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COVERING GAMES AND THE BANACH-MAZUR GAME: *K*-TACTICS.

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ABSTRACT Given a free ideal J of subsets of a set X, we consider games where player ONE plays an increasing sequence of elements of the σ-completion of J, and player TWO tries to cover the union of this sequence by playing one set at a time from J We describe various conditions under which player TWO has a winning strategy that uses only information about the most recent k moves of ONE, and apply some of these results to the Banach-Mazur game

1. **Introduction.** Let J be a free ideal of subsets of a given set. By $\langle J\rangle$ we denote the σ-ideal generated by J ($\langle J\rangle$ could turn out to be the power set of $\cup J$). Two concrete examples of ideals motivated much of our work. The one is $\mathcal{NWD}_{\mathbb{R}}$, the ideal of nowhere dense subsets of the real line $\mathbb{R}$. In this case $\langle\mathcal{NWD}_{\mathbb{R}}\rangle$ is the ideal of meager sets of reals. The other is $[\kappa]^{<\lambda}$ where $\omega = \mathrm{cof}(\lambda) \leq \lambda \leq \kappa$ are cardinal numbers.

We are interested in games of the following type: Player ONE plays a set $O_n \in \langle J\rangle$ during inning n, to which TWO responds with a set $T_n \in J$. ONE is required to play an increasing sequence of sets; TWO's objective is to cover $\bigcup_{n\in\omega} O_n$ with $\bigcup_{n\in\omega} T_n$. As long as TWO remembers the complete history of the game, this task is trivial. However, it often happens that TWO needs to know only the last k moves of the opponent in order to win. A strategy that accomplishes this is called a *winning k-tactic*.

We consider four such games, MG($\mathcal{A}, J$), MG(J), the "monotonic game", SMG(J), the "strongly monotonic game", and VSG(J), the "very strong game". The study of these games was initiated in [S1], and motivated by Telgarský's conjecture that for every $k > 0$ there exists a topological space (X, τ) such that TWO has a winning $k+1$-tactic but no winning k-tactic in the Banach-Mazur game on (X, τ) (see Section 4.4 for more information). However, we find the games considered here of interest independent of the original motivation. The game MG(J) was introduced in [S1], as was the game SMG(J); the games MG($\mathcal{A}, J$) and VSG(J) appear here for the first time.

In Sections 2 and 3, we introduce and discuss pseudo-Lusin sets, the irredundancy property and the coherent decomposition property of ideals. These properties, together with the ω-path partition relation, are the main tools for constructing winning k-tactics

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in our games These combinatorial properties of ideals are very likely of independent interest—they have already appeared in the literature in various guises

In Section 4 we apply the results of Sections 2 and 3 to give various conditions sufficient for the existence of winning k-tactics for TWO in the games mentioned above Not surprisingly, as the game becomes more favorable for TWO, weaker conditions suffice Among other things, our results show that in the Banach-Mazur game on the space that inspired the invention of meager-nowhere dense games, TWO has a winning 2-tactic

The appendix is devoted to a proof of an unpublished consistency result of Stevo Todorčević, which we use in Section 4

Our notation is mostly standard One important exception may be that we use the symbol $\subset$ exclusively to mean "is a proper subset of" Where we otherwise deviate from standard notation or terminology we explicitly alert the reader For convenience we also assume the consistency of traditional (Zermelo-Fraenkel) set theory All statements we make about the consistency of various mathematical assertions must be understood as consistency which can be proven by means of that theory

We are grateful to Stevo Todorčević for sharing with us his insights about the matters we study here, and for his kind permission to present in this paper some of his answers to our questions

2. **The irredundancy property.** For a partially ordered set $(P, <)$ which has no maximum element we let

$$\operatorname{add}(P, <)$$

be the least cardinal number, λ, for which there is a collection of cardinality λ of elements of P which do not have an upper bound in P This cardinal number is said to be the *additivity* of $(P, <)$ Note that $\operatorname{add}(P, <)$ is either 2, or else it is infinite In the latter case $(P, <)$ is said to be *directed* We attend exclusively to directed partially ordered sets in this paper

A free ideal J on a set S is partially ordered by $\subset$ The partially ordered set $(J, \subset)$ is directed The symbol $\langle J\rangle$ denotes the σ-completion of J (*i e* , the smallest collection which contains each union of countably many sets from J) We say that J is a *σ-complete ideal* if $J = \langle J\rangle$ Note that J is σ-complete exactly when $\operatorname{add}(J)$ is uncountable

The other important example for our study is the set ${}^{\omega}\omega$ of sequences of nonnegative integers, we use $\mathfrak{c}$ to denote the cardinality of this set We say g *eventually dominates* f and write $f \ll g$ if $\lim_{n\to\infty}\big(g(n) - f(n)\big) = \infty$ It is customary to denote $\operatorname{add}({}^{\omega}\omega, \ll)$ by $\mathfrak{b}$

A well known theorem of Miller ([M], p 94, Theorem 1 2) states that

$$\operatorname{add}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle, \subset) \leq \operatorname{add}({}^{\omega}\omega, \ll) \quad (= \mathfrak{b})$$

Again, for an arbitrary partially ordered set $(P, <)$ the symbol

$$\operatorname{cof}(P, <)$$

denotes the least cardinal number, κ, for which there is a collection X of cardinality κ of elements of P such that: for each $p \in P$ there is an $x \in X$ such that $p \leq x$. This cardinal number is said to be the *cofinality* of $(P, <)$. It is customary to denote $\mathrm{cof}({}^{\omega}\omega, \ll)$ by $\mathfrak{d}$.

Another well-known theorem (see *e.g.* [F], Proposition 13(b)) states that

$$(\mathfrak{d} =) \, \mathrm{cof}({}^{\omega}\omega, \ll) \leq \mathrm{cof}(\langle \mathcal{NWD}_{\mathbb{R}} \rangle, \subset).$$

This theorem, as well as Miller's theorem cited above, are consequences of the construction below for Example 2.

