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ANALYSIS OF AN EIGENVALUE PROBLEM ARISING IN A THERMAL WAVE PROPAGATION PROBLEM

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Abstract In this paper we study a non-standard eigenvalue problem which arises in the context of a thermal wave propagation problem, and some generalizations thereof. The eigenvalue distribution is fully explored and useful bounds on the location of the eigenvalues are obtained.

Keywords: eigenvalue problem; thermal wave propagation

2000 *Mathematics subject classification:* Primary 45C05

1. Introduction

There is a variety of industrial processes where a fuel, typically propane or butane, is burnt with the help of a catalyst. This usually results in a lower reaction temperature than is normal and will therefore produce less pollution (see [**2**,**3**], in which the modelling of the above processes is discussed in detail). If this takes place in a tube, with the catalyst coating the inner surface of the tube, then the governing equations for the gas flow and temperature (θ) can, in certain physical limits, be combined and cast into the form,

$$[c+2(1-x^2)]\frac{\partial\theta}{\partial\xi}=\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)+\frac{1}{q^2}\frac{\partial^2\theta}{\partial\xi^2}$$

for $0<x<1$, $-\infty<\xi<\infty$. The partial differentiation equation in which c and q are positive physical constants and x and ξ are coordinates along the radius and axis of the tube respectively, needs to be solved subject to the boundary conditions:

$$\frac{\partial\theta}{\partial x}=0 \quad \text{on } x=0,\ -\infty<\xi<\infty;$$
$$\frac{\partial\theta}{\partial x}=0 \quad \text{on } x=1,\ -\infty<\xi<0;$$
$$\frac{\partial\theta}{\partial x}=\mu(1-\theta) \quad \text{on } x=1,\ 0<\xi<\infty;$$
$$\theta\to 0 \quad \text{as } \xi\to-\infty,\ \theta\to 1 \quad \text{as } \xi\to+\infty \text{ for all } 0\leqslant x\leqslant 1.$$

The solution to this linear elliptic boundary value problem is complicated by the split boundary conditions on $x = 1$ and the variable coefficient in the θ_ξ term and the use of a Wiener–Hopf technique is hampered by this complication. If an eigenfunction expansion involving terms of the form $\phi(x, z) = \mathrm{e}^{\lambda\xi}\psi(x)$ is attempted in the domain $\xi < 0$, then one is faced with attempting to solve

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) + \left[\frac{\lambda^2}{q^2} - \lambda(c + 2(1 - x^2))\right]\psi = 0, \quad 0 < x < 1,$$

subject to $\psi_x(0) = \psi_x(1) = 0$.

This eigenvalue problem, which is the subject of this paper, is linear in the function ψ but nonlinear in the eigenvalue parameter λ and therefore is of a non-standard form and has some interesting and unusual properties. These are investigated in both a rigorous analytic and asymptotic manner in the first part of this paper including §§ 2 and 3. The second, rather shorter, part of the paper describes an efficient numerical method to construct the eigenvalues and eigenfunctions and confirms the asymptotic results, which are presented in § 4.

2. The eigenvalue problem

We now examine the eigenvalue problem,

$$[x\psi']' + \left[\frac{\lambda^2}{q^2} - \lambda G(x)\right]x\psi = 0, \quad x \in (0, 1), \tag{2.1}$$

$$\psi(x),\ \psi'(x) \text{ bounded as } x \to 0, \tag{2.2}$$

$$\psi'(1) = 0, \tag{2.3}$$

with λ being the eigenvalue parameter. Here $q > 0$ is a constant, and $G\colon [0, 1] \to \mathbb{R}$ is given by

$$G(x) = 2(1 - x^2) + c \quad \forall x \in [0, 1], \tag{2.4}$$

with the constant $c > 0$. We observe that

$$G(x) > 0 \quad \forall x \in [0, 1]. \tag{2.5}$$

We say that $\lambda = \lambda^* \in \mathbb{C}$ is an eigenvalue of (2.1)–(2.3) (which we henceforth refer to as (E)) if there exists a non-trivial function $\psi = \psi^*\colon [0, 1] \to \mathbb{C}$, with $\psi^* \in C[0, 1] \cap C^2(0, 1) \cap C^1(0, 1]$ such that $\psi = \psi^*$ solves (E) when $\lambda = \lambda^*$. We now examine the structure of the set of eigenvalues to (E). We begin with the following proposition.

