

FANO FOURFOLDS WITH LARGE ANTICANONICAL BASE LOCUS

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Abstract A famous theorem of Shokurov states that a general anticanonical divisor of a smooth Fano threefold is a smooth K3 surface. This is quite surprising since there are several examples where the base locus of the anticanonical system has codimension two. In this paper, we show that for four-dimensional Fano manifolds the behaviour is completely opposite: if the base locus is a normal surface, and hence has codimension two, all the anticanonical divisors are singular.

1. Introduction

1.1. Motivation and main result

The anticanonical system is arguably the most natural object attached to a Fano manifold, and it plays an important role in the classification of Fano manifolds of low dimension. While it is expected that the anticanonical bundle always has global sections [10, 1, 28], it is in general not globally generated. In dimension three, Shokurov’s theorem gives a complete description of the situation:

Theorem 1.1 [48]. *Let X be a smooth Fano threefold such that the base locus $\text{Bs}(|-K_X|)$ is not empty. Then the base locus is a smooth rational curve. Moreover, a general anticanonical divisor $Y \in |-K_X|$ is smooth.*

Keywords: Fano manifold; anticanonical divisor; base locus; MMP

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In dimension four, our current knowledge about the anticanonical system is very limited. An example by the first named author and Voisin shows that Shokurov's theorem does not generalise to higher dimension:

Example 1.2 [23, Example 2.12]. Let S be a smooth del Pezzo surface of degree one, and denote by $p \in S$ the unique base point of $|-K_S|$. Set $X := S \times S$, and let $Y \in |-K_X|$ be a general anticanonical divisor. Then Y is singular in the point (p, p) .

Indeed, the threefold Y contains the base locus $\text{Bs}(|-K_X|) = p \times S \cup S \times p$ which has embedding dimension four in the point (p, p) . The varieties in Example 1.2 belong to the unique family of Fano fourfolds having the maximal Picard number 18 [9, Theorem 1.1], so one might hope that the presence of singular general anticanonical divisors is a rare pathology. In this paper, we destroy this hope by showing that a sufficiently large base locus always leads to singularities:

Theorem 1.3. *Let X be a smooth Fano fourfold such that $h^0(X, \mathcal{O}_X(-K_X)) \geq 3$. Assume that the base locus $\text{Bs}(|-K_X|)$ is an irreducible normal surface. Then a general anticanonical divisor $Y \in |-K_X|$ is not \mathbb{Q} -factorial; in particular, it is singular.*

This theorem covers all the examples we know of smooth Fano fourfolds such that $\text{Bs}(|-K_X|)$ is an irreducible surface; cf. Examples 3.2, 3.3, 3.4. These examples differ from Example 1.2 in a significant way: they have moving singularities (i.e., the singular locus $Y_{\text{sing}} \subset \text{Bs}(|-K_X|)$ depends on the choice of $Y \in |-K_X|$). Theorem 1.3 is almost optimal: if the base scheme $\mathbf{Bs}(|-K_X|)$ is smooth of dimension at most one, a general anticanonical divisor Y is smooth by a strong version of Bertini's theorem [12, Corollary 2.4]. It seems likely that Theorem 1.3 still holds under the relaxed assumption that $\text{Bs}(|-K_X|)$ contains a surface. However, this additional generality leads to numerous case distinctions in our proofs, a technicality that we wanted to avoid.

While our main result states that the anticanonical geometry of a Fano fourfold is more complicated than in dimension three, the tools developed in this paper indicate that two-dimensional base loci lead to numerous restrictions on the geometry and the numerical invariants. Therefore, Theorem 1.3 is actually a first step towards the classification of Fano fourfolds with large anticanonical base locus. We plan to come back to this topic in the near future.

1.2. The setup

Let X be a smooth Fano fourfold such that $h^0(X, \mathcal{O}_X(-K_X)) \geq 3$ and $\text{Bs}(|-K_X|)$ is an irreducible normal surface B .

Let $Y \in |-K_X|$ be a general anticanonical divisor, so Y is a normal threefold (Corollary 2.8). By Kodaira, vanishing the restriction morphism

$$H^0(X, \mathcal{O}_X(-K_X)) \rightarrow H^0(Y, \mathcal{O}_Y(-K_X))$$

is surjective. Thus, we know that

$$\text{Bs}(|-K_X|_Y|) = \text{Bs}(|-K_X|) = B$$

and $h^0(Y, \mathcal{O}_Y(-K_X)) = h^0(X, \mathcal{O}_X(-K_X)) - 1 \geq 2$. Denote by $|M|$ the mobile part of the linear system $|-K_X|_Y$. Then we have a linear equivalence of effective Weil divisors

$$-K_X|_Y \simeq M + B.$$

The proof of Theorem 1.3 will be by contradiction, so we make the following assumption.

Assumption 1.4. We assume that Y is \mathbb{Q} -factorial.

Combined with the known results about anticanonical divisors (cf. Corollary 2.9), this assumption implies that the Weil divisors M and B are Cartier. By Grothendieck's Lefschetz hyperplane theorem [39, Example 3.1.25], the restriction morphism

$$\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y)$$

is an isomorphism, so there exists uniquely defined Cartier divisor classes $M_X \rightarrow X$ and $B_X \rightarrow X$ such that

$$M \simeq M_X|_Y, \quad B \simeq B_X|_Y.$$

Note that it is not clear whether the divisors M_X and B_X are effective, establishing this will be an important first step in our proof.

1.3. General strategy and structure of the proof

The proof of Shokurov's theorem 1.1 can be split into two steps, cf. [24, §2.3]: first, one shows that Y has canonical singularities. Then one applies Mayer's theorem to a minimal resolution $Y' \rightarrow Y$.

Theorem 1.5 [42, Proposition 5]. *Let S be a smooth K3 surface, and let A be a nef and big Cartier divisor on S such that $\mathrm{Bs}(|A|) \neq \emptyset$. Then we have a decomposition into fixed and mobile part*

$$A \simeq M + B,$$

where $B \simeq \mathbb{P}^1$ and M is a nef divisor that defines an elliptic fibration $\varphi_{|M|} : S \rightarrow \mathbb{P}^1$.

Since the first part of Shokurov's proof also works for Fano fourfolds [28, Theorem 5.2], the most natural approach would be to look for an analogue of Mayer's theorem for Calabi-Yau threefolds (with mild singularities). This approach immediately encounters two obstacles: while it is easy to classify fixed prime divisors on a K3 surface (they are (-2) -curves), there are many possibilities on a Calabi-Yau threefold (e.g., Enriques surfaces [7, Theorem 3.1]). Moreover, the mobile part of a linear system on a surface is always nef. This is not true on a threefold.

The second obstacle leads to a basic case distinction in our proof: if M_X is nef, the basepoint free theorem yields a morphism $X \rightarrow T$ that we can use to study the anticanonical system. This situation is close to the Examples 3.2, 3.3, 3.4, and we will need a series of rather specific classification results to settle this case in Section 7.

In the second case, M_X is not nef. We use the embedding $Y \subset X$ in a more direct way. We start by showing that $-K_X + M_X$ is nef and big (i.e., the anticanonical divisor

compensates the negativity of M_X). This allows us to show that the surface B is a complete intersection: we find an effective divisor B_X with canonical singularities such that

$$B = Y \cap B_X.$$

Moreover, using an extension theorem of Fujino, we obtain that the restriction map

$$H^0(X, \mathcal{O}_X(-K_X)) \rightarrow H^0(B_X, \mathcal{O}_{B_X}(-K_X))$$

is surjective. Combined with some classification results for linear systems on irregular surfaces (cf. Section 4), we exclude this possibility.

1.4. Future research

In Theorem 1.3, we make the assumption that $h^0(X, \mathcal{O}_X(-K_X)) \geq 3$, an assumption that is satisfied by all smooth Fano fourfolds that we are aware of.¹ Riemann-Roch computations only show that $h^0(X, \mathcal{O}_X(-K_X)) \geq 2$ (Theorem 2.8), so it would be highly desirable to have an answer to the following

Problem 1.6 [15, 37, 41]. Is there a smooth Fano fourfold X with $h^0(X, \mathcal{O}_X(-K_X)) = 2$?

Is there a smooth Calabi-Yau threefold Y with an ample Cartier divisor A such that $h^0(Y, \mathcal{O}_Y(A)) = 1$?

Beauville [6] gave an example of a numerical Calabi-Yau threefold with a fixed ample divisor. For our purpose, we are interested in strict Calabi-Yau's (i.e., Y is simply connected).

A significant problem in this theory is the lack of interesting examples. Our Example 3.4 generalises a construction from the threefold case, but it is still related to the del Pezzo surface of degree one.

Problem 1.7. Construct new examples of smooth Fano fourfolds X such that $\text{Bs}(|-K_X|)$ has dimension two.

2. Notation and basic facts

We work over \mathbb{C} ; for general definitions, we refer to [20]. All the schemes appearing in this paper are projective; manifolds and normal varieties will always be supposed to be irreducible. For notions of positivity of divisors and vector bundles, we refer to Lazarsfeld's book [39, 40]. Given two Cartier divisors D_1, D_2 on a projective variety, we denote by $D_1 \simeq D_2$ (resp. $D_1 \equiv D_2$) the linear equivalence (resp. numerical equivalence) of the Cartier divisor classes. Given a Cartier divisor D , we will denote by $\mathcal{O}_X(D)$ both the associated invertible sheaf and the corresponding line bundle. Somewhat abusively we will say that a Cartier divisor class is effective if it contains an effective divisor.

We use the terminology of [31] for birational geometry and of [34] for rational curves. We refer to [35] for the definitions and basic facts about singularities of the MMP.

¹There is an example with $h^0(X, \mathcal{O}_X(-K_X)) = 3$; see [37, Proposition 2.2][47, Table 2].

Given a normal projective surface S with Gorenstein singularities, we will denote by

$$q(S) := h^1(S, \mathcal{O}_S)$$

the irregularity, and by

$$p_g(S) := h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(K_S))$$

the geometric genus.

Definition 2.1. Let Y be a projective variety, and let D be a nef Cartier divisor on Y . The numerical dimension $\nu(Y, D)$ is defined as

$$\max\{k \in \mathbb{N} \mid D^k \neq 0\}.$$

We collect a number of basic facts for the convenience of the reader:

Fact 2.2. Let A be a Cartier divisor on a projective variety Y such that the complete linear system $|A|$ has a non-empty fixed part B . Then $h^0(Y, \mathcal{O}_Y(B)) = 1$.

Fact 2.3. Let X be a Fano manifold, and let D be a nef Cartier divisor on X . Then there exists a morphism with connected fibres $\varphi: X \rightarrow T$ and an ample Cartier divisor H_T on T such that $D \simeq \varphi^* H_T$.

Proof. By the basepoint free theorem, we have $mD \simeq \varphi^* H_m$ for some very ample divisor $H_m \rightarrow T$ for every $m \gg 0$. Thus, $H_T := H_{m+1} - H_m$ is a Cartier divisor such that $D \simeq \varphi^* H_T$. Since D is numerically the pull-back of an ample class on T , the divisor H_T is ample. \square

Fact 2.4. Let S be an irreducible projective surface with Gorenstein singularities, and let $C \subset S$ be an irreducible curve that is not contained in the singular locus of S . If $-K_S \cdot C \geq 2$, the curve C deforms in S in a positive-dimensional family.