Let $(P, <)$ be a directed partially ordered set. The *bursting number* of $(P, <)$ ([I], p. 401) is the smallest cardinal number which exceeds the cardinality of each of the bounded subsets of $(P, <)$. This cardinal number is denoted by $\mathrm{burst}(P, <)$. More important is the *principal bursting number* of $(P, <)$, denoted $\mathrm{bu}(P, <)$ and define as

$$\mathrm{bu}(P, <) = \min\{\mathrm{burst}(Q, <) : Q \text{ is a cofinal subset of } P\}$$

(following [I], p. 409). It is always the case that $\mathrm{add}(P, <) \leq \mathrm{bu}(P, <)$.

DEFINITION 1. A directed partially ordered set $(P, <)$ has the *irredundancy property* if:

$$\mathrm{bu}(P, <) = \mathrm{add}(P, <).$$

The cofinal subfamily $\mathcal{A}$ of $(P, <)$ is said to be *irredundant* if $\mathrm{burst}(\mathcal{A}, <) \leq \mathrm{add}(P, <)$.

Not all σ-complete ideals have the irredundancy property. Here is an ad hoc example. Let S_1 and S_2 be disjoint sets such that S_i has cardinality $\aleph_i$ for each i. Define an ideal J on the union of these sets by admitting a set Y into J if: $Y \cap S_1$ is countable and $Y \cap S_2$ has cardinality less than $\aleph_2$. Then $\mathrm{add}(J, \subset) = \aleph_1$ and $\mathrm{cof}(J, \subset) = \aleph_2$. No cofinal family of J is irredundant.

A refined version of the classical notion of a Lusin set is instrumental in verifying the presence of the irredundancy property in many directed partially ordered sets. Since what we'll define is not exactly the same as the classical notion, we call our "Lusin sets" *pseudo-Lusin sets*. Let $\kappa \leq \lambda$ be infinite cardinal numbers. Let $(P, <)$ be a directed set.

DEFINITION 2. A subset L of P is a (κ, λ) pseudo-Lusin set if:

(1) λ is the cardinality of L and

(2) for each $x \in P$ the cardinality of the set $\{y \in L : y \leq x\}$ is less than κ.

If a directed set $(P, <)$ has a (κ, λ) pseudo-Lusin set, then $\mathrm{add}(P, <) \leq \kappa$ and $\lambda \leq \mathrm{cof}(P, <)$. Moreover, every partially ordered set has an $\big(\mathrm{add}(P, <), \mathrm{add}(P, <)\big)$ pseudo-Lusin set. Thus, if $\mathrm{add}(P, <) = \mathrm{cof}(P, <)$, then these are the only types of pseudo-Lusin sets in $(P, <)$.

Let J be a free ideal on a set S. The uniformity number of J, written $\mathrm{unif}(J)$, is the minimal cardinal κ such that there is a subset of S of cardinality κ, which is not an element of J.

If $L \subset \mathbb{R}$ is a Lusin set in the classical sense (*i.e.*, L is uncountable and every meager set meets L in only countably many points), then $\big\{\{x\} : x \in L\big\}$ is an $(\omega_1, |L|)$ pseudo-Lusin set. There will be pseudo-Lusin sets even when there are no (classical) Lusin sets:

If $\mathrm{unif}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle) > \mathrm{add}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle, \subset)$ (see *e.g.* [M], §6 for a consistency proof of this inequality), then every set of real numbers of cardinality $\aleph_1$ is meager, whence there is no Lusin set in the classical sense.

The reader might compare our notion of a (κ, λ)-pseudo-Lusin set with Cichon's notion of a (κ, λ)-Lusin set (see [Ci]).

For $(P, <)$ a directed set, the connection between the irredundancy property and the existence of certain pseudo-Lusin sets is as follows: There is an $\big(\mathrm{add}(P, <), \mathrm{cof}(P, <)\big)$ pseudo-Lusin set for $(P, <)$ if, and only if, $(P, <)$ has the irredundancy property, if, and only if, $(P, <)$ has a cofinal $\big(\mathrm{add}(P, <), \mathrm{cof}(P, <)\big)$-pseudo-Lusin set. These equivalences could be proven by an argument as in the proof of 4.4 on p. 409 of [I].

COROLLARY 1. *Let $\kappa > \lambda \geq \aleph_0$ be cardinals, λ regular. If* $\mathrm{cof}([\kappa]^{<\lambda}, \subset) = \kappa$, *then* $([\kappa]^{<\lambda}, \subset)$ *has the irredundancy property.*

PROOF. Let $\{S_\alpha : \alpha < \kappa\}$ be a pairwise disjoint subcollection from $[\kappa]^{<\lambda}$. Then this family is a (λ, κ) pseudo-Lusin set for this ideal. Applying the cofinality hypothesis we conclude that this ideal has the irredundancy property. ■

The ideal of finite subsets of an infinite set has the irredundancy property; the set of one-element subsets of such an infinite set forms an appropriate pseudo-Lusin set for this ideal.

LEMMA 2. *Let $\kappa > \lambda$ be an uncountable cardinal numbers, λ regular. Then the following statements are equivalent:*

(1) The ideal $([\kappa]^{<\lambda}, \subset)$ *has cofinality κ.*

(2) There is a free ideal J such that:

(a) $\mathrm{add}(J, \subset) = \lambda$,

(b) $\mathrm{cof}(J, \subset) = \kappa$ *and*

(c) $(J, \subset)$ *has the irredundancy property.*

PROOF. The proof of $1 \Rightarrow 2$ is trivial. We show that 2 implies 1. Let J be a free ideal on the set S such that $\mathrm{cof}(J, \subset) = \kappa$ and $\mathrm{add}(J, \subset) = \lambda$, and $(J, \subset)$ has the irredundancy property. Let $L \subset J$ be an (λ, κ) pseudo-Lusin set for J. Also let $\mathcal{C} \subset J$ be a cofinal family of cardinality κ. For each $X \in \mathcal{C}$ define: $S_X = \{Y \in L : Y \subseteq X\}$. Then the collection $\mathcal{B} = \{S_X : X \in \mathcal{C}\}$ is cofinal in $([L]^{<\lambda}, \subset)$. ■

The following examples play an important role in our game-theoretic applications.

