 \\ & 3 & 1 & \\ & 3 & 1 & \end{array},
$$

$$
\begin{array}{c|cc|c} 2 & 3 \mid & 2 & 3 \\ & 3 & 2 & \\ & 3 & 2 & \end{array},\quad
\begin{array}{c|cc|c} 2 & 3 \mid & 1 & 2 \\ & 3 & 1 & \\ & 3 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 2 \mid & 1 & 3 \\ & 2 & 1 & \\ & 2 & 1 & \end{array},\quad
\begin{array}{c|cc|c} 1 & 3 \mid & 1 & 2 \\ & 3 & 1 & \\ & 3 & 1 & \end{array},
$$

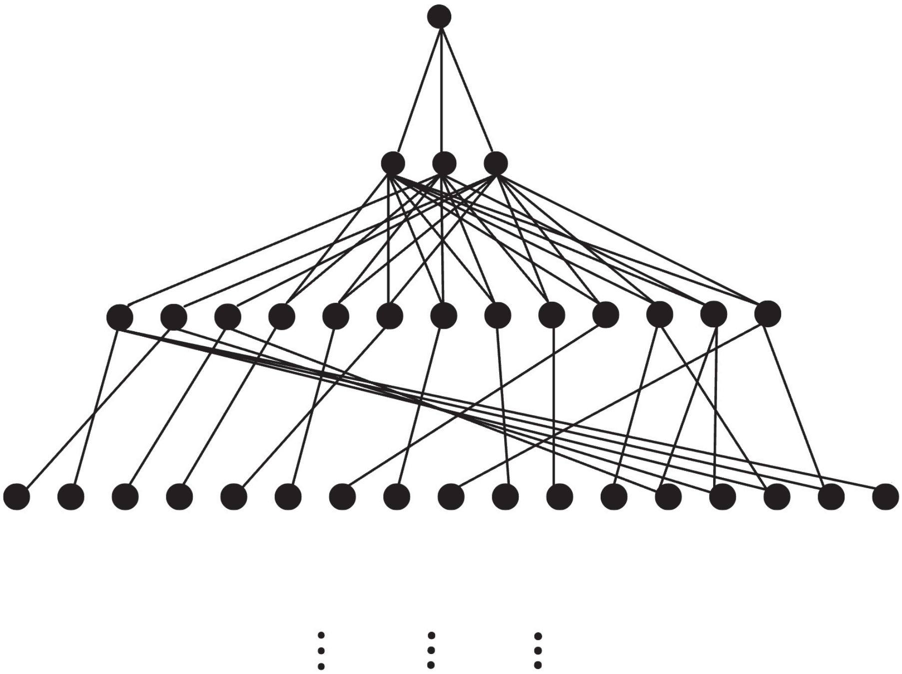

FIGURE 2. Bratteli diagram for billiards in ideal polygon with three sides.

$$3\left|\begin{array}{ccc}2 & | & 1\\ 2 & & 1\\ 2 & & 1\end{array}\right|\begin{array}{cc}2 & 1\\ \multicolumn{2}{c}{,}\\ & \end{array}\left|\begin{array}{ccc}3 & | & 2\\ 3 & & 2\\ 3 & & 2\end{array}\right|\begin{array}{cc}3 & 2\\ \multicolumn{2}{c}{,}\\ & \end{array}\left|\begin{array}{ccc}3 & | & 1\\ 3 & & 1\\ 3 & & 1\end{array}\right|\begin{array}{cc}3 & 1\\ \multicolumn{2}{c}{,}\\ & \end{array}\left|\begin{array}{ccccc}3 & | & 2 & | & 1\\ 3 & | & 2 & & 1\\ 3 & & 2 & & 1\end{array}\right|\begin{array}{c}3\\ ,\\ \end{array}$$

$$2\left|\begin{array}{ccccc}3 & | & 2 & | & 1\\ 3 & | & 2 & & 1\\ 3 & & 2 & & 1\end{array}\right|\begin{array}{cc}3 & 1\\ \multicolumn{2}{c}{,}\\ & \end{array}\left|\begin{array}{ccccc}2 & | & 3 & | & 1\\ 2 & | & 3 & & 1\\ 2 & & 3 & & 1\end{array}\right|\begin{array}{cc}3 & 1\\ \multicolumn{2}{c}{,}\\ & \end{array}\left|\begin{array}{ccccc}3 & | & 1 & | & 2\\ 3 & & 1 & | & 2\\ 3 & & 1 & & 2\end{array}\right|\begin{array}{c}3\\ ,\\ \end{array}$$

$$2\left|\begin{array}{ccccc}3 & | & 1 & | & 2\\ 3 & & 1 & | & 2\\ 3 & & 1 & & 2\end{array}\right|\begin{array}{c}3\\ \\ \end{array}$$

3.3. The case of semi-ideal rational polygons. The coding rules for the case of semi-ideal rational polygons are described in Theorem 2.4.

In this case as well, the map τ is a homeomorphism. Thus, we can 'pluck' the corresponding pointed geodesic trajectories from their base arcs and generate forward pointed geodesics. Therefore, we proceed towards the generation of the corresponding Bratteli diagram, according to the method discussed above. This becomes a convolution of the previous two cases. Thus, the forward iterates of the billiards corresponding to the semi-ideal polygons described in [11] and as considered above are also *Bratteli–Vershikisable*.

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