Proof. Let $\tau: S' \rightarrow S$ be the composition of normalisation and minimal resolution. Then we have $K_{S'} \simeq \tau^* K_S - E$, where E is an effective divisor that maps into the singular locus of S . Thus, if $C' \subset S'$ is the strict transform, it is not contained in the support of E . Thus, we have

$$-K_{S'} \cdot C' \geq -K_S \cdot C \geq 2.$$

The statement now follows from [34, Theorem 1.15]. \square

Fact 2.5 [8, Lemma 2.4.1]. Let X be a normal projective variety with rational singularities such that $q(X) > 0$. Then the Albanese map $\alpha: X \rightarrow A$ to the Albanese torus exists and is determined by the Albanese map of some resolution of singularities.

Fact 2.6. Let S be a normal projective surface with rational singularities, and let A be a Cartier divisor on S . Then the Riemann-Roch formula

$$\chi(S, \mathcal{O}_S(A)) = \frac{1}{2} A^2 + \frac{1}{2} (-K_S) \cdot A + \chi(S, \mathcal{O}_S)$$

holds.

Let Y be a normal projective threefold with terminal singularities, and let A be a Cartier divisor on Y . Then the Riemann-Roch formula

$$\chi(Y, \mathcal{O}_Y(A)) = \frac{1}{12} A \cdot (A - K_Y) \cdot (2A - K_Y) + \frac{1}{12} A \cdot c_2(Y) + \chi(Y, \mathcal{O}_Y)$$

holds.

Proof. It is sufficient to verify that all the terms are well-defined and can be calculated on a resolution of singularities. This is well-known; cf. [22, Section 2]. The statement then follows via the projection formula. \square

In the rest of the paper, we will refer to the following set of statements by inversion of adjunction:

Theorem 2.7 [31, Theorem 5.50] [29, Theorem] [35, Theorem 4.9]. *Let X be a normal projective variety, and let $S \subset X$ be a reduced Weil divisor that is Cartier in codimension two. Let Δ be an effective boundary divisor on X that has no common component with S such that $K_X + S + \Delta$ is \mathbb{Q} -Cartier. Then the following holds:*

- *The pair $(X, S + \Delta)$ is lc near S if and only if the pair $(S, \Delta|_S)$ is slc.*
- *The pair $(X, S + \Delta)$ is plt near S if and only if the pair $(S, \Delta|_S)$ is klt.*

2.1. Anticanonical divisors on Fano fourfolds

The following statement collects the known results about anticanonical divisors on smooth Fano fourfolds

Theorem 2.8 [28, Theorem 5.2], [23, Theorem 1.7], [21, Theorem 2]. *Let X be a smooth Fano fourfold. Then we have*

$$h^0(X, \mathcal{O}_X(-K_X)) \geq 2.$$

Let $Y \in |-K_X|$ be a general anticanonical divisor. Then Y is a normal prime divisor with terminal Gorenstein singularities. The variety Y is Calabi-Yau, that is, $K_Y \simeq \mathcal{O}_Y$ and

$$H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0.$$

In fact, the possible singularities of Y are completely described by [21, Theorem 4], but we will not need this description for our proof. By [25, Lemma 5.1], a terminal \mathbb{Q} -factorial Gorenstein threefold singularity is factorial, so Theorem 2.8 implies the following:

Corollary 2.9. *Let X be a smooth Fano fourfold. If a general anticanonical divisor Y is \mathbb{Q} -factorial, it is factorial.*

Theorem 2.10 [33, Theorem]. *Let X be a Fano manifold of dimension at least four, and let $Y \in |-K_X|$ be an irreducible² anticanonical divisor. Then the inclusion*

$$\overline{NE}(Y) \hookrightarrow \overline{NE}(X)$$

²The statement in [33] is for smooth divisors, but the proof works without this assumption.

is a bijection. Equivalently, a Cartier divisor $D \rightarrow X$ is nef if and only if its restriction $D|_Y$ is nef.

Theorem 2.11 [28, Proposition 4.2]. *Let Y be a normal projective threefold with canonical Gorenstein singularities such that $c_1(Y) = 0$. Let $A \rightarrow Y$ be an ample Cartier divisor, and let $D \in |A|$ be a general element. Then the pair (Y, D) is lc.*

Corollary 2.12. *In the situation of Setup 1.2, let M be a general element of the mobile part of $|-K_X|_Y|$. Then the pair $(Y, M+B)$ is lc, and hence, the pairs (Y, M) and (Y, B) are lc. In particular, B is a normal projective Gorenstein surface with at most lc singularities and $p_g(B) = 0$. Moreover,*

$$(B, M|_B)$$

is an lc pair such that

$$-K_X|_B \simeq K_B + M|_B \quad (1)$$

is ample with $(K_B + M|_B) \cdot M|_B > 0$.

Proof. Since M is a general divisor in the mobile part, the divisor $M+B$ is general in $|-K_X|_Y|$. Thus, $(Y, M+B)$ is lc by Theorem 2.11. The statement for (Y, M) and (Y, B) is now immediate since we assume that Y is \mathbb{Q} -factorial. By adjunction (Theorem 2.7), this implies that the surfaces M and B have at most slc singularities. Since B is normal by assumption, it has lc singularities. Consider the cohomology sequence associated to the exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y(B) \rightarrow \mathcal{O}_B(B) \simeq \mathcal{O}_B(K_B) \rightarrow 0.$$

The Calabi-Yau threefold Y has $h^1(Y, \mathcal{O}_Y) = 0$ and $h^0(Y, \mathcal{O}_Y(B)) = 1$ by Fact 2.2, so $p_g(B) = h^0(B, \mathcal{O}_B(K_B)) = 0$.

Again, by adjunction, the pair $(B, M|_B)$ is slc (hence lc) since $(Y, M+B)$ is lc. Since K_Y is trivial, the linear equivalence (1) follows from the adjunction formula. Finally, observe that the support of the divisor $M+B \simeq -K_X|_Y$ is connected since it is ample. Therefore, the intersection $M \cap B$ has positive dimension and $(K_B + M|_B) \cdot M|_B = -K_X|_B \cdot M|_B > 0$. \square

3. Examples and first observations

We start this section by collecting some arguments that give a moral explanation for Theorem 1.3.

Lemma 3.1. *Let X be a projective manifold, and let A be an ample divisor on X . Let $B \subset \text{Bs}(|A|)$ be a submanifold of dimension at least $\frac{\dim X}{2}$ such that the conormal bundle $\mathcal{N}_{B/X}^*$ is nef. Then every divisor $Y \in |A|$ has at least one singular point along B .*

Proof. Consider the natural morphism

$$\alpha : H^0(X, \mathcal{O}_X(A)) \simeq H^0(X, \mathcal{I}_B \otimes \mathcal{O}_X(A)) \rightarrow H^0(B, \mathcal{I}_B / \mathcal{I}_B^2 \otimes \mathcal{O}_X(A)).$$

By [8, Lemma 1.7.4], the divisor Y is smooth along B if and only if the section $\alpha(s) \in H^0(B, \mathcal{I}_B/\mathcal{I}_B^2 \otimes \mathcal{O}_X(A))$ is nowhere zero, where $0 \neq s \in H^0(X, \mathcal{O}_X(A))$ is a section vanishing on Y . Since $\mathcal{I}_B/\mathcal{I}_B^2 \otimes \mathcal{O}_X(A) \simeq \mathcal{N}_{B/X}^* \otimes \mathcal{O}_X(A)$ is a tensor product of a nef vector bundle and an ample line bundle, it is ample [40, Theorem 6.2.12.(iv)]. Set $d = \operatorname{codim}_X B$. If $\alpha(s)$ does not vanish in any point of B , the top Chern class $c_d(\mathcal{N}_{B/X}^* \otimes \mathcal{O}_X(A))$ is zero. Yet since $d \leq \dim B$, this is a contradiction to the positivity of Chern classes of ample vector bundles [40, Theorem 8.3.9]. \square

Remark. The key point of the proof above is that the vector bundle $\mathcal{N}_{B/X}^* \otimes \mathcal{O}_X(A)$ is ample, and one may wonder if this holds without any assumption on $\mathcal{N}_{B/X}^*$: as a toy case, assume that the *base scheme* $\mathbf{Bs}(|A|)$ is the submanifold B . We can resolve the indeterminacies of $\varphi_{|A|}$ by blowing up B and compute that $\mathcal{N}_{B/X}^* \otimes \mathcal{O}_X(A)$ is nef. However, it is not clear if the vector bundle is ample; in fact, this is impossible if $h^0(X, \mathcal{O}_X(A)) \leq \dim X - 1$.

The lemma allows to cover all the known examples of Fano fourfolds with a two-dimensional base locus:

Example 3.2. Let $X = S_1 \times S$ be the product of smooth del Pezzo surface S_1 of degree one and S a smooth del Pezzo surface of degree at least two. Then every anticanonical divisor $Y \in |-K_X|$ is singular.

Proof. The base locus is $B = p \times S$, where $p = \mathbf{Bs}(|-K_{S_1}|)$. Thus, $\mathcal{N}_{B/X}^* \simeq \mathcal{O}_B^{\oplus 2}$ is trivial, and Lemma 3.1 applies.

Applied to an irreducible component of the base locus, the argument above also works for [23, Example 2.12].

Example 3.3. Let $X = Z \times \mathbb{P}^1$, where Z is a smooth Fano threefold such that $C := \mathbf{Bs}(|-K_Z|)$ is not empty. Then every anticanonical divisor $Y \in |-K_X|$ is singular.

Proof. The base locus is $B = C \times \mathbb{P}^1$, and we know that $\mathcal{N}_{C/Z}^* \simeq \mathcal{O}_C^{\oplus 2}$ or $\mathcal{N}_{C/Z}^* \simeq \mathcal{O}_C \oplus \mathcal{O}_C(1)$ (cf. the proof of [24, Lemma 2.4.4]). Thus, $\mathcal{N}_{B/X}^*$ is nef, and Lemma 3.1 applies.

The next manifold is a generalisation of a well-known example in the threefold case [44, Table 2, No 1].

Example 3.4. For $n \geq 4$, let M be a general sextic hypersurface in the weighted projective space $\mathbb{P}(1^n, 2, 3)$. Then M is a del Pezzo manifold of dimension n and degree 1 [16, Theorem 8.11] (i.e., we have $-K_M \simeq (n-1)A$ with A an ample Cartier divisor such that $A^n = 1$). The del Pezzo manifold M has a ladder [16, Theorem 3.5]; that is, we can choose $n-1$ general elements D_1, \dots, D_{n-1} in the linear system $|A|$ such that

$$C := D_1 \cap \dots \cap D_{n-1}$$

is a smooth elliptic curve. Moreover, $|A|$ has a unique base-point p .