EXAMPLE 1. The ideal of countable subsets of an infinite set.

Let κ be an uncountable cardinal number. Then $\mathrm{add}([\kappa]^{\leq\aleph_0}, \subset) = \aleph_1$ and $\mathrm{bu}([\kappa]^{\leq\aleph_0}, \subset) \geq \aleph_1$. For uncountable cardinal numbers κ it is always the case that $\kappa \leq \mathrm{cof}([\kappa]^{\leq\aleph_0}, \subset)$. A set of the form $\big\{\{\alpha_\xi\} : \xi < \kappa\big\}$ (where this enumeration is bijective and $\omega_1 \leq \kappa$) is an (ω_1, κ) pseudo-Lusin set for $[\kappa]^{\leq\aleph_0}$. The only difficult cases to decide whether or not the irredundancy property is present are those where $\kappa < \mathrm{cof}([\kappa]^{\leq\aleph_0}, \subset)$; this occurs for example when κ has countable cofinality. It turns out that for these the irredundancy property is not decidable by the axioms of traditional set theory:

A family $\mathcal{A}$ of countable subsets of κ is *locally countable* if for each $A \in \mathcal{A}$ the set $\{B \in \mathcal{A} : B \subseteq A\}$ is countable. A family $\mathcal{K} \subset [\kappa]^{\aleph_0}$ which satisfies the stronger property that $|\{A \cap X : X \in \mathcal{K}\}| \leq \aleph_0$ for any countable subset A of κ, is said to be a *Kurepa family*. Note that the existence of a cofinal Kurepa family witnesses that $\mathrm{bu}([\kappa]^{\aleph_0}, \subset) = \aleph_1 = \mathrm{add}([\kappa]^{\aleph_0}, \subset)$.

(1) Todorčević has shown (p. 843 of [To4] or [To2]) that $([\kappa]^{\aleph_0}, \subset)$ has the irredundancy property if, and only if, there is a Kurepa family in $[\kappa]^{\aleph_0}$ of cardinality $\kappa^{\aleph_0}$. In §2 of [To5] he showed that if κ has countable cofinality then $\square_\kappa$ implies that there is a Kurepa family of cardinality κ^+ in $[\kappa]^{\aleph_0}$. Thus, if $\square_\kappa$ and moreover $\mathrm{cof}([\kappa]^{\aleph_0}, \subset) = \kappa^+$ is true for each uncountable cardinal κ of countable cofinality, then there is for each uncountable cardinal λ a *cofinal* Kurepa family in $[\lambda]^{\aleph_0}$. These hypotheses hold in **L**, the constructible universe.

One might ask if any hypotheses beyond ZFC are necessary to obtain the conclusion that $([\kappa]^{\leq\aleph_0}, \subset)$ has a cofinal Kurepa family. Todorčević also noted (p. 843 of [To4]) that the version

$$(\aleph_{\omega+1}, \aleph_\omega) \longrightarrow (\omega_1, \omega)$$

of Chang's Conjecture implies that $\aleph_1 < \mathrm{bu}([\aleph_\omega]^{\leq\aleph_0}, \subset)$ (and thus this ideal does not have the irredundancy property). Now [L-M-S] established the consistency of the above version of Chang's Conjecture modulo the consistency of the existence of a fairly large cardinal.

(2) This takes care of uncountable cardinals of countable cofinality. What is the situation for those of uncountable cofinality? It is clear that $([\kappa]^{\leq\aleph_0}, \subset)$ has the irredundancy property if κ is $\aleph_n$ for some finite n (a result of Isbell, [I]) or if, for some $m < \omega$, κ is the m-th successor of a singular strong limit cardinal of uncountable cofinality. In fact, the axiomatic system of traditional set theory has to be strengthened fairly dramatically before one could create circumstances where there is a cardinal number of uncountable cofinality which is strictly less than the cofinality of its ideal of countable sets; it follows from Lemma 4.10 of [J-M-P-S] that if there is a cardinal number of uncountable cofinality which is smaller than the cofinality of its ideal of countable sets, then there is an inner model with many measurable cardinal numbers.

Information about the ideal of countable subsets of some infinite set can be used to gain information about some other ideals, using the notion of a locally small family.

DEFINITION 3. A family $\mathcal{F}$ of subsets of a set S is *locally small* if:

$$|\{Y \in \mathcal{F} : Y \subseteq X\}| \leq \aleph_0$$

for each X in $\mathcal{F}$.

If the ideal of countable subsets of an infinite set has an irredundant cofinal family then that cofinal family is ipso facto locally small. If there is an $\big(\omega_1, \mathrm{cof}(J, \subset)\big)$ pseudo-Lusin set for the σ-complete free ideal J on the set S, then J contains a locally small cofinal family.

EXAMPLE 2 The ideal of meager subsets of the real line

Assume that $\mathrm{add}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle, \subset) = \mathrm{cof}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle, \subset)$ (This equation is for example implied by Martin's Axiom) Then $\langle \mathcal{NWD}_{\mathbb{R}}\rangle$ has the irredundancy property In this case one may insure that the cofinal family which witnesses the irredundancy is a well-ordered chain of meager sets

Irredundancy does not require having a well ordered cofinal chain of meager sets For let an initial ordinal be given According to a theorem of Kunen ([K], p 906, Theorem 3 18) it is consistent that the cardinality of the real line is regular and larger than that initial ordinal, and at the same time there is an $(\omega_1, \mathfrak{c})$ pseudo-Lusin set It follows that $\langle \mathcal{NWD}_{\mathbb{R}}\rangle$ has a locally small cofinal family of cardinality $\mathfrak{c}$ In particular, $\langle J_{\mathbb{R}}\rangle$ has the irredundancy property If the continuum is larger than $\aleph_1$ it also follows that this ideal has no cofinal well-ordered chain