Proposition 2.1. *The eigenvalues of* (E) *are all real.*

Proof. Let $\lambda^* \in \mathbb{C}$ be an eigenvalue of (E) with eigenfunction, $\psi^*\colon [0, 1] \to \mathbb{C}$. Then $\psi^*(x) \not\equiv 0$ on $[0, 1]$. We normalize ψ^* so that

$$\int_0^1 x\psi^*(x)\bar{\psi}^*(x)\,\mathrm{d}x = 1. \tag{2.6}$$

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Now from Equation (2.1) we obtain

$$[x\psi^{*\prime}\bar{\psi}^*]_0^1 - \int_0^1 x\psi^{*\prime}(\bar{\psi}^*)'\,\mathrm{d}x + \frac{\lambda^{*2}}{q^2}\int_0^1 x\psi^*\bar{\psi}^*\,\mathrm{d}x - \lambda^*\int_0^1 xG(x)\psi^*\bar{\psi}^*\,\mathrm{d}x = 0. \quad (2.7)$$

However, $(\bar{\psi}^*)' = \overline{(\psi^{*\prime})}$, and so, using conditions (2.2), (2.3) and (2.6), we find that λ^* must be a root of the quadratic equation

$$\Lambda^2 + b\Lambda + c = 0, \quad (2.8)$$

where

$$b = -q^2\int_0^1 xG(x)|\psi^*(x)|^2\,\mathrm{d}x < 0 \quad \text{and} \quad c = -q^2\int_0^1 x|\psi^*(x)'|^2\,\mathrm{d}x < 0 \quad (2.9)$$

Now using (2.9) we have

$$b^2 - 4c = q^4\left\{\int_0^1 xG(x)|\psi^*(x)|^2\,\mathrm{d}x\right\}^2 + 4q^2\int_0^1 x|\psi^*(x)'|^2\,\mathrm{d}x > 0,$$

and so any root $\Lambda \in \mathbb{C}$ of (2.8) must be real, and hence $\lambda^* \in \mathbb{R}$, as required. □

Thus, we may now restrict attention to (E) with $\lambda \in \mathbb{R}$. To begin with, we consider the modified problem:

$$[x\phi']' + [\gamma - \lambda G(x)]x\phi = 0, \quad x \in (0,1), \quad (2.10)$$

$$\phi(x),\ \phi'(x) \text{ bounded as } x \to 0, \quad (2.11)$$

$$\phi'(1) = 0, \quad (2.12)$$

which we will refer to as P$[\lambda]$. Here, $\gamma \in \mathbb{C}$, $\lambda \in \mathbb{R}$, and for each fixed $\lambda \in \mathbb{R}$, we will regard P$[\lambda]$ as an eigenvalue problem with eigenvalue parameter $\gamma \in \mathbb{C}$. An examination of P$[\lambda]$ shows that for each fixed $\lambda \in \mathbb{R}$, then P$[\lambda]$ is a singular Sturm–Louiville eigenvalue problem, with eigenvalue parameter $\gamma \in \mathbb{C}$. In particular, for any $\gamma \in \mathbb{C}$, let $\phi_+(x)$ and $\phi_-(x)$ be two linearly independent solutions of Equation (2.10) on $[0,1]$. Then $\phi_+(x)$ may be chosen so that it is analytic at $x = 0$, while $\phi_-(x)$ is singular at $x = 0$. Specifically, $\phi_+(x)$ may be chosen so that

$$\phi_+(x) = 1 + \frac{1}{4}\left[\lambda(2-c) - \frac{\lambda^2}{q}\right]x^2 + O(x^3) \quad \text{as } x \to 0^+,$$

while $\phi_-(x)$ has

$$\phi_-(x) = \log x + O(x^2 \log x) \quad \text{as } x \to 0^+.$$

The principle solution of Equation (2.10) is thus $\phi_+(x)$, and it is readily verified that the singular Sturm–Louiville problem P$[\lambda]$ is in the limit circle non-oscillatory class. Thus, the classical Sturm–Louiville theory (see, for example, [**1**, Chapter 8]) applies to P$[\lambda]$ for each fixed $\lambda \in \mathbb{R}$, which establishes the following statements for P$[\lambda]$.

(i) All of the eigenvalues of $\mathrm{P}[\lambda]$ are real, and form a countably infinite set, say, $\gamma_0(\lambda), \gamma_1(\lambda), \dots$, with

$$-\infty < \gamma_0(\lambda) < \gamma_1(\lambda) < \gamma_2(\lambda) < \cdots$$

and

$$\gamma_n(\lambda) \to \infty \quad \text{as } n \to \infty.$$

(ii) Each eigenvalue $\gamma = \gamma_n(\lambda)$ has a one-dimensional eigenspace with normalized eigenfunction $\phi_n \colon [0,1] \to \mathbb{R}$, such that

$$\int_0^1 x\phi_n(x,\lambda)^2 \,\mathrm{d}x = 1.$$

(iii) Distinct eigenvalues $\gamma = \gamma_n(\lambda)$ and $\gamma = \gamma_m(\lambda)$ have orthogonal eigenfunctions, that is, for $n \neq m$,

$$\int_0^1 x\phi_n(x,\lambda)\phi_m(x,\lambda) \,\mathrm{d}x = 0.$$

Now, for each fixed $n = 0, 1, 2, \dots$, we may regard $\gamma = \gamma_n(\lambda)$ as a function of $\lambda \in \mathbb{R}$, and it follows from the analytic dependence of Equation (2.10) on λ and γ that $\gamma_n \colon \mathbb{R} \to \mathbb{R}$ is continuous and has continuous derivative, that is $\gamma_n \in C^1(\mathbb{R})$. Moreover, via (i), the curves $\gamma = \gamma_n(\lambda)$ are non-intersecting in the (λ,γ)-plane. In addition, it is also straightforward to establish, via Froebenius theory, that for each $n = 0, 1, 2, \dots$, then $\phi_n(x,\lambda)$ is such that $\phi_n \in C^3([0,1] \times \mathbb{R})$, and, moreover,