Let now $\mu: X \rightarrow M$ be the blow-up along C , and denote by E the exceptional divisor. Then μ resolves the base locus of the linear system $|V|$ generated by D_1, \dots, D_{n-1} , so we have a fibration

$$f: X \rightarrow \mathbb{P}^{n-2}$$

such that $\mu^*A \simeq f^*H + E$, where H is the hyperplane divisor on \mathbb{P}^{n-2} . Using the Nakai-Moishezon criterion, we obtain immediately that

$$-K_X \simeq \mu^*(-K_M) - (n-2)E \simeq (n-1)\mu^*A - (n-2)E \simeq (n-2)f^*H + \mu^*A$$

is ample, so X is a Fano manifold. The general f -fibre is a surface S obtained as the complete intersection of $n-2$ elements in $|V|$, so it is a del Pezzo surface of degree one. In particular, $|-K_S|$ has a unique base point p_S such that $\mu(p_S) = p$. This implies that $\mu^{-1}(p) \simeq \mathbb{P}^{n-2}$ is contained in $\text{Bs}(|-K_X|)$.

Since $\mathcal{N}_{\mathbb{P}^{n-2}/X}^* \simeq \mathcal{O}_{\mathbb{P}^{n-2}} \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ is nef, we can apply Lemma 3.1 to see that all the anticanonical divisors in X are singular.

Remark 3.5. Let X be a complex manifold, and let $B \subset X$ be a submanifold of codimension two. Let $B \subset Y \subset X$ be a normal prime divisor. Then Y is singular in a point $p \in B$ if and only if Y is not factorial in p . Indeed, if Y is factorial in p , the Weil divisor $B \subset Y$ is Cartier near the point p . Thus, if Y is singular in p , so is the Cartier divisor B .

The next statement shows that for most prime Fano manifolds (i.e., the prime Fanos with index one), the presence of a codimension two base locus is incompatible with the smoothness of the anticanonical divisor.

Proposition 3.6. *Let X be a Fano manifold of dimension at least four with Picard number one such that $h^0(X, \mathcal{O}_X(-K_X)) \geq 3$. Assume that the base locus $\text{Bs}(|-K_X|)$ has codimension two and a general anticanonical divisor $Y \in |-K_X|$ is smooth. Then X has Fano index at least three.*

Proof. Let

$$-K_X|_Y \simeq M + B$$

be the decomposition into fixed and mobile part. Since Y is smooth, the Weil divisors M and B are Cartier. Since $\dim X \geq 4$ and $\rho(X) = 1$, the Lefschetz hyperplane theorem implies $\text{Pic } Y \simeq \text{Pic } X \simeq \mathbb{Z}H$, where H is the ample generator of the Picard group. Thus, we have

$$M \simeq aH|_Y, \quad B \simeq bH|_Y$$

with $a, b \in \mathbb{N}^*$. Since $h^0(Y, \mathcal{O}_Y(B)) = 1$ and $h^0(Y, \mathcal{O}_Y(M)) \geq 2$, we have $a > b$, and therefore, $a + b \geq 3$. Since

$$-K_X|_Y \simeq M + B \simeq (a + b)H|_Y,$$

the statement follows from $\text{Pic } Y \simeq \text{Pic } X$. □

Lemma 3.7. *Let X be a smooth Fano fourfold, and let D be a non-zero nef Cartier divisor on X . Then $h^0(X, \mathcal{O}_X(D)) \geq 2$.*

Proof. If D has numerical dimension one, there exists a fibration $\varphi: X \rightarrow \mathbb{P}^1$ such that $D \simeq \varphi^*H$ with H an ample Cartier divisor on \mathbb{P}^1 (cf. Fact 2.3). Thus, $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(H)) \geq 2$ yields the claim.

If D has numerical dimension at least two, we have $D^2 \cdot K_X^2 = D^2 \cdot (-K_X)^2 > 0$. Moreover, $h^0(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D))$ by Kodaira vanishing, and by Riemann-Roch,

$$\begin{aligned} \chi(X, \mathcal{O}_X(D)) &= \frac{D^4}{4!} - \frac{D^3 \cdot K_X}{2 \cdot 3!} + \frac{D^2 \cdot (K_X^2 + c_2(X))}{12 \cdot 2!} - \frac{D \cdot K_X \cdot c_2(X)}{24 \cdot 1!} + \chi(X, \mathcal{O}_X) \\ &\geq \frac{D^2 \cdot (-K_X)^2}{24} + \frac{D^2 \cdot c_2(X)}{24} - \frac{D \cdot K_X \cdot c_2(X)}{24} + 1. \end{aligned}$$

Since $-K_X \cdot D \cdot c_2(X) \geq 0$ and $D^2 \cdot c_2(X) \geq 0$ by [46, Theorem 1.3], [43, Theorem 6.1] we obtain that the second line is at least two. \square

Lemma 3.8. *Let X be a smooth Fano fourfold, and let L be a line bundle on X such that $h^0(X, L) \neq 0$. Let $D \in |L|$ be a general divisor, and assume that for a general anticanonical divisor $Y \in |-K_X|$, the pair $(Y, D \cap Y)$ is log-canonical. Then the pair (X, D) is log-canonical.*

Proof. Since $\dim X \leq 4$, the anticanonical system $|-K_X|$ has no fixed component; cf. Theorem 2.8. In particular, Y has no common component with D .

We argue by contradiction and assume that the non-lc locus Z is not empty.

1st step. Assume that Z has positive dimension. Since Y is an ample divisor, it intersects Z in at least one point, so the pair (X, D) is not lc near Y . Thus, the pair $(X, D + Y)$ is not lc near Y . By Theorem 2.7, this is a contradiction to Corollary 2.12.

2nd step. Assume that Z has dimension zero. We endow $Z \subset X$ with its natural scheme structure; cf. [17, §7]. Since

$$L - (K_X + D) \simeq -K_X$$

is ample, we can apply Fujino's extension theorem (with W the empty set; cf. [17, Theorem 8.1]) to see that the restriction map

$$H^0(X, L) \rightarrow H^0(Z, L)$$

is surjective. Since Z is a finite scheme, the restriction $L|_Z$ is globally generated. Thus, L is globally generated near Z ; in particular, a general $D \in |L|$ is smooth near Z . Hence, the pair (X, D) is lc, a contradiction. \square

4. Auxiliary results about linear systems, part I

In Section 5, our study of the Fano fourfold X will lead to several restrictions on the geometry of the base locus $\text{Bs}(|-K_X|)$. These restrictions can be strengthened when combined with the technical results of this section.

Lemma 4.1. *Let S be a normal projective surface with canonical singularities such that $p_g(S) = 0$. Assume that the Albanese map induces a fibration $\alpha: S \rightarrow C$ onto a curve of genus $g = q(S) > 0$. Then the general α -fibre is \mathbb{P}^1 or $g = 1$.*

Proof. Since canonical surface singularities are rational, we can replace S with a resolution of singularities. Thus, S is a smooth surface with $p_g(S) = 0$ and $q(S) \geq 1$.

If the general α -fibre is not \mathbb{P}^1 , the surface S is not uniruled. Thus, we can apply [5, Theorem VI.13] to the minimal model of S to obtain that $q(S) = 1$. \square

Proposition 4.2. *Let S be a normal projective Gorenstein surface with $p_g(S) = 0$, and let A be a nef and big Cartier divisor on S such that*

$$A \simeq K_S + \Delta_S,$$

where Δ_S is an effective Weil divisor such that the pair (S, Δ_S) is lc and $A \cdot \Delta_S > 0$. Assume that we have

$$h^0(S, \mathcal{O}_S(A)) \leq q(S).$$

Then S has canonical singularities, and the Albanese map induces a fibration

$$\alpha : S \rightarrow C$$

onto a curve of genus $q(S) > 0$. Set $r := \text{rk}(\alpha_ \mathcal{O}_S(A))$. Then we have*

$$(r - 1)(q(S) - 1) \leq 1.$$

Moreover, we have

$$h^0(S, \mathcal{O}_S(A)) = q(S).$$

Remark. In the second step of the proof, we will use the following fact that follows easily from the Leray spectral sequence: let S be a normal projective surface with Gorenstein singularities such that $p_g(S) = 0$. Assume that S has irrational (hence non-canonical [31, Corollary 5.24]) singularities, and let $\tau : S_c \rightarrow S$ be the canonical modification [35, Theorem 1.31]. Then we have

$$p_g(S_c) = p_g(S) = 0, \quad q(S_c) > q(S).$$

Proof of Proposition 4.2. Since $A \cdot \Delta_S > 0$, we have $\Delta_S \neq 0$. Thus,

$$H^2(S, \mathcal{O}_S(K_S + \Delta_S)) \simeq H^0(S, \mathcal{O}_S(-\Delta_S)) = 0,$$

and we have the inequalities

$$q(S) \geq h^0(S, \mathcal{O}_S(K_S + \Delta_S)) \geq \chi(S, \mathcal{O}_S(K_S + \Delta_S)).$$

1st step. Assume that S has canonical singularities. By Fact 2.6, we have the Riemann-Roch formula

$$\chi(S, \mathcal{O}_S(K_S + \Delta_S)) = \frac{1}{2}(K_S + \Delta_S) \cdot \Delta_S + \chi(\mathcal{O}_S).$$

Hence, $A \cdot \Delta_S > 0$ implies that $q(S) > \chi(S, \mathcal{O}_S) = 1 - q(S)$. In particular, we have $q(S) > 0$, so there is a non-trivial Albanese morphism α . If $\dim \alpha(S) = 2$, the ramification formula shows that $p_g(S) > 0$ which we excluded. Thus, the Albanese morphism gives a fibration [8, Lemma 2.4.5]

$$\alpha : S \rightarrow C$$

onto a curve C of genus $g := q(S) > 0$. Since the pair (S, Δ_S) is lc, the direct image sheaf

$$\alpha_* \mathcal{O}_S(K_{S/C} + \Delta_S) \simeq \alpha_* \mathcal{O}_S(A) \otimes \mathcal{O}_C(-K_C)$$

is a nef vector bundle [19, Theorem 1.1] of rank r . Thus,

$$V := \alpha_* \mathcal{O}_S(A) \simeq \alpha_* \mathcal{O}_S(K_{S/C} + \Delta_S) \otimes \mathcal{O}_C(K_C)$$

has $c_1(V) \geq r(2g - 2)$. By the Riemann-Roch formula on curves, we have

$$h^0(S, \mathcal{O}_S(A)) = h^0(C, V) \geq \chi(C, V) = c_1(V) + r\chi(C, \mathcal{O}_C) \geq r(g - 1). \quad (2)$$

Now observe that $r(g - 1) > g$ unless $(r - 1)(g - 1) \leq 1$.

Finally, let us show that we have $h^0(S, \mathcal{O}_S(A)) = q(S)$: if $h^0(S, \mathcal{O}_S(A)) \leq q(S) - 1$, the Riemann-Roch estimate (2) becomes $g - 1 \geq r(g - 1)$.

Subcase a) Assume that $g > 1$. Then the unique possibility $r = 1$. Since A is nef and big, this implies that the general α -fibre is not \mathbb{P}^1 . Yet this contradicts Lemma 4.1.