Stevo Todorčević has informed us that it is also consistent, modulo the consistency of a form of Chang's Conjecture that $\langle J_{\mathbb{R}}\rangle$ does not have the irredundancy property Actually, something apparently weaker than that form of Chang's Conjecture is used we present this result of Todorčević's in Theorem 3, which he kindly permitted us to include in this paper

THEOREM 3 (TODORČEVIĆ) *If* "ZFC + $\mathrm{MA}_{\aleph_1}$ + there is no Kurepa family in $[\aleph_\omega]^{\aleph_0}$ of cardinality larger than $\aleph_\omega$" *is a consistent theory, then so is the theory* "ZFC + $\mathrm{bu}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle, \subset) > \mathrm{add}(\langle \mathcal{NWD}_{\mathbb{R}}\rangle, \subset) = \aleph_1$"

PROOF Let $\mathbf{P}$ be the set of finite functions with domain a subset of $\aleph_\omega$ and range a subset of ω and let $\mathbf{P}$ be partially ordered by reverse inclusion (in other words, $\mathbf{P}$ is the standard set for adding $\aleph_\omega$ Cohen reals) For D a countable subset of $\aleph_\omega$ we write $\mathbf{P}(D)$ for the set of elements of $\mathbf{P}$ whose domains are subsets of D

Suppose we have a sequence $\{N_\xi \quad \xi < \theta\}$ $(\theta > \aleph_\omega)$ of $\mathbf{P}$-names for meager sets of reals Let $D_\xi \in [\aleph_\omega]^{\aleph_0}$ be the support of N_ξ *i e*, $N_\xi \in \mathbf{V}^{\mathbf{P}(D_\xi)}$ By the hypothesis of the theorem and by [To4], p 843, there is an uncountable set $A \subset \theta$ such that $D = \bigcup_{\xi\in A} D_\xi$ is countable Thus, $N_\xi \in \mathbf{V}^{\mathbf{P}(D)}$ for each $\xi \in A$ Since $\mathbf{P}(D)$ is essentially the poset for adding 1 Cohen real and since $\mathrm{MA}_{\aleph_1}$ holds, $\mathbf{V}^{\mathbf{P}(D)} \models$ "$\bigcup_{\xi\in A} N_\xi$ is meager" (because $\mathbf{V}^{\mathbf{P}(D)} \models$ "MA(σ-centered)") ■

The hypothesis of Theorem 3 is consistent modulo the consistency of the relevant form of Chang's Conjecture, because that form of the conjecture is preserved by c c c generic extensions

3. **The coherent decomposition property.** Let J be a free ideal on a set S and let $\langle J\rangle$ be its σ-completion Let $\mathcal{A}$ be a subcollection of $\langle J\rangle$

DEFINITION 4 (1) $\mathcal{A}$ has a coherent decomposition if there is for each $A \in \mathcal{A}$ a sequence $(A^n \quad n < \omega)$ such that

(a) $A^n \in J$ for each n,

(b) $A^n \subseteq A^m$ whenever $n < m < \omega$, and

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(c) For all A and B in $\mathcal{A}$ such that $A \subset B$, there is an m such that $A^n \subseteq B^n$ whenever $n \geq m$.

The collection $\{(A^n : n < \omega) : A \in \mathcal{A}\}$ is said to be a coherent decomposition for $\mathcal{A}$.

(2) The ideal J has the coherent decomposition property if some cofinal subset of $\langle J \rangle$ has a coherent decomposition.

It is worth mentioning that if J has the coherent decomposition property and if $\langle J \rangle$ has a cofinal chain, than the family $\langle J \rangle$ itself has a coherent decomposition. We now explore the coherent decomposition property for our examples.

EXAMPLE 1 (CONTINUED).

THEOREM 4. *Let $\mathcal{A}$ be a locally small family of countable sets such that $(\mathcal{A}, \subset)$ is a well-founded partially ordered set. Then $\mathcal{A}$ has a coherent decomposition.*

PROOF. Let $\Phi: \mathcal{A} \longrightarrow \alpha$ be the rank function for the well-founded set $(\mathcal{A}, \subset)$. Since $\mathcal{A}$ is locally small we may assume that α is ω_1.

For A in $\mathcal{A}$ with $\Phi(A) = 0$, choose a sequence $(A^n : n < \omega)$ of finite subsets of A such that $A = \bigcup_{n<\omega} A^n$ and $A^n \subseteq A^{n+1}$ for all n.

Let $0 < \beta < \omega_1$ be given and assume that we have already assigned to each A in $\mathcal{A}$ for which $\Phi(A) < \beta$, a sequence $(A^n : n < \omega)$ in compliance with 1 and 2. Now Let B be an element of $\mathcal{A}$ such that $\Phi(B) = \beta$. Write $F(B) = \{A \in \mathcal{A} : A \subset B\}$.

To begin, arbitrarily choose a sequence $(S_n : n < \omega)$ of finite sets such that $B = \bigcup_{n<\omega} S_n$. For each $A \in F(B)$, define $g_A : \omega \longrightarrow \omega$ such that for each $n < \omega$,

$$g_A(n) = \min\{k < \omega : A^n \subseteq S_0 \cup \cdots \cup S_k\}.$$

Then $\{g_A : A \in F(B)\}$ is countable since $\mathcal{A}$ is locally small. Let $f \in {}^{\omega}\omega$ be a strictly increasing function such that $g_A \ll f$ for each A in $F(B)$. Define:

$$B^n = S_0 \cup \cdots \cup S_{f(n)}$$

for each n. Then $(B^n : n < \omega)$ is as required. ∎

COROLLARY 5. *Let J be a free ideal on a set S and let $\mathcal{A}$ be a locally small family of sets in $\langle J \rangle$ such that $(\mathcal{A}, \subset)$ is a well-founded partially ordered set. Then $\mathcal{A}$ has a coherent decomposition.*