$$\phi_{nx}(0,\lambda) = 0 \quad \forall \lambda \in \mathbb{R}. \tag{2.13}$$

Now, for each fixed $n = 0, 1, 2, \dots$, we examine the properties of the corresponding function $\gamma = \gamma_n(\lambda)$ in more detail. For $\gamma = \gamma_n(\lambda)$, the corresponding eigenfunction $\phi = \phi_n(x,\lambda)$ then satisfies

$$[x\phi_{nx}]_x + [\gamma_n(\lambda) - \lambda G(x)]x\phi_n = 0, \quad x \in (0,1), \tag{2.14}$$

$$\phi_{nx}(0,\lambda) = \phi_{nx}(1,\lambda) = 0. \tag{2.15}$$

Since $\phi_n \in C^3([0,1] \times \mathbb{R})$ and $\gamma_n \in C^1[\mathbb{R}]$, we can differentiate through both (2.14) and (2.15) with respect to λ to obtain

$$[x\chi_{nx}]_x + [\gamma_n(\lambda) - \lambda G(x)]x\chi_n = -[\gamma_n'(\lambda) - G(x)]x\phi_n, \quad x \in (0,1), \tag{2.16}$$

$$\chi_{nx}(0,\lambda) = \chi_{nx}(1,\lambda) = 0, \tag{2.17}$$

where $\chi_n(x,\lambda) = \phi_{n\lambda}(x,\lambda)$ for all $(x,\lambda) \in [0,1] \times \mathbb{R}$. Therefore, $\chi_n(x,\lambda)$ provides a solution to (2.16), (2.17) for any $\lambda \in \mathbb{R}$. However, we can regard (2.16), (2.17) as an inhomogeneous boundary value problem for $\chi_n(x,\lambda)$ on $x \in [0,1]$, and since $\phi_n(x,\lambda)$ is a non-trivial solution of the homogeneous part, and $\chi_n(x,\lambda) = \phi_{n\lambda}(x,\lambda)$ solves (2.16), (2.17), then the Fredholm theory (see, for example, [**4**]) demands that

$$\int_0^1 [\gamma_n'(\lambda) - G(x)]x\phi_n^2(x,\lambda)\,\mathrm{d}x = 0 \tag{2.18}$$

and so, using (ii) in (2.18), we must have

$$\gamma_n'(\lambda) = \int_0^1 G(x)x\phi_n^2(x,\lambda)\,\mathrm{d}x > 0 \tag{2.19}$$

for each $\lambda \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. Thus we have the following proposition.

Proposition 2.2. *The functions $\gamma_n\colon \mathbb{R} \to \mathbb{R}$, for each $n = 0, 1, 2, \dots$, are such that $\gamma_n(\lambda)$ are strictly monotone increasing with $\lambda \in \mathbb{R}$, and satisfy the inequalities*

$$c\lambda + \mu_n^2 \leqslant \gamma_n(\lambda) \leqslant (2+c)\lambda + \mu_n^2 \quad \textit{for all } \lambda \geqslant 0,$$

while

$$(2+c)\lambda + \mu_n^2 \leqslant \gamma_n(\lambda) \leqslant c\lambda + \mu_n^2 \quad \textit{for all } \lambda \leqslant 0.$$

Moreover,

$$\gamma_n(0) = \mu_n^2 \quad \textit{for } n = 0, 1, 2, \dots,$$

where

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots$$

are the non-negative roots of the equation $J_1(X) = 0$, with $J_1(X)$ being the Bessel function of order one.

Proof. For fixed $n = 0, 1, 2, \dots$, strict monotonicity of $\gamma_n(\lambda)$ with $\lambda \in \mathbb{R}$ follows from (2.19). We next observe that

$$c \leqslant G(x) \leqslant 2 + c \quad \text{for all } x \in [0,1]. \tag{2.20}$$

Thus, it follows from (ii) and (2.19) that

$$c \leqslant \gamma_n'(\lambda) \leqslant 2 + c \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.21}$$

Application of $\int_0^\lambda \cdots \mathrm{d}s$ to (2.21) establishes that, for $\lambda \geqslant 0$,

$$c\lambda + \gamma_n(0) \leqslant \gamma_n(\lambda) \leqslant (2+c)\lambda + \gamma_n(0). \tag{2.22}$$

Similarly, application of $\int_\lambda^0 \cdots \mathrm{d}s$ to (2.21) establishes that, for $\lambda \leqslant 0$,

$$(2+c)\lambda + \gamma_n(0) \leqslant \gamma_n(\lambda) \leqslant c\lambda + \gamma_n(0). \tag{2.23}$$