Subcase b) Assume that $g = 1$. In this case, the Riemann-Roch inequality becomes

$$h^0(S, \mathcal{O}_S(K_S + \Delta_S)) \geq \frac{1}{2}(K_S + \Delta_S) \cdot \Delta_S > 0,$$

so $h^0(S, \mathcal{O}_S(K_S + \Delta_S)) \geq 1 = g$.

2nd step. We show that S has canonical singularities. Since S is Gorenstein, it has canonical singularities if and only if it has rational singularities. Arguing by contradiction, we assume that S has non-canonical singularities and denote by $\tau: S_c \rightarrow S$ the canonical modification. Then we have $K_{S_c} \simeq \tau^* K_S - E$ with E an effective Weil divisor. Observe that

$$K_{S_c} + E + \tau^* \Delta_S \simeq \tau^*(K_S + \Delta_S),$$

so the pair $(S_c, \Delta_{S_c}) := (S_c, E + \tau^* \Delta_S)$ is lc. Moreover,

$$A_c := \tau^* A \simeq K_{S_c} + \Delta_{S_c}$$

is a nef and big Cartier divisor with $A_c \cdot \Delta_{S_c} = A \cdot \Delta_S > 0$ and $h^0(S, \mathcal{O}_S(A)) = h^0(S_c, \mathcal{O}_{S_c}(A_c))$. Finally, by the remark before the proof, one has $p_g(S_c) \leq p_g(S) = 0$ and $q(S_c) > q(S)$. Therefore, by Step 1 of the proof,

$$h^0(S, \mathcal{O}_S(A)) = h^0(S_c, \mathcal{O}_{S_c}(A_c)) = q(S_c) > q(S),$$

a contradiction to our assumption. □

The conditions in Proposition 4.2 are very restrictive; however, there is a classical example that will play a prominent role in Section 7:

Example 4.3. Let C be an elliptic curve, and for some point $p \in C$, let

$$0 \rightarrow \mathcal{O}_C \rightarrow V \rightarrow \mathcal{O}_C(p) \rightarrow 0$$

be an unsplit extension. Set $\alpha: S := \mathbb{P}(V) \rightarrow C$ and denote by ζ_S the tautological class on S . Set $\Delta_S := 3\zeta_S - \alpha^*p$. Then Δ_S is an ample divisor [20, V, Proposition 2.21] such

that $|\Delta_S|$ contains an element with normal crossing singularities. Thus, the pair (S, Δ_S) is lc, and

$$A := K_S + \Delta_S \simeq \zeta_S$$

is an ample divisor with $h^0(S, \mathcal{O}_S(A)) = 1 = q(S)$.

Since it would be tedious to go into the classification of surfaces with small invariants, we want a convenient criterion to exclude this kind of exceptional surfaces in the proof of Theorem 1.3:

Proposition 4.4. *Let S be a normal projective Gorenstein surface with $p_g(S) = 0$, and let A be an ample Cartier divisor on S such that*

$$A \simeq K_S + \Delta_S,$$

where Δ_S is an effective Weil divisor such that the pair (S, Δ_S) is lc. Assume that we have

$$h^0(S, \mathcal{O}_S(A)) \leq q(S).$$

Then Δ_S does not contain any smooth rational curve.

Proof. We argue by contradiction, so let $l \subset \Delta_S$ be a smooth rational curve. In particular, $\Delta_S \neq 0$, and therefore, $A \cdot \Delta_S > 0$. By Proposition 4.2, the surface has canonical singularities, and we have an Albanese fibration $\alpha : S \rightarrow C$ onto a smooth curve C of genus $g = q(S)$. Moreover, we have

$$(r-1) \cdot (q(S)-1) \leq 1, \quad (3)$$

where $r = \text{rk}(\alpha_* \mathcal{O}_S(A))$.

1st case. Assume that $g > 1$. By Lemma 4.1, the general α -fibre F is \mathbb{P}^1 . Since $A \cdot F \geq 1$, we have $r \geq 2$, and thus, (3) yields $g = 2$ and $r = 2$. Moreover, $A \cdot F = 1$ implies that $\alpha : S \rightarrow C$ is a \mathbb{P}^1 -bundle over the genus two curve C [34, II, Theorem 2.8]. In particular, the rational curve l is a fibre of α , and thus a nef divisor on S . The pair $(S, \Delta_S - l)$ is lc, so the direct image

$$\alpha_* \mathcal{O}_S(K_{S/C} + \Delta_S - l) \simeq \alpha_* \mathcal{O}_S(A) \otimes \mathcal{O}_C(-K_C - l)$$

is a nef vector bundle [19, Theorem 1.1]. Thus, $\alpha_* \mathcal{O}_S(A)$ has degree at least six. By the Riemann-Roch formula on the curve C , this implies $h^0(S, \mathcal{O}_S(A)) = h^0(C, \alpha_* \mathcal{O}_S(A)) \geq 4$, a contradiction.

2nd case. Assume that $g = 1$. By Proposition 4.2, we have $h^0(S, \mathcal{O}_S(A)) = 1$, and thus by Riemann-Roch,

$$1 \geq \chi(S, \mathcal{O}_S(A)) = \frac{1}{2}(K_S + \Delta_S) \cdot \Delta_S > 0.$$

Therefore, $A \cdot \Delta_S = (K_S + \Delta_S) \cdot \Delta_S = 2$, and Lemma 4.5 applies: all the irreducible components of Δ_S are rational curves, so they are contracted by the Albanese map. Moreover, the support of Δ_S is connected, so it is contained in an α -fibre F_0 . Since $p_a(\Delta_S) = 2$, we deduce that $p_a(F_0) \geq 2$. Since the arithmetic genus is constant in

the flat family $\alpha: S \rightarrow C$, we obtain that S is not uniruled. Let $\tau: S' \rightarrow S$ be the minimal resolution, and let $\nu: S' \rightarrow S_m$ the minimal model of S' . Since S has canonical singularities, we have

$$p_g(S_m) = p_g(S') = p_g(S) = 0, \quad q(S_m) = q(S') = q(S) > 0.$$

By [5, Proposition VI.6], the minimal surface S_m does not contain any rational curves. Since the exceptional divisors of the resolution τ are rational curves, the rigidity lemma [11, Lemma 1.15] yields a birational map $\nu: S \rightarrow S_m$ that contracts all the rational curves on S . In particular, it contracts the divisor Δ_S . Yet $p_a(\Delta_S) = 2 > 0$, a contradiction to the fact that the smooth surface S_m has rational singularities. \square

Lemma 4.5. *Let S be a normal projective surface with canonical singularities and $p_g(S) = 0$, and let A be an ample Cartier divisor on S such that*

$$A \simeq K_S + \Delta_S,$$

where Δ_S is an effective Weil divisor such that the pair (S, Δ_S) is lc and $A \cdot \Delta_S = 2$. Assume that Δ_S has an irreducible component that is a smooth rational curve l . Then the support of Δ_S is connected, and all its irreducible components are rational curves.

Proof. Recall first that canonical surface singularities are Gorenstein, so both K_S and $\Delta_S = A - K_S$ are Cartier. We have $p_a(\Delta_S) = 2$, so the inclusion $l \subset \Delta_S$ must be strict. Since A is ample with $A \cdot \Delta_S = 2$, we obtain

$$\Delta_S = l + R$$

with R an irreducible curve such that $A \cdot R = 1$. In particular, we have by subadjunction

$$\deg K_R \leq (K_S + R) \cdot R \leq (K_S + l + R) \cdot R = A \cdot R = 1,$$

so $p_a(R) = \frac{1}{2} \deg K_R + 1 \leq 1$. In particular, Δ_S is connected because otherwise

$$2 = p_a(\Delta_S) = p_a(R) + p_a(l) \leq 1$$

yields a contradiction.

Let $\tau: S' \rightarrow S$ be the minimal resolution and set

$$\Delta_{S'} := \tau^* \Delta_S.$$

Since Δ_S is connected, the cycle $\Delta_{S'}$ is connected. Denote by $R' \subset \Delta_{S'}$ the strict transform of R . We claim that R' is a rational curve, and this concludes the proof.

Proof of the claim. Since S has canonical singularities, the minimal resolution τ is crepant, so we have $K_{S'} \simeq \tau^* K_S$ and

$$K_{S'} + \Delta_{S'} = \tau^*(K_S + \Delta_S) = \tau^* A.$$

Thus, we have

$$1 = \tau^* A \cdot R' = (K_{S'} + \Delta_{S'}) \cdot R' = (K_{S'} + R') \cdot R' + (\Delta_{S'} - R') \cdot R'.$$

If we show that $(\Delta_{S'} - R') \cdot R' \geq 2$, then the adjunction formula yields the claim. Since $\Delta_{S'}$ is connected, we have $(\Delta_{S'} - R') \cdot R' \geq 1$. We will argue by contradiction to exclude the case where we have an equality.

In this case, R' is a curve of arithmetic genus one that meets $\Delta_{S'} - R'$ transversally exactly in one point. In particular, we have

$$2 = p_a(\Delta_{S'}) = p_a(R') + p_a(\Delta_{S'} - R') = 1 + p_a(\Delta_{S'} - R').$$

The curve $l \subset S$ being smooth, it meets the exceptional divisor over every point $p \in l \cap S_{\text{sing}}$ in exactly one point, and the intersection is transverse. Thus, we have

$$p_a(\Delta_{S'} - R') = p_a(l) + \sum_{p \in l \cap S_{\text{sing}}} \tau^{-1}(p) = 0$$

since the exceptional divisors of the minimal resolution of an ADE-singularity have arithmetic genus zero. This gives the final contradiction. \square

5. Positivity arguments

This section is the technical core of the paper. We will study the positivity properties of the Cartier classes M_X and B_X and successively improve our knowledge about the existence and singularities of effective divisors contained in these classes.

Proposition 5.1. *In the situation of Setup 1.2, the divisor B_X is not nef. If B_X is effective, then $h^0(X, \mathcal{O}_X(B_X)) = 1$ and B_X is a normal prime divisor.*

Proof. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-M_X) \rightarrow \mathcal{O}_X(B_X) \rightarrow \mathcal{O}_Y(B) \rightarrow 0.$$

We have $h^0(X, \mathcal{O}_X(-M_X)) = 0$ since otherwise $-M_X$, and thus its restriction $-M$ to the general anticanonical divisor Y , is effective. Thus, we have an injection

$$H^0(X, \mathcal{O}_X(B_X)) \hookrightarrow H^0(Y, \mathcal{O}_Y(B)).$$

Since $h^0(Y, \mathcal{O}_Y(B)) = 1$ by Fact 2.2, we have $h^0(X, \mathcal{O}_X(B_X)) \leq 1$. By Lemma 3.7, this implies that B_X is not nef.