PROOF. For each B in $\mathcal{A}$, let $\big(S_n(B) : n < \omega\big)$ be a sequence from J such that $B = \bigcup_{n<\omega} S_n(B)$. Also write $\Gamma(B) = \{A \in \mathcal{A} : A \subseteq B\}$. Then $\mathcal{B} = \{\Gamma(A) : A \in \mathcal{A}\}$ is a well-founded, locally small collection of countable subsets of $\mathcal{A}$. Choose, by Theorem 4, for each $A \in \mathcal{A}$ a sequence $\big(\Gamma(A)^n : n < \omega\big)$ of finite subsets of $\Gamma(A)$ such that:

(1) $\Gamma(A) = \bigcup_{n<\omega} \Gamma(A)^n$ where $\Gamma(A)^n \subseteq \Gamma(A)^{n+1}$ for each n, and

(2) for all A and B in $\mathcal{A}$ with $A \subset B$ there exists an m such that:

$$\Gamma(A)^n \subseteq \Gamma(B)^n$$

for all $n \geq m$.

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For each A in $\mathcal{A}$ and each $n < \omega$ define:

$$A^n = \bigcup\{S_j(B) : j \leq n \text{ and } B \in \Gamma(A)^n\}.$$

Then the sequences $(A^n : n < \omega)$ are as required. ■

COROLLARY 6. *If $([\kappa]^{\leq\aleph_0}, \subset)$ has the irredundancy property, then it has the coherent decomposition property.*

PROOF. An irredundant cofinal family is necessarily locally small. We may thin out any cofinal family to a well-founded cofinal family. Now apply Theorem 4. ■

Here is a result whose proof is quite analogous to that of Theorem 4. We state it in the present form because we'll use it in this form.

THEOREM 7. *Let λ be an uncountable cardinal number which has countable cofinality. Let $\lambda_0 < \lambda_1 < \cdots$ be a sequence of infinite regular cardinal numbers which converges to λ. Let $(\mathcal{A}, \subset)$ be a well-founded family of sets, each of cardinality λ, such that*

$$|\{Y \in \mathcal{A} : Y \subseteq X\}| \leq \lambda$$

for each X in $\mathcal{A}$. Then $\mathcal{A}$ has the coherent decomposition property. In particular:

There exists for each $A \in \mathcal{A}$ a sequence $(A^n : n < \omega)$ such that:

(1) $|A^n| \leq \lambda_n$ for all n,
(2) $A^n \subseteq A^{n+1}$ for all n,
(3) $A = \bigcup_{n=0}^{\infty} A^n$ and
(4) if $A \subset B$, then there is an $m < \omega$ such that $A^n \subseteq B^n$ for all $n \geq m$.

COROLLARY 8. *Let λ be a cardinal number of countable cofinality. If $([\kappa]^{\leq\lambda}, \subset)$ has the irredundancy property then it has the coherent decomposition property.*

EXAMPLE 2 (CONTINUED). We show that the ideal of meager sets of the real line has the coherent decomposition property, and also that it has a second combinatorial property which plays an important role in our game-theoretic applications. It is convenient, for this section, to work with the set ${}^{\omega}2$, with the usual Tychonoff product topology ($2 = \{0, 1\}$ is taken to have the discrete topology) in place of $\mathbb{R}$. For a subset S of the domain of a function g, the symbol $g\lceil_S$ denotes the restriction of g to the set S. For s an element of ${}^{<\omega}2$, the symbol $[s]$ denotes the set of all those x in ${}^{\omega}2$ for which $x\lceil_{\text{length}(s)} = s$. Subsets of ${}^{\omega}2$ of the form $[s]$ where s ranges over ${}^{<\omega}2$, form a base for the topology of ${}^{\omega}2$. Let $f \in {}^{\omega}\omega$ be a strictly increasing sequence and let x be an element of ${}^{\omega}2$. Define:

$$B_{x,f} = \{z \in {}^{\omega}2 : \forall_n^{\infty}(z\lceil_{[f(n),f(n+1))} \neq x\lceil_{[f(n),f(n+1))})\}.$$

Now also fix an $n \in \omega$ and define

$$B_{x,f}^n = \{z \in {}^{\omega}2 : (\forall k \geq n)(z\lceil_{[f(k),f(k+1))} \neq x\lceil_{[f(k),f(k+1))})\}.$$

Then $B_{x,f}^m \subseteq B_{x,f}^n$ whenever $m < n < \omega$; also, $B_{x,f} = \bigcup_{n<\omega} B_{x,f}^n$.

PROPOSITION 9. *For $x, y \in {}^{\omega}2$ and strictly increasing $f, g \in {}^{\omega}\omega$, the following assertions are equivalent:*

(1) $B_{x,f} \subset B_{y,g}$.

(2) *(1)* $B_{x,f} \neq B_{y,g}$ *and*

(2) $(\forall_n^{\infty})(\exists k)\big(g(n) \leq f(k) < f(k+1) \leq g(n+1)$ *and* $x\lceil_{[f(k),f(k+1))} = y\lceil_{[f(k),f(k+1))}\big)$

PROOF. Only the implication $1 \Rightarrow 2(\mathrm{b})$ requires a proof. If 1 holds, then (a) of 2 holds. Assume the negation of 2(b). It reads:

$$(\exists_n^{\infty}) \quad (\forall k) \quad \Big(\neg\big(g(n) \leq f(k) < f(k+1) \leq g(n+1)\big) \text{ or } \neg(x\lceil_{[f(k),f(k+1))} = y\lceil_{[f(k),f(k+1))})\Big)$$

Put $S = \big\{n < \omega : (\forall k)(\neg([f(k), f(k+1)] \subseteq [g(n), g(n+1)]) \text{ or } \neg(x\lceil_{[f(k),f(k+1))} = y\lceil_{[f(k),f(k+1))})\big\}$. Our hypothesis is that S is an infinite set.