Now, when $\lambda = 0$, P[0] is

$$\begin{gathered}[x\phi']' + \gamma x\phi = 0, \quad x \in (0,1),\\ \phi(x),\ \phi'(x) \text{ bounded as } x \to 0,\\ \phi'(1) = 0,\end{gathered}$$

and it is readily established that the eigenvalues of P[0] are given by $\gamma = \mu_n^2$, $n = 0, 1, 2, \dots$, where

$$0 = \mu_0^2 < \mu_1^2 < \mu_2^2 < \mu_3^2 < \cdots$$

and μ_n, $n = 0, 1, 2, \dots$, are the non-negative zeros of the Bessel function $J_1(\mu)$. Hence, we conclude that

$$\gamma_n(0) = \mu_n^2, \quad n = 0, 1, 2, \dots,$$

and substitution into (2.22) and (2.23) completes the proof. □

Proposition 2.3. *For each $n = 0, 1, 2, \dots$, the function $\gamma_n : \mathbb{R} \to \mathbb{R}$ has a unique zero (which is simple) at, say, $\lambda = \sigma_n$, with*

$$0 = \sigma_0 > \sigma_1 > \sigma_2 > \cdots$$

and $\sigma_n \to -\infty$ as $n \to \infty$.

Proof. Fix $n = 0, 1, 2, \dots$, then it follows from Proposition 2.2 that $\gamma_n(\lambda)$ is strictly monotone increasing with $\lambda \in \mathbb{R}$, and that

$$\gamma_n(\lambda) \to \begin{cases} +\infty & \text{as } \lambda \to +\infty, \\ -\infty & \text{as } \lambda \to -\infty. \end{cases}$$

We conclude that $\gamma_n(\lambda)$ has a unique (simple, since $\gamma_n'(\lambda) > 0$) zero at, say, $\lambda = \sigma_n$. It also follows from the inequalities in Proposition 2.2 that $\gamma_0(0) = 0$ and $\gamma_n(0) > 0$ for $n = 1, 2, \dots$, and so $\sigma_0 = 0$ and $\sigma_n < 0$ for all $n = 1, 2, \dots$. Moreover, the ordering in (i) requires that

$$0 = \sigma_0 > \sigma_1 > \sigma_2 > \sigma_3 > \cdots.$$

In addition, the inequalities in Proposition 2.2 give

$$-\frac{\mu_n^2}{c} \leqslant \sigma_n \leqslant -\frac{\mu_n^2}{2+c} \quad \text{for each } n = 0, 1, 2, \dots,$$

and so

$$\sigma_n \to -\infty \quad \text{as } n \to \infty.$$

The proof is complete. □

We also have the following proposition.

Proposition 2.4. *For each $n = 0, 1, 2, \dots$, the function $\gamma_n : \mathbb{R} \to \mathbb{R}$ is such that*

$$\gamma_n(\lambda) \to \begin{cases} +\infty & \text{as } \lambda \to +\infty, \\ -\infty & \text{as } \lambda \to -\infty. \end{cases}$$

and

$$\gamma_n(\lambda) = O(\lambda) \quad \text{as } |\lambda| \to \infty.$$

Proof. This follows directly from the inequalities in Proposition 2.2. □

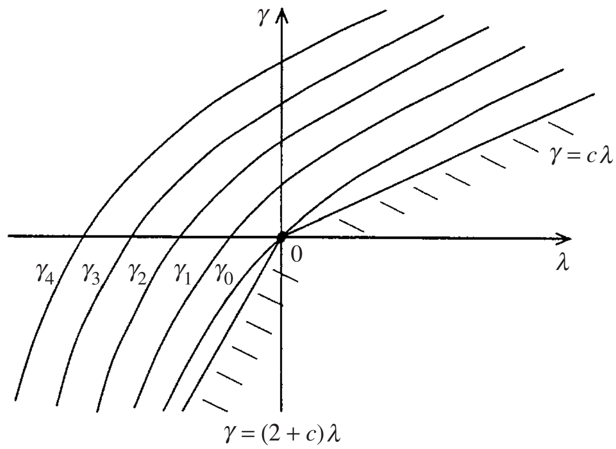

Figure 1. The functions $\gamma_n(\lambda)$, $n = 0, 1, 2, \dots$, in the (λ, γ)-plane.

We can now sketch the functions $\gamma_n(\lambda)$ for each $n = 0, 1, 2, \dots$, and this is illustrated in Figure 1.

We now state the following theorem.

Theorem 2.5. *The two-parameter eigenvalue problem (2.10)–(2.12) has a non-trivial solution if and only if*

$$(\lambda, \gamma) \in \bigcup_{n=0}^{\infty} \Lambda_n,$$

where $\Lambda_n = \{(\lambda, \gamma) \in \mathbb{R}^2 \colon \gamma = \gamma_n(\lambda)\}$.

Proof.