If B_X is effective, it is a normal prime divisor since its restriction to the ample divisor Y is the normal prime divisor B . \square

Proposition 5.2. *Let Y be a \mathbb{Q} -factorial threefold with canonical Gorenstein singularities such that $c_1(Y) = 0$. Let A be an ample Cartier divisor with $h^0(Y, \mathcal{O}_Y(A)) \geq 2$ such that $\text{Bs}(|A|)$ has pure dimension two. Let $A \simeq M + B$ be the decomposition into mobile part and fixed part B . Then $A + M$ is a nef and big divisor.*

Proof. We choose a general element M in the mobile part, so by Theorem 2.11, the pair $(Y, M + B)$ is lc. If $A + M$ is not nef, there exists an irreducible curve γ such that $(A + M) \cdot \gamma < 0$. Since A is ample, we have $M \cdot \gamma \leq -2$. In particular, we have $\gamma \subset \text{Bs}(|M|)$. Since $|M|$ is the mobile part of $|A|$, we have $\text{Bs}(|M|) \subset \text{Bs}(|A|)$. By assumption, $\text{Bs}(|A|)$

has pure dimension two, so it coincides with B . Thus, we have $\gamma \subset B$, and hence,

$$\gamma \subset M \cap B.$$

The surface $M + B$ is slc by Theorem 2.7, so it has normal crossing singularities in codimension one. Therefore, B and M are smooth in the generic point of γ ; in particular, γ is not contained in the singular locus of M . Since

$$-K_M \cdot \gamma = -M \cdot \gamma \geq 2,$$

we know by Fact 2.4 that the curve γ deforms in M . In particular, M contains a positive-dimensional family of curves γ_t such that $M \cdot \gamma_t < 0$. Yet M is a mobile divisor and $\dim Y = 3$, so this is impossible. \square

Corollary 5.3. *In the situation of Setup 1.2, the divisor $-K_X + M_X$ is nef and has numerical dimension at least three. In particular,*

$$H^q(X, \mathcal{O}_X(M_X)) = 0 \quad \forall q \geq 2.$$

Proof. By Theorem 2.10, we know that $-K_X + M_X$ is nef if and only if the restriction to Y is nef. Thus, nefness follows from Proposition 5.2. The numerical dimension is at least three since the restriction to Y is big, so of numerical dimension three. Finally, the vanishing follows from

$$H^q(X, \mathcal{O}_X(M_X)) = H^q(X, \mathcal{O}_X(K_X + (-K_X + M_X)))$$

and the numerical Kawamata-Viehweg vanishing theorem [39, Example 4.3.7]. \square

Proposition 5.4. *In the situation of Setup 1.2, assume that M_X is not nef. Then we have*

$$H^q(X, \mathcal{O}_X(K_X + M_X)) = 0 \quad \forall q \geq 3.$$

For the proof, we start with a general lemma which is essentially contained in Küronya's paper [38]:

Lemma 5.5. *Let Y be a normal projective threefold with canonical Gorenstein singularities, and let D be an effective Cartier divisor on Y such that for every irreducible component $D' \subset D$, the restriction $D|_{D'}$ is pseudoeffective. Then*

$$H^q(Y, \mathcal{O}_Y(K_Y + D)) = 0 \quad \forall q \geq 2$$

unless (maybe) D is nef of numerical dimension at most one.

Proof. Since the restriction $D|_{D'}$ is pseudoeffective and $\dim D' = 2$, there are at most finitely many curves $\gamma \subset Y$ such that $D \cdot \gamma < 0$. In particular, for a general very ample divisor $A \subset Y$, the restriction $D|_A$ is nef. Moreover,

$$(D|_A)^2 = D^2 \cdot A = D|_D \cdot A|_D \geq 0,$$

and equality holds if and only if the pseudoeffective class $D|_D$ is zero. Hence, if $(D|_A)^2 = 0$, the restriction $D|_D$ is numerically trivial and D is nef with numerical dimension at most

one. If $(D|_A)^2 > 0$, the restriction $D|_A$ is nef and big and we obtain the vanishing by [38, Theorem C].³ \square

Proof of Proposition 5.4. We twist the exact sequence

$$0 \rightarrow \mathcal{O}_X(-Y) \simeq \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$$

with M_X to get

$$0 \rightarrow \mathcal{O}_X(K_X + M_X) \rightarrow \mathcal{O}_X(M_X) \rightarrow \mathcal{O}_Y(M) \rightarrow 0.$$

Since M is mobile, it satisfies the conditions of Lemma 5.5; by assumption, M_X (and thus M) is not nef, so we have $H^q(Y, \mathcal{O}_Y(M)) = 0$ for $q \geq 2$. By the exact sequence in cohomology, this implies

$$H^q(X, \mathcal{O}_X(K_X + M_X)) = H^q(X, \mathcal{O}_X(M_X)) \quad \forall q \geq 3.$$

Yet the right-hand side vanishes by Corollary 5.3. \square

Corollary 5.6. *In the situation of Setup 1.2, assume that M_X is not nef. Then the divisor class B_X is effective.*

Proof. We twist the restriction sequence

$$0 \rightarrow \mathcal{O}_X(-Y) \simeq \mathcal{O}_X(-(M_X + B_X)) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$$

with B_X to get

$$0 \rightarrow \mathcal{O}_X(-M_X) \rightarrow \mathcal{O}_X(B_X) \rightarrow \mathcal{O}_Y(B) \rightarrow 0.$$

By Serre duality and Proposition 5.4, one has

$$H^1(X, \mathcal{O}_X(-M_X)) = H^3(X, \mathcal{O}_X(K_X + M_X)) = 0.$$

Therefore, $h^0(Y, \mathcal{O}_Y(B)) = 1$ implies that B_X is effective. \square

Proposition 5.7. *In the situation of Setup 1.2, the divisor class M_X is effective and mobile.*

Remark. If M_X is nef, the linear system $|M_X|$ might have fixed components. Nevertheless, the class being nef, it is mobile. If M_X is not nef, the proof will show that

$$h^0(X, \mathcal{O}_X(M_X)) = h^0(X, \mathcal{O}_X(-K_X)) - 1.$$

Proof. If M_X is nef, this is immediate from Lemma 3.7. If M_X is not nef, we know by Corollary 5.6 that B_X is effective. Thus, B_X is normal prime divisor by Proposition 5.1, and we can twist the restriction sequence for B_X by $K_X + B_X$ to get an exact sequence

$$0 \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_X(K_X + B_X) \rightarrow \mathcal{O}_{B_X}(K_X + B_X) \simeq \mathcal{O}_{B_X}(K_{B_X}) \rightarrow 0.$$

³The statement in [38] is for a manifold, but it is straightforward to see that the proof works in our setup.

Taking cohomology, we get a sequence

$$\begin{aligned} \dots \rightarrow H^3(X, \mathcal{O}_X(K_X)) \rightarrow H^3(X, \mathcal{O}_X(K_X + B_X)) \rightarrow H^3(B_X, \mathcal{O}_{B_X}(K_{B_X})) \\ \rightarrow H^4(X, \mathcal{O}_X(K_X)) \rightarrow H^4(X, \mathcal{O}_X(K_X + B_X)) \rightarrow 0. \end{aligned}$$

Using Serre duality on X and B_X , this transforms into

$$\begin{aligned} \dots \rightarrow H^1(X, \mathcal{O}_X) = 0 \rightarrow H^3(X, \mathcal{O}_X(K_X + B_X)) \rightarrow H^0(B_X, \mathcal{O}_{B_X}) = \mathbb{C} \\ \rightarrow H^0(X, \mathcal{O}_X) = \mathbb{C} \rightarrow H^0(X, \mathcal{O}_X(-B_X)) = 0 \rightarrow 0. \end{aligned}$$

In conclusion, we get

$$H^q(X, \mathcal{O}_X(K_X + B_X)) = 0 \quad \forall q \geq 3.$$

Now we twist the restriction sequence to Y by M_X to get

$$0 \rightarrow \mathcal{O}_X(-B_X) \rightarrow \mathcal{O}_X(M_X) \rightarrow \mathcal{O}_Y(M) \rightarrow 0.$$

By Serre duality and what precedes,

$$H^1(X, \mathcal{O}_X(-B_X)) = H^3(X, \mathcal{O}_X(K_X + B_X)) = 0.$$

Thus, the restriction map

$$H^0(X, \mathcal{O}_X(M_X)) \rightarrow H^0(Y, \mathcal{O}_Y(M))$$

is surjective. In particular,

$$\text{Bs}(|M|) = \text{Bs}(|M_X|) \cap Y.$$

Since Y is ample and M is mobile, this shows that $\dim \text{Bs}(|M_X|) \leq 2$. Thus, $|M_X|$ is mobile. \square

The divisor M_X being effective by Proposition 5.7, the divisor class $-K_X + M_X$ is big. Since $-K_X + M_X$ is also nef by Corollary 5.3, we finally obtain the following:

Corollary 5.8. *In the situation of Setup 1.2, the divisor $-K_X + M_X$ is nef and big.*

Proposition 5.9. *In the situation of Setup 1.2, assume that $h^0(X, \mathcal{O}_X(B_X)) \neq 0$. Then the pair (X, B_X) is log-canonical. Moreover, the log-canonical centres are the prime divisor B_X and (possibly) some smooth curves $C \subset B_X$ such that $B_X \cdot C < 0$.*

Proof. We have $h^0(X, \mathcal{O}_X(B_X)) = 1$ by Proposition 5.1. Thus, B_X is the unique effective divisor in its linear system, and hence general. Since $(Y, B) = (Y, Y \cap B_X)$ is log-canonical by Corollary 2.12, we know by Lemma 3.8 that (X, B_X) is lc.

Let us now describe its lc centres: since $B_X - (K_X + B_X) = -K_X$ is ample, we know by [17, Theorem 8.1] that for every lc centre $Z \subset X$, the restriction morphism

$$H^0(X, \mathcal{O}_X(B_X)) \rightarrow H^0(Z, \mathcal{O}_Z(B_X))$$

is surjective. Since $Z \subset B_X$ and $h^0(X, \mathcal{O}_X(B_X)) = 1$, we obtain that $H^0(Z, \mathcal{O}_Z(B_X)) = 0$.

This immediately excludes the possibility that $\dim Z = 0$. Since B_X is normal by Proposition 5.1, there are no two-dimensional lc centres.

The unique remaining possibility is that $\dim Z = 1$ and the centre is minimal. Thus, by Kawamata subadjunction [27], the curve Z is normal and there exists an effective divisor Δ_Z on Z such that

$$K_Z + \Delta_Z \sim_{\mathbb{Q}} (K_X + B_X)|_Z.$$

In particular,

$$B_X|_Z - (K_Z + \Delta_Z) \sim_{\mathbb{Q}} -K_X|_Z$$

is ample. Thus, if $B_X \cdot Z \geq 0$, a Riemann-Roch computation shows that $h^0(Z, \mathcal{O}_Z(B_X)) > 0$, a contradiction. \square

We now come to the key technical result of this section:

Lemma 5.10. *In the situation of Setup 1.2, assume that $h^0(X, \mathcal{O}_X(B_X)) \neq 0$. Then we have a surjective restriction morphism*

$$H^0(X, \mathcal{O}_X(-K_X)) \rightarrow H^0(B_X, \mathcal{O}_{B_X}(-K_X)).$$

Proof. By Proposition 5.9, the pair (X, B_X) is lc, and B_X is an lc centre for this pair. By Fujino's extension theorem [18, Theorem 1.11], we have to show that

$$-K_X - (K_X + B_X) = -K_X + M_X$$

is nef and log big (i.e., the restriction of $-K_X + M_X$ to every lc centre of (X, B_X) is big).