Consider an n in S. For each k, there are the following possibilities:

(1) $\neg\big([f(k), f(k+1)] \subseteq [g(n), g(n+1)]\big)$

(2) $[f(k), f(k+1)] \subseteq [g(n), g(n+1)]$, but $x\lceil_{[f(k),f(k+1))} \neq y\lceil_{[f(k),f(k+1))}$.

Put $S_n = \{k : 2 \text{ holds for } k\}$. We consider two cases.

CASE 1. There are infinitely many n for which S_n is nonempty.

Choose an infinite sequence $(n_1, n_2, n_3, \ldots)$ from S such that:

(1) $S_{n_m} \neq \emptyset$,

(2) $n_{m+1} > g(n_m + 1)$, and

(3) $(\exists k)\big(g(n_m + 1) < f(k) < g(n_{m+1})\big)$, for each m, and

(4) $f(1) < g(n_1)$.

This is possible because f and g are increasing, and S is infinite. Put $T = \bigcup_{j=1}^{\infty}[g(n_j), g(n_j + 1))$. Define z, an element of ${}^{\omega}2$, so that $z\lceil_T = y\lceil_T$ and $z(n) = 1 - x(n)$ for each $n \in \omega \setminus T$. Then $z \in B_{x,f}$ while $z \notin B_{y,g}$. Thus 1 fails in this case.

CASE 2. There are only finitely many $n \in S$ for which S_n is nonempty.

We may assume that $S_n = \emptyset$ for each $n \in S$. Consider $n \in S$. We then have that $[f(k), f(k+1)) \nsubseteq [g(n), g(n+1))$ for each $k \in \omega$. We distinguish between two possibilities:

(1) $(\exists k)\big(g(n) \leq f(k) < g(n+1)\big)$ or

(2) $(\forall k)\big(f(k) \notin [g(n), g(n+1))\big)$

CASE 2(A). Possibility 1 occurs for infinitely many $n \in S$:

Choose $n_1 < n_2 < n_3 < \cdots$ from S such that

- $2 \cdot n_j \leq n_{j+1}$ for each j,
- for each j there is a k such that $g(n_j + 1) < f(k) < g(n_{j+1})$,
- for each j there is a k such that $f(k) \in [g(n_j), g(n_{j+1}))$, and
- $f(1) < g(n_1)$.

Put $T = \bigcup_{j=1}^{\infty}[g(n_j), g(n_j + 1))$ and define z so that $z\lceil_T = y\lceil_T$, and $z(n) = 1 - x(n)$ for each $n \in \omega \setminus T$. From the hypothesis of Case 2(A) it follows that $z \in B_{x,f}$, but $z \notin B_{y,g}$. Thus, 1 fails also in this case.

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CASE 2(B) Possibility 1 occurs for only finitely many $n \in S$

We may assume that possibility 2 occurs for each $n \in S$ Choose $k_1 < k_2 < k_3 <$ such that for each j there is an $n \in S$ with $[g(n), g(n+1)) \subset [f(k_j), f(k_j+1))$ For each j choose $n_j \in S$ such that $[g(n_j), g(n_j+1)) \subset [f(k_j), f(k_j+1))$ As before define $T = \bigcup_{j=1}^{\infty} [g(n_j), g(n_j+1))$ Finally, define z so that $z\lceil_T = y\lceil_T$ and $z(n) = 1 - x(n)$ for each $n \in \omega \setminus T$ Then $z \in B_{x,f}$ and $z \notin B_{y\,g}$, showing that 1 fails also in this case

This completes the proof ∎

LEMMA 10 *Let f and g be strictly increasing elements of ${}^{\omega}\omega$ for which there is some $k < \omega$ such that $g(n+k) = f(n)$ for all but finitely many n If $B_{x\,f} \subseteq B_{y\,g}$, then $B_{x\,f} = B_{y\,g}$*

PROOF Assume that $B_{x,f} \neq B_{y\,g}$ and suppose that $B_{y\,g} \not\subseteq B_{x\,f}$ We show that $B_{x\,f} \not\subseteq B_{y\,g}$ Let z be an element of $B_{y\,g} \setminus B_{x,f}$ Fix N such that

(1) $z\lceil_{[g(n+k)\,g(n+k+1))} \neq y\lceil_{[g(n+k)\,g(n+k+1))}$ and

(2) $f(n) = g(n+k)$

for each $n \geq N$

Since z is not an element of $B_{x,f}$, there are infinitely many $n \geq N$ for which $z\lceil_{[f(n)\,f(n+1))} = x\lceil_{f(n),f(n+1))}$ Consequently the set $S = \{n \geq N \quad x\lceil_{[f(n)\,f(n+1))} \neq y\lceil_{[f(n),f(n+1))}\}$ is infinite Now define t such that $t\lceil_{[f(n),f(n+1))} = y\lceil_{[f(n)\,f(n+1))}$ for each $n \in S$, and $t(m) = 1 - x(m)$ for each $m \in \omega \setminus \bigcup_{n \in S} [f(n), f(n+1))$ Then t is in $B_{x,f}$ but not in $B_{y\,g}$ ∎

Under the hypothesis of Lemma 10, $x(n) = y(n)$ for all but finitely many n

PROPOSITION 11 *Let x, y be elements of ${}^{\omega}2$ and let f, g be increasing elements of ${}^{\omega}\omega$ Of the following two assertions, 1 implies 2*

(1) $B_{x,f} \subset B_{y\,g}$

(2) $f \ll g$

PROOF Assume that $B_{x\,f} \subset B_{y\,g}$ Fix, by Proposition 9, an N such that

$$(\forall n \geq N)\,(\exists k)\,\Big([f(k), f(k+1)] \subseteq [g(n), g(n+1)] \text{ and } x\lceil_{[f(k)\,f(k+1))} = y\lceil_{[f(k)\,f(k+1))}\Big)$$

For each $n \geq N$ choose k_n such that $[f(k_n), f(k_n+1)] \subseteq [g(n), g(n+1)]$ It follows that $k_n + 1 \leq k_{n+1}$ for each $n \geq N$ (since f and g are increasing)