($\Leftarrow$) Suppose that $(\lambda^*, \gamma^*) \in \bigcup_{n=0}^{\infty} \Lambda_n$, then $\gamma^* = \gamma_n(\lambda^*)$ for some $n = 0, 1, 2, \dots$. However, $\gamma_n(\lambda^*)$ is an eigenvalue of $\mathrm{P}[\lambda^*]$, and so (2.10)–(2.12) has a non-trivial solution when $(\lambda, \gamma) = (\lambda^*, \gamma^*)$, as required.

($\Rightarrow$) Suppose for $(\lambda, \gamma) = (\lambda^*, \gamma^*)$ that (2.10)–(2.12) has a non-trivial solution. Then γ^* must be an eigenvalue of $\mathrm{P}[\lambda^*]$, and so $\gamma^* = \gamma_n(\lambda^*)$ for some $n = 0, 1, 2, \dots$, which requires $(\lambda^*, \gamma^*) \in \Lambda_n$ for some $n = 0, 1, 2, \dots$ and hence $(\lambda^*, \gamma^*) \in \bigcup_{n=0}^{\infty} \Lambda_n$.

□

Remark 2.6. For each $(\lambda, \gamma) \in \bigcup_{n=0}^{\infty} \Lambda_n$, (2.10)–(2.12) has a one-dimensional family of non-trivial solutions, which, for $(\lambda, \gamma) \in \Lambda_n$ $(n = 0, 1, 2, \dots)$, is spanned by $\psi = \phi_n(x, \lambda) \colon [0, 1] \to \mathbb{R}$.

We are now in a position to locate the eigenvalues of (E). We first require three preliminary results.

Lemma 2.7. $\lambda^* \in \mathbb{R}$ *is an eigenvalue of* (E) *if and only if* $(\lambda^*, \lambda^{*2}/q^2) \in \bigcup_{n=0}^{\infty} \Lambda_n$.

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Proof.

($\Rightarrow$) Suppose $\lambda^* \in \mathbb{R}$ is an eigenvalue of (E), then there exists a non-trivial solution $\psi^*\colon [0,1] \to \mathbb{C}$ of (2.1)–(2.3) when $\lambda = \lambda^*$. However, $\phi = \psi^*$ provides a non-trivial solution to (2.10)–(2.12) when $(\lambda, \gamma) = (\lambda^*, \lambda^{*2}/q^2)$, and so, via Theorem 2.5, $(\lambda^*, \lambda^{*2}/q^2) \in \bigcup_{n=0}^{\infty} \Lambda_n$, as required.

($\Leftarrow$) Suppose $(\lambda^*, \lambda^{*2}/q^2) \in \bigcup_{n=0}^{\infty} \Lambda_n$, then there exists a non-trivial solution $\phi^*\colon [0,1] \to \mathbb{C}$ of (2.10)–(2.12), when $\lambda = \lambda^*$ and $\gamma = \lambda^{*2}/q^2$, after which it is readily verified that $\psi = \phi^*$ provides a non-trivial solution to (2.1)–(2.3) when $\lambda = \lambda^*$. Hence, $\lambda = \lambda^*$ is an eigenvalue of (E), as required.

□

Lemma 2.8. *For each $n = 0, 1, 2, \dots$, the function $f_n\colon \mathbb{R} \to \mathbb{R}$, given by*

$$f_n(\lambda) = \gamma_n(\lambda) - \frac{\lambda^2}{q^2}, \quad \lambda \in \mathbb{R},$$

has exactly two zeros, say, $\lambda = \lambda_n^+$ and $\lambda = \lambda_n^-$, with

$$0 < \lambda_0^+ < \lambda_1^+ < \lambda_2^+ < \cdots$$

and

$$0 = \lambda_0^- > \lambda_1^- > \lambda_2^- > \cdots .$$

Moreover,

$$\lambda_n^+ \to +\infty \quad \text{and} \quad \lambda_n^- \to -\infty \quad \text{as } n \to \infty$$

while

$$\frac{q^2 c + [q^4 c^2 + 4q^2 \mu_n^2]^{1/2}}{2} \leqslant \lambda_n^+ \leqslant \frac{q^2(2+c) + [q^4(2+c)^2 + 4q^2\mu_n^2]^{1/2}}{2} \tag{2.24}$$

and

$$\frac{q^2(2+c) - [q^4(2+c)^2 + 4q^2\mu_n^2]^{1/2}}{2} \leqslant \lambda_n^- \leqslant \frac{q^2 c - [q^4 c^2 + 4q^2 \mu_n^2]^{1/2}}{2} \tag{2.25}$$

for each $n = 0, 1, 2, \dots$.