By Corollary 5.8, the nef divisor $-K_X + M_X$ is big on X . Since M_X is mobile by Proposition 5.7, the restriction $M_X|_{B_X}$ is pseudoeffective, so $(-K_X + M_X)|_{B_X}$ is big. By Proposition 5.9, an lc centre Z that is distinct from B_X is a curve such that $B_X \cdot Z < 0$. Therefore, $M_X \cdot Z > 0$, and hence $(-K_X + M_X)|_Z$ is ample. \square

Proposition 5.11. *In the situation of Setup 1.2, assume that $h^0(X, \mathcal{O}_X(B_X)) \neq 0$. Then we have an injection*

$$H^0(B, \mathcal{O}_B(-K_X)) \hookrightarrow H^1(B_X, \mathcal{O}_{B_X}) \cong H^1(B, \mathcal{O}_B).$$

Moreover, we have

$$h^0(B_X, \mathcal{O}_{B_X}(-K_X)) = 1.$$

Proof. We have $B = Y \cap B_X$, so $B \subset B_X$ is a Cartier divisor that is linearly equivalent to $-K_X|_{B_X}$. Thus, we have an exact sequence

$$0 \rightarrow \mathcal{O}_{B_X} \rightarrow \mathcal{O}_{B_X}(-K_X) \rightarrow \mathcal{O}_B(-K_X) \rightarrow 0.$$

Let us first show that the restriction map

$$H^0(B_X, \mathcal{O}_{B_X}(-K_X)) \rightarrow H^0(B, \mathcal{O}_B(-K_X))$$

is zero. Since $B \subset \text{Bs}(|-K_X|)$, the restriction map

$$H^0(X, \mathcal{O}_X(-K_X)) \rightarrow H^0(B, \mathcal{O}_B(-K_X))$$

is zero. Since $H^0(X, \mathcal{O}_X(-K_X)) \rightarrow H^0(B_X, \mathcal{O}_{B_X}(-K_X))$ is surjective by Lemma 5.10, we get the claim.

This already implies the second statement since the kernel of the restriction map is $H^0(B_X, \mathcal{O}_{B_X}) = \mathbb{C}$. Since the restriction map is zero, we have an injection

$$H^0(B, \mathcal{O}_B(-K_X)) \rightarrow H^1(B_X, \mathcal{O}_{B_X}).$$

Now consider the exact sequence

$$0 \rightarrow \mathcal{O}_{B_X}(K_X) \rightarrow \mathcal{O}_{B_X} \rightarrow \mathcal{O}_B \rightarrow 0.$$

The pair (X, B_X) is lc by Proposition 5.9, and B_X is normal by Proposition 5.1, so B_X is a threefold with lc singularities. Since $K_X|_{B_X}$ is an antiample Cartier divisor, we can apply Kodaira vanishing [36, Corollary 6.6] to get

$$H^1(B_X, \mathcal{O}_{B_X}(K_X)) = 0 = H^2(B_X, \mathcal{O}_{B_X}(K_X)).$$

Thus, we have an isomorphism $H^1(B_X, \mathcal{O}_{B_X}) \simeq H^1(B, \mathcal{O}_B)$, and the first statement follows. \square

Corollary 5.12. *In the situation of Setup 1.2, assume that $h^0(X, \mathcal{O}_X(B_X)) \neq 0$. Then the pair (X, B_X) is plt (i.e., the threefold B_X has canonical Gorenstein singularities).*

Proof. By Proposition 5.11, we have $h^0(B, \mathcal{O}_B(-K_X)) \leq q(B)$. By Corollary 2.12, the conditions of Lemma 4.2 are satisfied. Thus, we know that B has canonical singularities. By inversion of adjunction (Theorem 2.7), the pair $(B_X, B) = (B_X, Y \cap B_X)$ is plt near B . Thus, $(B_X, 0)$ is plt (i.e., klt) near B . Since B_X is Gorenstein, this implies that B_X has canonical singularities near B . Since $B = B_X \cap Y$ is an ample divisor, the non-canonical locus of B_X is at most a finite set.

By Proposition 5.9, the pair (X, B_X) is lc, and the lc centres of dimension at most two are irreducible curves C . Again by inversion of adjunction, the non-canonical locus of B_X coincides with the union of lower-dimensional lc centres which has pure dimension one. By the first paragraph, the non-canonical locus has dimension at most zero, so it is empty. \square

We are now ready for the first reduction step in the proof of Theorem 1.3:

Theorem 5.13. *In the situation of Setup 1.2, the divisor M_X is nef.*

Proof. We argue by contradiction and assume that M_X is not nef. By the cone theorem, there exists a K_X -negative extremal ray $\mathbb{R}^+ \gamma$ such that $M_X \cdot \gamma < 0$. Since M_X is mobile by Proposition 5.7, the extremal ray is small. Thus, by Kawamata's classification [26, Theorem 1.1], a connected component of the exceptional locus is a $\mathbb{P}^2 \subset X$ such that $\mathcal{O}_{\mathbb{P}^2}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. Thus, the intersection $Y \cap \mathbb{P}^2$ is either a line or the whole surface \mathbb{P}^2 . In the latter case, we would have $\mathbb{P}^2 \subset M \subset Y$, in contradiction to the mobility of M . Therefore, $Y \cap \mathbb{P}^2$ is a smooth rational curve l such that $M \cdot l < 0$. Thus,

$$l \subset \text{Bs}(|M|) \subset \text{Bs}(|-K_X|_Y) = B$$

shows that the effective divisor $M|_B$ contains a smooth rational curve.

Since M_X is not nef, the divisor B_X is effective by Corollary 5.6. Thus, Proposition 5.11 implies that we have an injection

$$H^0(B, \mathcal{O}_B(-K_X)) \hookrightarrow H^1(B, \mathcal{O}_B).$$

By Corollary 2.12, the surface B satisfies the conditions of Proposition 4.4. Thus, the support of $M|_B$ does not contain a smooth rational curve, in contradiction to the first paragraph. \square

6. Auxiliary results about linear systems, part II

Theorem 5.13 shows the nefness of the divisor class M_X which by Fact 2.3 will give us a morphism $X \rightarrow T$ that adds additional structure to all the varieties appearing in our setup. In this section, we prove further technical results that use these structures.

We start with a statement that is a variation of Kollár's injectivity theorem [32, Theorem 2.2].

Proposition 6.1. *Let Y be a normal \mathbb{Q} -factorial projective variety with klt singularities such that $c_1(Y) = 0$. Assume that Y admits a fibration $\psi: Y \rightarrow C$ onto a smooth projective curve, and let F be a general fibre. Let A be a nef and big Cartier divisor such that*

$$A \equiv mF + B$$

with B an effective divisor such that (Y, B) is lc. If $m > 1$, the restriction map

$$H^0(Y, \mathcal{O}_Y(A)) \rightarrow H^0(F, \mathcal{O}_F(A))$$

is surjective.

Proof. It is clearly sufficient to show that

$$H^1(Y, \mathcal{O}_Y(A - F)) = 0.$$

Since Y has klt singularities and (Y, B) is lc, the pair $(Y, \epsilon B)$ is klt for every $\epsilon < 1$ [31, Corollary 2.35(5)]. In particular, $(Y, \frac{1}{m}B)$ is klt. Now we write

$$A - F \equiv K_Y + (m-1)F + B = (K_Y + \frac{1}{m}B) + (m-1)(F + \frac{1}{m}B).$$

Since $A \equiv m(F + \frac{1}{m}B)$ and $m > 1$, the \mathbb{Q} -divisor class $(m-1)(F + \frac{1}{m}B)$ is nef and big. Now we conclude with Kawamata-Viehweg vanishing [11, Theorem 7.26]. \square

Lemma 6.2. *Let Y be a normal projective \mathbb{Q} -factorial threefold with terminal singularities such that $K_Y \simeq \mathcal{O}_Y$. Suppose that Y admits an elliptic fibration*

$$\varphi: Y \rightarrow T$$

*onto a surface T . Let A be a nef and big Cartier divisor such that we have $A \simeq M + B$, where $M \simeq \varphi^*M_T$ with M_T a nef and big Cartier divisor on T and B is an effective divisor such that $B \subset \text{Bs}(|A|)$.*

Then we have an injection

$$H^0(B, \mathcal{O}_B(K_B + M)) \hookrightarrow H^0(T, \mathcal{O}_T(K_T + M_T)). \quad (4)$$

Proof. Since Y is a terminal, Gorenstein, \mathbb{Q} -factorial threefold, it is factorial [25, Lemma 5.1]. Thus, all the Weil divisors on Y are Cartier.

Consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(M) \rightarrow \mathcal{O}_Y(A) \rightarrow \mathcal{O}_B(A) \rightarrow 0$$

and the long exact sequence in cohomology

$$H^0(Y, \mathcal{O}_Y(A)) \rightarrow H^0(B, \mathcal{O}_B(A)) \rightarrow H^1(Y, \mathcal{O}_Y(M)) \rightarrow H^1(Y, \mathcal{O}_Y(A)).$$

By Kawamata-Viehweg vanishing, we have $H^1(Y, \mathcal{O}_Y(A)) = 0$. Since $B \subset \text{Bs}(|A|)$, the restriction map $H^0(Y, \mathcal{O}_Y(A)) \rightarrow H^0(B, \mathcal{O}_B(A))$ is zero, so we have an isomorphism

$$H^0(B, \mathcal{O}_B(K_B + M)) \simeq H^0(B, \mathcal{O}_B(A)) \simeq H^1(Y, \mathcal{O}_Y(M)). \quad (5)$$

By the Leray spectral sequence, we have an exact sequence

$$0 \rightarrow H^1(T, \varphi_* \mathcal{O}_Y(M)) \rightarrow H^1(Y, \mathcal{O}_Y(M)) \rightarrow H^0(T, R^1 \varphi_* \mathcal{O}_Y(M)).$$

By the canonical bundle formula [2], there exists a boundary divisor Δ_T on T such that (T, Δ_T) is klt and $K_T + \Delta_T \sim_{\mathbb{Q}} 0$. Since $\varphi_* \mathcal{O}_Y(M) \simeq \mathcal{O}_T(M_T)$ is nef and big, we can apply the Kawamata-Viehweg vanishing theorem [11, Theorem 7.26] to see that

$$H^1(T, \varphi_* \mathcal{O}_Y(M)) = 0.$$

By the projection formula, we have

$$R^1 \varphi_* \mathcal{O}_Y(M) \simeq R^1 \varphi_* \mathcal{O}_Y \otimes \mathcal{O}_T(M_T),$$

so $R^1 \varphi_* \mathcal{O}_Y(M)$ is torsion-free by [32, Theorem 2.1].