CLAIM $[f(k_n), f(k_n+1)] \subset [g(n), g(n+1)]$ *for infinitely many n*

PROOF OF THE CLAIM For otherwise, fix $M \geq N$ such that $[f(k_n), f(k_{n+1}] = [g(n), g(n+1)]$ for each $n \geq M$ Then we have $k_{n+1} = k_n + 1$ for each $n \geq M$ It follows that $g(n) = f\big(n + (k_M - M)\big)$ for all $n \geq M$ Then Lemma 10 implies that $B_{x,f} = B_{y\,g}$, contrary to the fact that $B_{x\,f}$ is a proper subset of $B_{y\,g}$ This completes the proof of the claim

Thus, there are infinitely many n for which $k_{n+1} > k_n + 1$ Let $m > 1$ be given, and fix $L \geq M$ such that $|\{n < L \quad k_{n+1} > k_n + 1\}| \geq k_1 + m$ Then $k_n > (n+m)$ for each $n \geq L$, we have

$$f(n+1) < f(n+m) \leq f(k_n) < g(n+1)$$

for each $n \geq L$ In particular, $m \leq g(n+1) - f(n+1)$ for each $n \geq L$ This completes the proof that $f \ll g$ ∎

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PROPOSITION 12. *Let x and y be elements of* ${}^{\omega}2$ *and let f and g be increasing elements of* ${}^{\omega}\omega$. *If* $B_{x,f} \subset B_{y,g}$, *then there is an* $m < \omega$ *such that* $B^n_{x,f} \subseteq B^n_{y,g}$ *whenever* $n \geq m$.

PROOF. From our hypotheses and Proposition 9 there is an m such that for each $n \geq m$ there is a k such that $[f(k), f(k+1)) \subseteq [g(n), g(n+1))$ and $x\lceil_{[f(k),f(k+1))} = y\lceil_{[f(k),f(k+1))}$. By Proposition 11 there is an $M > m$ such that $f(j) \leq g(j)$ for each $j \geq M$. We show that $B^n_{x,f} \subseteq B^n_{y,g}$ for each $n \geq M$.

Let z be an element of $B^n_{x,f}$. Then $z\lceil_{[f(j),f(j+1))} \neq x\lceil_{[f(j),f(j+1))}$ for each $j \geq n$. But consider any $j \geq n$. Then there is a k such that $[f(k), f(k+1)) \subset [g(j), g(j+1))$; $k \geq j$ for any such k, by the choice of M. It follows that $z\lceil_{[g(j),g(j+1))} \neq y\lceil_{[g(j),g(j+1))}$. Thus, z is also an element of $B^n_{y,g}$. ■

PROPOSITION 13. *For each* $X \in \langle \mathcal{NWD}_{\mathbb{R}} \rangle$ *there are an x in* ${}^{\omega}2$ *and an increasing f in* ${}^{\omega}\omega$ *such that* $X \subset B_{x,f}$.

PROOF. Let X be a meager set. We may assume that $X = \bigcup_{n=0}^{\infty} X_n$ where $X_n \subseteq X_{n+1}$ and X_n is closed, nowhere dense for each n. Fix a well-ordering of ${}^{<\omega}2$, and define $(s_n : n < \omega)$ and f in ${}^{\omega}\omega$ as follows:

Take $s_0 = \emptyset$ and $f(0) = 0$. Assume that $s_1, s_2, \ldots, s_n$ and $f(1), \ldots, f(n)$ have been defined so that:

(1) s_1 is the first element of ${}^{<\omega}2$ such that $[s_1] \cap X_1 = \emptyset$ and $f(1) = \text{length}(s_1)$,

(2) s_{j+1} is the first element of ${}^{<\omega}2$ such that $[t^\frown s_{j+1}] \cap X_j = \emptyset$ for each t in ${}^{\leq f(j)}2$, and $f(j+1) = \sum_{i=0}^{j+1} \text{length}(s_i)$ for each $j < n$.

Then let s_{n+1} be the first element of ${}^{<\omega}2$ such that $[t^\frown s_{n+1}] \cap X_n = \emptyset$ for each t in ${}^{\leq f(n)}2$; put $f(n+1) = f(n) + \text{length}(s_{n+1})$.

Finally, set $x = s_1^\frown s_2^\frown s_3^\frown \cdots$.

CLAIM. $X \subseteq B_{x,f}$.

For suppose that z is not an element of $B_{x,f}$. Then there are infinitely many n for which $z\lceil_{[f(n),f(n+1))} = x\lceil_{[f(n),f(n+1))}$; in other words, there are infinitely many n for which $z\lceil_{[f(n),f(n+1))} = s_{n+1}$. Now fix an m. Choose an $n > m$ such that $z\lceil_{[f(n),f(n+1))} = s_{n+1}$. From the choice of s_{n+1} it follows that $[z\lceil_{f(n+1)}] \cap X_m = \emptyset$; in particular, $z \notin X_m$. Consequently, z is not an element of X. ■

PROPOSITION 14. *Each* $B^n_{x,f}$ *is in* $\mathcal{NWD}_{\mathbb{R}}$.

PROOF. Consider an s from ${}^{\omega}2$ for which $[s] \cap B^n_{x,f} \neq \emptyset$. Choose m such that $f(m) >$ length(s) and $m > n$. Then choose t from ${}^{<\omega}2$ such that length$(s^\frown t) \geq f(m+1)$ and $s^\frown t\lceil_{[f(m),f(m+1))} = x\lceil_{[f(m),f(m+1))}$. Then $[s^\frown t] \cap B^n_{x,f} = \emptyset$. It follows that $B^n_{x,f}$ is nowhere dense. ■

Consequently, $B_{x,f}$ is a meager set for each x in ${}^{\omega}2$ and for each increasing f from ${}^{\omega}\omega$.