Proof. For $n = 0, 1, 2, \dots$, consider $f_n\colon \mathbb{R} \to \mathbb{R}$, given by

$$f_n(\lambda) = \gamma_n(\lambda) - \frac{\lambda^2}{q^2}, \quad \lambda \in \mathbb{R}.$$

It follows directly from Propositions 2.2–2.4 that $f_n(\lambda)$ has exactly two zeros, at, say, $\lambda = \lambda_n^+$ and $\lambda = \lambda_n^-$, with $\lambda_0^- = 0$ and $\lambda_0^+ > 0$, while $\lambda_n^- < 0$ and $\lambda_n^+ > 0$ for $n = 1, 2, \dots$. Moreover, the orderings in (i) require that

$$\cdots < \lambda_2^- < \lambda_1^- < \lambda_0^- = 0 < \lambda_0^+ < \lambda_1^+ < \lambda_2^+ \cdots .$$

In addition, the inequalities (2.24) and (2.25) are direct consequences of the inequalities in Proposition 2.2, while (2.24) and (2.25) establish that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. □

Remark 2.9. We observe from (2.24) and (2.25) that

$$\lambda_n^+ \sim q\mu_n \quad \text{as } n \to \infty,$$
$$\lambda_n^- \sim -q\mu_n \quad \text{as } n \to -\infty.$$

Lemma 2.10. *Let $A = \{(\lambda, \gamma) \in \mathbb{R}^2 \colon \gamma = \lambda^2/q^2\}$. Then*

$$A \cap \left[\bigcup_{n=0}^{\infty} \Lambda_n\right] = \{(\lambda_n^+, \lambda_n^{+2}/q^2) \colon n = 0, 1, 2, \dots\} \cup \{(\lambda_n^-, \lambda_n^{-2}/q^2) \colon n = 0, 1, 2, \dots\}.$$

Proof. Let $(\lambda^*, \gamma^*) \in A \cap [\bigcup_{n=0}^{\infty} \Lambda_n]$. Then $\gamma^* = \lambda^{*2}/q^2$ and $\gamma^* = \gamma_r(\lambda^*)$ for some $r = 0, 1, 2, \dots$. Hence,

$$\frac{\lambda^{*2}}{q^2} = \gamma_r(\lambda^*)$$

for some $r = 0, 1, 2, \dots$, and so $f_r(\lambda^*) = 0$. Thus $\lambda^* = \lambda_r^+$ or λ_r^-, so that $\gamma^* = \lambda_r^{+2}/q^2$ or λ_r^{-2}/q^2 accordingly, and we conclude that

$$(\lambda^*, \gamma^*) \in \{(\lambda_n^+, \lambda_n^{+2}/q^2) \colon n = 0, 1, 2, \dots\} \cup \{(\lambda_n^-, \lambda_n^{-2}/q^2) \colon n = 0, 1, 2, \dots\}$$

and so

$$A \cap \left[\bigcup_{n=0}^{\infty} \Lambda_n\right] \subseteq \{(\lambda_n^+, \lambda_n^{+2}/q^2) \colon n = 0, 1, 2, \dots\} \cup \{(\lambda_n^-, \lambda_n^{-2}/q^2) \colon n = 0, 1, 2, \dots\}. \tag{2.26}$$

Now consider $(\lambda_r^+, \lambda_r^{+2}/q^2)$ for any $r = 0, 1, 2, \dots$. Then clearly $(\lambda_r^+, \lambda_r^{+2}/q^2) \in A$. Also $\lambda_r^{+2}/q^2 = \gamma_r(\lambda_r^+)$, by definition, so that $(\lambda_r^+, \lambda_r^{+2}/q^2) \in \Lambda_r$. Hence,

$$(\lambda_r^+, \lambda_r^{+2}/q^2) \in A \cap \left[\bigcup_{n=0}^{\infty} \Lambda_n\right].$$

A similar conclusion follows if we start with $(\lambda_r^-, \lambda_r^{-2}/q^2)$ for any $r = 0, 1, 2, \dots$. We conclude that

$$\{(\lambda_n^+, \lambda_n^{+2}/q^2) \colon n = 0, 1, 2, \dots\} \cup \{(\lambda_n^-, \lambda_n^{-2}/q^2) \colon n = 0, 1, 2, \dots\} \subseteq A \cap \left[\bigcup_{n=0}^{\infty} \Lambda_n\right]. \tag{2.27}$$

The result follows immediately from (2.26) and (2.27). □

We now have the following theorem.

Theorem 2.11. *The set of eigenvalues of* (E) *is given by*

$$\mathcal{E} = \{\lambda \in \mathbb{R} \colon \lambda = \lambda_n^+,\ n = 0, 1, 2, \dots\} \cup \{\lambda \in \mathbb{R} \colon \lambda = \lambda_n^-,\ n = 0, 1, 2, \dots\}.$$

Proof.

($\Rightarrow$) Let λ^* be an eigenvalue of (E), then $(\lambda^*, \lambda^{*2}/q^2) \in \bigcup_{n=0}^{\infty} \Lambda_n$, via Lemma 2.7. Hence, $(\lambda^*, \lambda^{*2}/q^2) \in A \cap [\bigcup_{n=0}^{\infty} \Lambda_n]$, and so, via Lemma 2.10, $\lambda^* \in \mathcal{E}$.