Since $K_Y \simeq \mathcal{O}_Y$ and φ is flat over a big open subset of T , we have by relative duality [30]

$$(R^1 \varphi_* \mathcal{O}_Y) \simeq (\varphi_* \mathcal{O}_Y(K_{Y/T}))^* \simeq \mathcal{O}_T(K_T)$$

in the complement of finitely many points. Since $R^1 \varphi_* \mathcal{O}_Y(M)$ is torsion-free, the morphism

$$R^1 \varphi_* \mathcal{O}_Y(M) \simeq \mathcal{O}_T(K_T) \otimes \mathcal{O}_T(M_T) \rightarrow \mathcal{O}_T(K_T + M_T)$$

is thus injective and yields an inclusion

$$H^0(T, R^1 \varphi_* \mathcal{O}_Y(M)) \rightarrow H^0(T, \mathcal{O}_T(K_T + M_T)).$$

The statement now follows from (5). □

We need the following characterisation of Example 4.3, a singular variant of [14, Theorem 0.2] and its proof.

Proposition 6.3. *Let S be a normal projective surface with canonical singularities such that $H^1(S, \mathcal{O}_S) \neq 0$. Let A be an ample, effective Cartier divisor such that $A^2 = 1$ and $K_S \cdot A = -1$. Then S is a \mathbb{P}^1 -bundle $f: \mathbb{P}(V) \rightarrow C$ over an elliptic curve C such that A is the tautological divisor.*

Proof. Since $K_S \cdot A = -1$, the canonical bundle is not pseudoeffective, so S is uniruled. Since $H^1(S, \mathcal{O}_S) \neq 0$, the surface is not rationally connected, and the MRC fibration coincides with the Albanese map $f: S \rightarrow \text{Alb}(S)$. In particular, f contracts all the rational curves on S and $\pi_1(S) \simeq \pi_1(C)$, where C is the image of f . Our goal is to show that $q(S) = 1$ and f is a \mathbb{P}^1 -bundle.

Since the Cartier divisor A is effective and $A^2 = 1$, the divisor A is an integral curve. By the adjunction formula, the arithmetic genus of A is one, so either A is smooth or its normalisation is \mathbb{P}^1 . Since the ample divisor A is not contracted by the fibration f , we see that A is a smooth elliptic curve. Now recall that canonical surface singularities are hypersurface singularities, so by Goresky-MacPherson's Lefschetz theorem for homotopy groups [39, Theorem 3.1.21, Remark 3.1.41], the morphism

$$\mathbb{Z}^2 \simeq \pi_1(A) \rightarrow \pi_1(S)$$

is surjective; in particular, $q(S) = 1$ and C is also an elliptic curve. Since

$$\pi_1(A) \rightarrow \pi_1(S) \simeq \pi_1(C)$$

is surjective, the étale map $A \rightarrow C$ is an isomorphism. Thus, A is a section of f . Since A is ample and has degree one on the fibres, all the f -fibres are integral curves. Thus, f is a \mathbb{P}^1 -bundle by [34, II, Theorem 2.8]. Since $A \cdot f = 1$, we have $S \simeq \mathbb{P}(f_* \mathcal{O}_S(A))$. \square

Remark 6.4. An elementary computation [20, V, Proposition 2.21] shows that in the situation of Proposition 6.3, the divisor A is adjoint (i.e., we have

$$A \simeq K_S + \Delta_S$$

with Δ_S an ample divisor with $\Delta_S^2 = 3$).

Proposition 6.5. *Let Z be a normal projective threefold with canonical Gorenstein singularities such that $q(Z) = 1$ and $-K_Z$ is nef. Let $S \subset Z$ be a normal surface with canonical singularities such that S is an ample Cartier divisor in Z . Suppose that*

$$h^0(Z, \mathcal{O}_Z(S)) = 1, \quad \text{and} \quad S \cdot (-K_Z)^2 \geq 2.$$

Then either $K_S \equiv 0$ or S is a ruled surface over an elliptic curve.

Proof. Let $\tau: Z' \rightarrow Z$ be a terminalisation of Z . Then $-K_{Z'} \simeq \tau^*(-K_Z)$ is nef and $h^i(Z', \mathcal{O}_{Z'}) = h^i(Z, \mathcal{O}_Z)$ for all $i \in \mathbb{N}$. Since $\tau^*S \cdot (-K_{Z'})^2 = S \cdot (-K_Z)^2 > 0$, the nef divisor $-K_{Z'}$ is not trivial. Thus, Z' is uniruled, and hence, $h^3(Z', \mathcal{O}_{Z'}) = 0$. Since $q(Z) = 1$ by assumption, we obtain $\chi(Z', \mathcal{O}_{Z'}) \geq 0$. Since τ^*S is nef and big and $-K_{Z'}$ is nef, we have

$$H^j(Z', \mathcal{O}_{Z'}(\tau^*S)) = H^j(Z', \mathcal{O}_{Z'}(K_{Z'} + (-K_{Z'} + \tau^*S))) = 0 \quad \forall j \geq 1$$

by Kawamata-Viehweg vanishing. Thus, the Riemann-Roch formula (Fact 2.6) yields

$$\begin{aligned} 1 = h^0(Z', \mathcal{O}_{Z'}(\tau^* S)) &= \chi(Z', \mathcal{O}_{Z'}(\tau^* S)) \\ &\geq \frac{1}{12} \tau^* S \cdot \tau^*(S - K_Z) \cdot \tau^*(2S - K_Z) + \frac{1}{12} \tau^* S \cdot c_2(Z'). \end{aligned} \quad (6)$$

Since $-K_{Z'}$ is nef and $\tau^* S$ is nef, we have $\tau^* S \cdot c_2(Z') \geq 0$ by [45, Corollary 1.5]. Applying the projection formula, we obtain

$$S \cdot (S - K_Z) \cdot (2S - K_Z) \leq 12.$$

Set $A := S|_S$ and $H = -K_Z|_S$. Then A is an ample Cartier divisor on S , and H is a nef and big Cartier divisor on S such that $H^2 \geq 2$. By the Hodge index inequality $(A \cdot H)^2 \geq A^2 \cdot H^2$ [8, Proposition 2.5.1], this implies $A \cdot H \geq 2$. Since

$$12 \geq S \cdot (S - K_Z) \cdot (2S - K_Z) = 2A^2 + 3A \cdot H + H^2 \geq 4 + 3A \cdot H,$$

we actually have $A \cdot H = 2$. Moreover, we have $A^2 \leq 2$.

1st case. Suppose that $A^2 = 2$. Then we have equality in the Hodge index inequality, and therefore, $A \equiv H$ by [8, Corollary 2.5.4]. Since

$$K_S \simeq (K_Z + S)|_S \simeq -H + A,$$

we obtain $K_S \equiv 0$.

2nd case. Suppose that $A^2 = 1$. Since

$$K_S \simeq (K_Z + S)|_S \simeq -H + A,$$

we have $K_S \cdot A = -1$. Finally, we have the Riemann-Roch inequality

$$h^0(S, \mathcal{O}_S(A)) \geq \frac{1}{2} A^2 + \frac{1}{2} (-K_S \cdot A) + \chi(\mathcal{O}_S).$$

Since S is an ample divisor in a threefold and Z has canonical singularities, we can use Kodaira vanishing to show that $q(S) = q(Z) = 1$. In particular, $\chi(\mathcal{O}_S) \geq 0$, and by the Riemann-Roch inequality, $h^0(S, \mathcal{O}_S(A)) > 0$, so A is an effective divisor. Thus, the polarised surface (S, A) satisfies the conditions of Proposition 6.3. \square

7. The nef case

The goal of this section is to show the following.

Theorem 7.1. *In the situation of setup 1.2, the divisor M_X is not nef.*

Setup 7.2. For the proof of Theorem 7.1, we will argue by contradiction and assume that M_X is nef. By the Fact 2.3, there exists a morphism with connected fibres

$$\varphi: X \rightarrow T \quad (7)$$

such that $M_X \simeq \varphi^* M_T$ for some ample Cartier divisor $M_T \rightarrow T$. Note that B_X is φ -ample since $B_X \sim_{\varphi} -K_X$.

Since X is a Fano, there exists a boundary divisor such that (X, Δ) is klt and $\Delta \sim_{\mathbb{Q}} -K_X$. Thus, we apply Ambro's theorem [2] to see that T is klt. Note that it is not clear whether T is \mathbb{Q} -factorial.

Lemma 7.3. *In the situation of Setup 7.2, we have $H^1(X, \mathcal{O}_X(-M_X)) = 0$ unless $T \simeq \mathbb{P}^1$ and $M_T \simeq \mathcal{O}_{\mathbb{P}^1}(m)$ with $m \geq 2$.*

Proof. Since $-K_X$ is φ -ample, we have $R^j \varphi_* \mathcal{O}_X = 0$ for every $j \geq 1$ by relative Kodaira vanishing. Since $M_X \simeq \varphi^* M_T$, we deduce that $R^j \varphi_* \mathcal{O}_X(-M_X) = 0$ for $j \geq 1$ by the projection formula. Thus, we have

$$H^1(X, \mathcal{O}_X(-M_X)) \simeq H^1(T, \mathcal{O}_T(-M_T)),$$

and the latter is zero by Kodaira vanishing on the klt space T unless T is a curve. The rest is now straightforward. \square

Corollary 7.4. *In the situation of Setup 7.2, we have $H^0(X, \mathcal{O}_X(B_X)) \neq 0$ unless $T \simeq \mathbb{P}^1$ and $M_T \simeq \mathcal{O}_{\mathbb{P}^1}(m)$ with $m \geq 2$.*

Proof. Immediate from Lemma 7.3 and the exact sequence

$$0 \rightarrow \mathcal{O}_X(-M_X) \rightarrow \mathcal{O}_X(B_X) \rightarrow \mathcal{O}_Y(B) \rightarrow 0.$$

\square

We will prove Theorem 7.1 by making a case distinction in terms of the dimension of the base T .

Proposition 7.5. *In the situation of Setup 7.2, we have $\dim T > 1$.*

Proof. If $\dim T = 1$, the fibration φ has as general fibre a smooth Fano threefold F (the divisor $-K_F \simeq -K_X|_F$ is ample) and $T \simeq \mathbb{P}^1$.

Choose $Y \in |-K_X|$ a general element, and denote by $\psi: Y \rightarrow \mathbb{P}^1$ the restriction of φ to Y . Denote by F_Y the general ψ -fibre; that is, $F_Y = F \cap Y$. We have a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(-K_X)) & \longrightarrow & H^0(Y, \mathcal{O}_Y(-K_X)) \\ \downarrow & & \downarrow r_Y \\ H^0(F, \mathcal{O}_F(-K_X)) & \longrightarrow & H^0(F_Y, \mathcal{O}_{F_Y}(-K_X)), \end{array}$$

and the horizontal maps are surjective since $q(X) = 0 = q(F)$.