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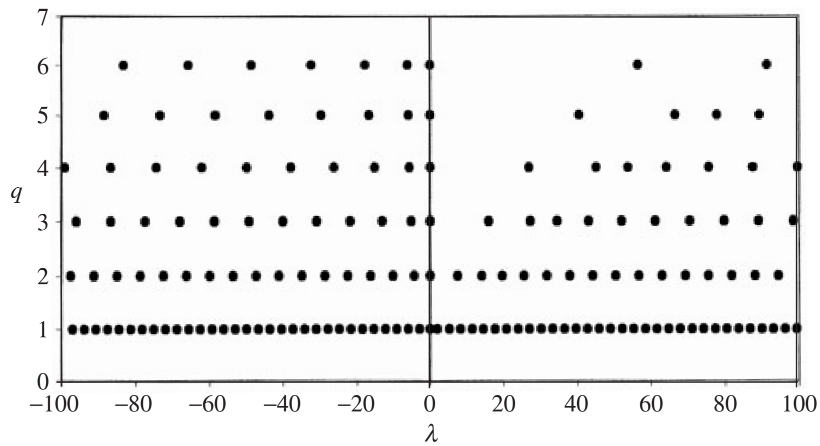

Figure 2. Eigenvalue distribution for $c = 1$ and $q = 1, 2, 3, 4, 5, 6$.

($\Leftarrow$) Let $\lambda^* \in \mathcal{E}$, then, without loss of generality, we have $\lambda^* = \lambda_r^+$ for some $r = 0, 1, 2, \dots$. It follows from Lemma 2.10 that

$$(\lambda^*, \lambda^{*2}/q^2) \in A \cap \left[\bigcup_{n=0}^{\infty} \Lambda_n\right]$$

and then, via Lemma 2.7, that λ^* is an eigenvalue of (E).

$\square$

Remark 2.12. Each eigenvalue $\lambda = \lambda_r^{\pm}$ $(r = 0, 1, 2, \dots)$ of (E) has a one-dimensional space of eigenfunctions spanned by $\psi = \phi_r(x, \lambda_r^{\pm})\colon [0, 1] \to \mathbb{R}$. We also note that the eigenfunctions $\phi_0(x, \lambda_0^{\pm})$ both have no zeros for $x \in [0, 1]$, while the eigenfunctions $\phi_r(x, \lambda_r^{\pm})$ have exactly r zeros for $x \in [0, 1]$ $(r = 1, 2, \dots)$.

3. Generalizations

The approach of the previous section can be applied directly to the following generalization of the eigenvalue problem (E), namely

$$[x\psi']' + [H(\lambda) - \lambda G(x)]x\psi = 0, \quad x \in (0, 1),$$
$$\psi(x),\ \psi'(x) \text{ bounded as } x \to 0,$$
$$\psi'(1) = 0,$$

which we will denote as (E′). Here, $G\colon [0, 1] \to \mathbb{R}$ is now such that G is analytic and strictly positive on $[0, 1]$, while $H\colon \mathbb{C} \to \mathbb{C}$ is such that $H(\mathbb{R}) \subseteq \mathbb{R}$. As in § 3, we can define functions $\gamma_n\colon \mathbb{R} \to \mathbb{R}$ $(0, 1, 2, \dots)$, which inherit all the properties of the corresponding functions defined in § 3. Following § 3, it is now straightforward to conclude that the set

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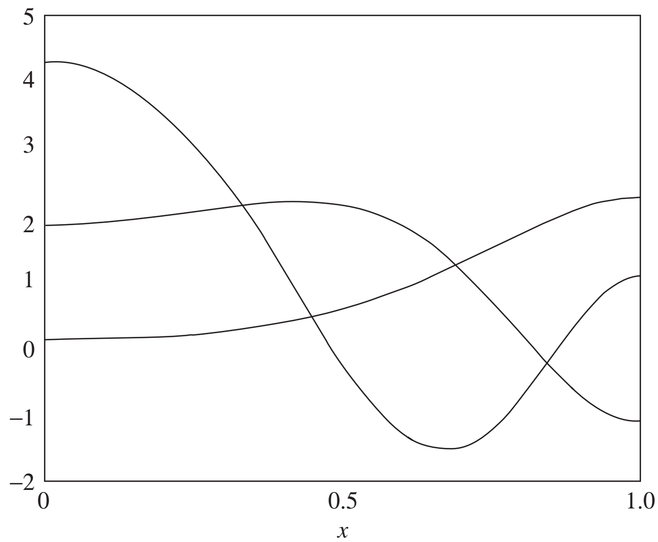

Figure 3. The normalized eigenfunctions associated with eigenvalues λ_0^+, λ_1^+ and λ_2^+ when $c = 1$ and $q = 4$.

of real eigenvalues of (E′) are precisely those values of $\lambda \in \mathbb{R}$ such that

$$\lambda \in \bigcup_{n=0}^{\infty} \mathcal{K}_n,$$

where

$$\mathcal{K}_n = \{\lambda \in \mathbb{R} \colon \gamma_n(\lambda) = H(\lambda)\}.$$

An interesting point to note is that should $H(\lambda) \equiv \gamma_r(\lambda)$, for some $r = 0, 1, 2, \dots,$ on $\lambda \in [\alpha, \beta]$, say, then λ^* is an eigenvalue of (E′) for each $\lambda^* \in [\alpha, \beta]$, and the set of real eigenvalues of (E′) is no longer discrete. Indeed, if it happens that $H(\lambda) \equiv \gamma_r(\lambda)$ for all $\lambda \in \mathbb{R}$ (some $r = 0, 1, 2, \dots$), then λ^* is an eigenvalue of (E′) for each $\lambda^* \in \mathbb{R}$. In addition, it is clearly possible, for suitable choices of the function $H(\lambda)$, to construct examples where the set $\mathcal{E}'$ of real eigenvalues of (E′) has a number of the following properties:

(i) $\mathcal{E}'$ is unbounded above and below;

(ii) $\mathcal{E}'$ is bounded below and unbounded above;

(iii) $\mathcal{E}'$ is bounded above and unbounded below;

(iv) $\mathcal{E}'$ is bounded both above and below;

(v) $\mathcal{E}'$ contains an interval of the real line;

(vi) $\mathcal{E}'$ is wholly discrete;

Table 1. *Computed eigenvalues* (λ) *for* $c = 1$ *and* $q = 4, 5, 6$ *in the range* $(-100, 100)$.

$q = 4$	$q = 5$	$q = 6$
-99.0847		
	-88.4678	
-89.6966		-83.2243
-74.3559	-73.3269	
		-65.6560
-62.0843	-58.3841	
-49.9184		-48.5985
	-43.7603	
-37.9265		
		-32.4021
	-29.6925	
-26.2477		
		-17.7634
	-16.6899	
-15.2044		
		-6.1012
	-5.9278	
-5.6526		
0	0	0
26.9486		
	40.4813	
45.0086		
53.7315		
		56.4381
64.0854		
	66.5173	
75.5442		
	77.8091	
87.4354		
	89.2618	
		91.3499
99.5462		

(vii) $\mathcal{E}'$ is wholly an interval of the real line;

(viii) $\mathcal{E}'$ has an accumulation point.

4. Numerical method and results

To numerically find the eigenvalues and eigenfunctions of (E), we first use the invariance of the differential equation under scaling transformations in ψ and the boundedness of

Table 2. *Convergence of ratio of nth eigenvalues.*

n	500	1000	2000	3000	4000
λ_n^+/λ_n^-	$-0.999\,30$	$-0.999\,64$	$-0.999\,81$	$-0.999\,88$	$-0.999\,91$

solution near the origin to turn it into an initial value problem of the form

$$\left.\begin{aligned} (xy')' + \left(\frac{\lambda^2}{q^2} - \lambda G(x)\right)xy = 0, \quad x \in (0,1),\\ y(0) = 1, \qquad y'(0) = 0. \end{aligned}\right\} \tag{4.1}$$

This can be readily integrated, using the NAG routine D02BBF, for any given λ to find $y(1)$ and $y'(1)$. An interval search for changes in sign of $y'(1)$ will give rough estimates of the positions of the eigenvalues which can be then found more precisely by Newtonian iteration. For any particular eigenvalue the eigenfunction is easily found and a simple quadrature will give $\int_0^1 xy^2(x)\,\mathrm{d}x = a$. The normalized eigenfunction can then be defined as $\tilde{y} = y/\sqrt{a}$. This scheme was efficient and successful in finding the eigenvalues for any positive values of q and c. The results of computations for the eigenvalues for q varying and c fixed at unity are shown in Figure 2 and are typical of the results we found. This figure, which should be interpreted qualitatively, demonstrates that as q increases the number of positive eigenvalues within a fixed range, $-100 \leqslant \lambda \leqslant +100$, decreases with the first strictly positive eigenvalues growing in size. There is a slight decrease in the number of negative eigenvalues as q increases but this is not so marked as with the positive ones. There is always an $O(1)$ negative eigenvalue whose size does not vary significantly as q varies. Furthermore, the negative eigenvalues are far more uniformly spaced than the positive ones. This is in line with the inequalities (2.24) and (2.25) when $q \gg 1$.

Table 1 gives precise quantitative information about the size of the eigenvalues for $q = 4, 5, 6$. The numerical results can also be used to confirm the asymptotic results $\lambda_n^{\pm} \sim \pm q\sqrt{\mu_n}$ as $n \to \infty$ to three decimal places. In Table 2 we have displayed λ_n^+/λ_n^- as a function of n when $q = 4$, and the approach is accurate to four decimal places for $n = 4000$.

Finally, Figure 3 shows the first three normalized eigenfunctions associated with eigenvalues λ_0^+, λ_1^+ and λ_2^+ for $c = 1$, $q = 4$. It is worth pointing out that the first eigenfunction has no zeros and that the second and third have one and two zeros, respectively, in $[0, 1]$, in accordance with Remark 2.12.

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