1st case. Suppose that $\mathcal{O}_T(M_T) \simeq \mathcal{O}_{\mathbb{P}^1}(m)$ with $m \geq 2$. We have $-K_X|_Y \simeq \psi^* M_T + B$; moreover, the pair (Y, B) is lc by Corollary 2.12. Thus, by Proposition 6.1, the restriction map r_Y is surjective. Since $-K_X|_Y \simeq \psi^* M_T + B$ and B is a fixed component of the linear system $|-K_X|_Y$, the image of r_Y is generated by a section vanishing on $B \cap F_Y$. Thus, r_Y is a surjective map of rank one, and therefore, $h^0(F_Y, \mathcal{O}_{F_Y}(-K_X)) = 1$. The restriction map $H^0(F, \mathcal{O}_F(-K_X)) \rightarrow H^0(F_Y, \mathcal{O}_{F_Y}(-K_X))$ has kernel $H^0(F, \mathcal{O}_F) \simeq \mathbb{C}$, so we deduce

$h^0(F, \mathcal{O}_F(-K_X)) = 2$. Yet $-K_F \simeq -K_X|_F$, and for a smooth Fano threefold, we always have $h^0(F, \mathcal{O}_F(-K_F)) \geq 3$ by [24, Corollary 2.1.14].

2nd case. Suppose that $\mathcal{O}_T(M_T) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. By Corollary 7.4, this implies that B_X is an effective divisor. Since $-K_X \simeq \varphi^*M_T + B_X \simeq F + B_X$, the divisor B_X is relatively ample. We claim that

$$H^1(Y, \mathcal{O}_Y(-K_X|_Y - F_Y)) = H^1(Y, \mathcal{O}_Y(B)) = 0.$$

As in the first case, this yields the surjectivity of $H^0(Y, \mathcal{O}_Y(-K_X)) \rightarrow H^0(F_Y, \mathcal{O}_{F_Y}(-K_X))$ and the desired contradiction.

Proof of the claim. By Proposition 5.1, we know that B_X is not nef, so there exists a K_X -negative extremal ray $\mathbb{R}^+\gamma$ such that $B_X \cdot \gamma < 0$. Since B_X is effective, the corresponding contraction

$$\mu: X \rightarrow X'$$

must be birational with exceptional locus contained in B_X . Since B_X is φ -ample and μ -antiample, the intersection of any μ -fibre with a φ -fibre must be finite. Thus, the fibres of μ have dimension at most one. By Ando's theorem [3, Theorem 2.3], this implies that μ is a smooth blowup along a surface. In particular, we have $(-K_X + B_X) \cdot \gamma = 0$, so $-K_X + B_X$ is non-negative on the extremal ray $\mathbb{R}^+\gamma$. Since γ was an arbitrary B_X -negative extremal ray, the cone theorem implies that $-K_X + B_X$ is nef. Since B_X is effective, the divisor $-K_X + B_X$ is nef and big. Therefore,

$$H^j(X, \mathcal{O}_X(B_X)) = H^j(X, K_X + \mathcal{O}_X(-K_X + B_X)) = 0$$

for $j \geq 1$ by Kawamata-Viehweg vanishing. Now consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-M_X) \rightarrow \mathcal{O}_X(B_X) \rightarrow \mathcal{O}_Y(B) \rightarrow 0.$$

Since $M_X \simeq \varphi^*M_T$ and $\dim T = 1$, we have $H^2(X, \mathcal{O}_X(-M_X)) \simeq H^2(T, \mathcal{O}_T(-M_T)) = 0$. By the long exact sequence in cohomology, the map $0 = H^1(X, \mathcal{O}_X(B_X)) \rightarrow H^1(Y, \mathcal{O}_Y(B))$ is surjective, and we are finally done. \square

Proposition 7.6. *In the situation of Setup 7.2, we have $\dim T < 3$.*

Proof. Assume that $\dim T \geq 3$. Since Y is an ample divisor, the morphism $\varphi|_Y: Y \rightarrow T$ is generically finite onto its image, so $M \simeq \varphi^*M_T|_Y$ is nef and big. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(M) \rightarrow \mathcal{O}_Y(-K_X) \rightarrow \mathcal{O}_B(-K_X) \rightarrow 0.$$

Since $H^1(Y, \mathcal{O}_Y(M)) = 0$, by Kawamata-Viehweg vanishing, the restriction morphism

$$H^0(Y, \mathcal{O}_Y(-K_X)) \rightarrow H^0(B, \mathcal{O}_B(-K_X))$$

is surjective. Since B is in the base locus of $| -K_X|_Y|$, we obtain $H^0(B, \mathcal{O}_B(-K_X)) = 0$.

Since $\dim T \geq 3$, we have $h^0(X, \mathcal{O}_X(B_X)) \neq 0$ by Corollary 7.4. Thus, Proposition 5.11 shows that we have an injection

$$H^0(B, \mathcal{O}_B(-K_X)) \hookrightarrow H^1(B, \mathcal{O}_B).$$

By Corollary 2.12, the surface B satisfies the conditions of Proposition 4.2. Thus, we have $q(B) > 0$, and the inclusion $H^0(B, \mathcal{O}_B(-K_X)) \hookrightarrow H^1(B, \mathcal{O}_B)$ is an equality. Since the first space has dimension zero, this is a contradiction. \square

Proposition 7.7. *In the situation of Setup 7.2, we have $\dim T \neq 2$*

This is the part of the proof that requires the most work. We will use properties of the anticanonical divisor Y and the effective divisor B_X to determine some of the invariants of the base T . Then we will use the smoothness of X to reach a contradiction.

Proof. Assume that $\dim T = 2$. By Corollary 7.4, the divisor B_X is effective. By Corollary 5.12, the divisor B_X has canonical Gorenstein singularities, and by Proposition 5.11, we have

$$h^0(B_X, \mathcal{O}_{B_X}(-K_X)) = 1 \quad \text{and} \quad q(B_X) = q(B).$$

Since $Y \subset X$ is an ample divisor, the fibration φ induces an elliptic fibration

$$\psi: Y \rightarrow T$$

such that $-K_X|_Y \simeq \psi^* M_T + B$. By Theorem 2.8 and Corollary 2.9, the threefold Y satisfies the conditions of Lemma 6.2. Hence, we have an injection

$$H^0(B, \mathcal{O}_B(K_B + M)) \hookrightarrow H^0(T, \mathcal{O}_T(K_T + M_T)). \quad (8)$$

Observe that the induced morphism $\varphi|_{B_X}$ is surjective onto T since B_X is φ -ample.

Step 1. We show that $h^0(T, \mathcal{O}_T(K_T + M_T)) = 0$ unless $K_T \simeq -M_T$. By the adjunction formula,

$$-K_{B_X} \simeq M_X|_{B_X} \simeq (\varphi^* M_T)|_{B_X}$$

is nef with numerical dimension two. Since B_X has canonical singularities, we can apply [13, Theorem 3.1] to see that there exists a boundary divisor Δ_T on T such that (T, Δ_T) is klt and

$$K_{B_X} \sim_{\mathbb{Q}} (\varphi^*(K_T + \Delta_T))|_{B_X}.$$

Thus, we have $K_T + \Delta_T \equiv -M_T$, and hence, $K_T + M_T = -\Delta_T$ is not pseudoeffective unless $\Delta_T = 0$. Therefore, $h^0(T, \mathcal{O}_T(K_T + M_T)) = 0$ unless $\Delta_T = 0$. In the latter case, $h^0(T, \mathcal{O}_T(K_T + M_T)) \neq 0$ implies that $K_T \simeq -M_T$; in particular, K_T is Cartier.

Step 2. We reach a contradiction. By Proposition 5.11, we have an injection

$$H^0(B, \mathcal{O}_B(-K_X)) \hookrightarrow H^1(B, \mathcal{O}_B).$$

By Corollary 2.12, the surface B satisfies the conditions of Proposition 4.2. In particular, B has canonical singularities, positive irregularity $q(B) > 0$, and the inclusion above is an equality. Since

$$H^0(B, \mathcal{O}_B(-K_X)) = H^0(B, \mathcal{O}_B(K_B + M)),$$

the injection (8) and the first step show that

$$H^0(B, \mathcal{O}_B(-K_X)) = H^1(B, \mathcal{O}_B) = H^0(T, \mathcal{O}_T(K_T + M_T)) = \mathbb{C}.$$

In particular, $K_T \simeq -M_T$ by the first step (i.e., T is a del Pezzo surface with at most canonical singularities).

Since B is the complete intersection of B_X and Y , the surface $B \subset B_X$ is an ample Cartier divisor, and the restriction of $\varphi|_{B_X}$ to B is still surjective onto the surface T . Since $q(T) = 0$ and $q(B) = 1$, and the irregularity is a birational invariant of varieties with rational singularities, the map

$$\tau := \varphi|_T : B \rightarrow T$$

is not birational. Thus, τ is *generically* finite of degree at least two and

$$(M_X|_B)^2 = (\tau^* M_T)^2 = \deg \tau \cdot M_T^2 \geq 2.$$

Thus, $q(B_X) = q(B) = 1$, $-K_{B_X} \simeq M_X|_{B_X}$ is nef, $B \subset B_X$ is an ample Cartier divisor with $h^0(B_X, \mathcal{O}_{B_X}(B)) = h^0(B_X, \mathcal{O}_{B_X}(-K_X)) = 1$ and $B \cdot (-K_{B_X})^2 \geq 2$. Thus, the threefold B_X and the surface $B \subset B_X$ satisfy the conditions of Proposition 6.5, and we have two cases:

Case a) We have $K_B \equiv 0$. Since $K_B \simeq B|_B$, this implies that B is nef. Thus, B_X is nef by Theorem 2.10, a contradiction to Proposition 5.1.

Case b) B is a ruled surface over an elliptic curve. Since $\rho(\mathbb{P}(V)) = 2$ and the map τ is surjective, the del Pezzo surface T has Picard number at most two (canonical singularities are \mathbb{Q} -factorial, [31, Proposition 4.11]). By Remark 6.4, the nef and big class $M_X|_B$ is ample and $(M_X|_B)^2 = 3$. Since $M_X|_B \simeq \tau^* M_T \simeq \tau^*(-K_T)$ is ample, the generically finite map τ is actually finite. Moreover, τ having degree at least two, we deduce that τ has degree three and $(-K_T)^2 = 1$. Thus, T is a del Pezzo surface of degree one and Picard number at most two; in particular, it is not smooth. By [4, Proposition 1.3], this implies that the fibration $\varphi : X \rightarrow T$ is not equidimensional, so there exists a prime divisor $D \subset X$ that is contracted onto a point in T .

Since the effective divisors B_X and Y are φ -ample, the intersection

$$D \cap B_X \cap Y = D \cap B$$

is non-empty. Hence, there exists a curve $E \subset B$ that is contracted by τ . Yet we showed above that τ is finite, a contradiction. \square

Remark. It may seem annoying that the case of a fibration $X \rightarrow T$ with T a del Pezzo surface of degree one requires so much additional effort. Note, however, that this situation is very close to Example 3.2, so our arguments must be specific enough to rule out this situation.

8. The conclusion.

Proof of Theorem 1.3. Assume that a general anticanonical divisor $Y \in |-K_X|$ is \mathbb{Q} -factorial, so we satisfy the Assumption 1.4 from the Setup 1.2. By Theorem 5.13, the divisor M_X is nef, yet this contradicts Theorem 7.1. \square

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