THEOREM 15. $\langle \mathcal{NWD}_{\mathbb{R}} \rangle$ *has a cofinal family which embeds in a cofinal subset of* $({}^{\omega}\omega, \ll)$ *and which has the coherent decomposition property.*

PROOF. By Propositions 14 and 13 the family of sets of the form $B_{x,f}$ where f is an increasing element of ${}^{\omega}\omega$ and x is an element of ${}^{\omega}2$, is a cofinal family of meager sets. By Proposition 12, this family has the coherent decomposition property. Also, the mapping which assigns f to $B_{x,f}$ is, according to Proposition 11, an order preserving mapping. ■

4. **Applications.** The ω-path partition relation is the one other combinatorial ingredient in our technique for constructing winning k-tactics, or for defeating a given k-tactic for TWO. For a positive integer n, infinite cardinal number λ and a partially ordered set $(P, <)$, the symbol

$$(P, <) \longrightarrow (\omega\text{- path})^n_{\lambda/<\omega}$$

means that for every function F: $[P]^n \longrightarrow \lambda$ there is an increasing ω-sequence

$$p_1 < p_2 < \cdots < p_m < \cdots$$

such that the set $\left\{F(\{p_{j+1}, \ldots, p_{j+n}\}) : j < \omega\right\}$ is finite. The negation of this assertion is denoted by the symbol

$$(P, <) \not\longrightarrow (\omega\text{- path})^n_{\lambda/<\omega}.$$

This partition relation has been studied in [S2], where various facts used below are proved. In particular, we often use the fact that $(P, <) \not\longrightarrow (\omega\text{- path})^2_{\omega/<\omega}$ for every partially ordered set $(P, <)$ for which there is a strict order-preserving map into $({}^{\omega}\omega, \ll)$.

4.1. *The game* MG($\mathcal{A}, J$). Let J be a free ideal on an infinite set S and let $\mathcal{A} \subset \langle J \rangle$ be a family with the property that for each $X \in \mathcal{A}$ there is a $Y \in \mathcal{A}$ such that $X \subset Y$. The game MG($\mathcal{A}, J$) is defined so that an ω-sequence $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ is a play if for each n,

(1) $O_n \in \mathcal{A}$ is player ONE's move in inning n,
(2) $T_n \in J$ is player TWO's move in inning n, and
(3) $O_n \subset O_{n+1}$.

Player TWO wins this play if $\bigcup_{n=1}^{\infty} O_n \subseteq \bigcup_{n=1}^{\infty} T_n$.

THEOREM 16. *Let J be a free ideal on a set S. If $\mathcal{A}$ is a family of sets in $\langle J \rangle$ such that:*
(1) for each $X \in \mathcal{A}$ there is a $Y \in \mathcal{A}$ such that $X \subset Y$,
(2) $(\mathcal{A}, \subset) \not\longrightarrow (\omega\text{-path})^k_{\omega/<\omega}$ for some $k \geq 2$, and
(3) $\mathcal{A}$ has a coherent decomposition
then TWO has a winning k-tactic in MG($\mathcal{A}, J$).

PROOF. Choose a function F: $[\mathcal{A}]^k \longrightarrow \omega$ which witnesses hypothesis 2. Also associate with each A in $\mathcal{A}$ a sequence $(A^n : n < \omega)$ such that hypothesis 3 is satisfied.

Define a k-tactic, Υ for TWO as follows. Let $(X_1, \ldots, X_j)$ be given such that $j \leq k$, $X_1 \subset \cdots \subset X_j$ and $X_i \in \mathcal{A}$ for $i \leq j$.

(1) If $j < k$: Then put $\Upsilon(X_1, \ldots, X_j) = X_1^1 \cup \cdots \cup X_j^1$.
(2) If $j = k$: Let m be such that
- $m \geq F(\{X_1, \ldots, X_k\})$ and
- $X_1^n \subseteq \cdots \subseteq X_k^n$ for all $n \geq m$.

Put $\Upsilon(X_1, \ldots, X_k) = X_1^m \cup \cdots \cup X_k^m$.

Then Υ is a winning k-tactic for TWO. For let $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ be a play of $\mathrm{MG}(\mathcal{A}, J)$ where:

- $T_j = \Upsilon(O_1, \ldots, O_j)$ for each $j \leq k$
- $T_{n+k} = \Upsilon(O_{n+1}, \ldots, O_{n+k})$ for each $n < \omega$.

For each $t \geq 1$ let m_t be the number associated with $(O_t, \ldots, O_{t+k-1})$ in part 2 of the definition of Υ. By the properties of F, the set $\{m_t : t = 1, 2, 3, \ldots\}$ is infinite. Thus choose $t_1 < t_2 < \cdots$ such that $m_j < m_{t_r}$ for all $j < t_r$. It follows from the criteria used in the choices of the numbers m_t that

$$O_1^{m_{t_r}} \subseteq \cdots \subseteq O_{m_{t_r}}^{m_{t_r}}$$

for all r. But $O_{m_{t_r}}^{m_{t_r}} \subseteq T_{m_{t_r}}$ for all r, according to the definition of Υ. It follows that $\bigcup_{n=1}^{\infty} O_n \subseteq \bigcup_{n=1}^{\infty} T_n$. ∎

COROLLARY 17. *There is a cofinal family $\mathcal{A} \subset \langle J_{\mathbb{R}} \rangle$ such that TWO has a winning 2-tactic in* $\mathrm{MG}(\mathcal{A}, J_{\mathbb{R}})$.

PROOF. Let $\mathcal{A}$ be the family of meager sets provided by Theorem 15. Thus, $\mathcal{A}$ has a coherent decomposition and there is an order preserving function from $(\mathcal{A}, \subset)$ to $({}^{\omega}\omega, \ll)$. But then $(\mathcal{A}, \subset) \not\to (\omega\text{-path})^2_{\omega/<\omega}$ holds, since $({}^{\omega}\omega, \ll) \not\to (\omega\text{-path})^2_{\omega/<\omega}$ holds. ∎

COROLLARY 18. *Let J be a free ideal on an infinite set. If $\mathcal{A}$ is a family of sets in $\langle J \rangle$ such that:*

(1) $\mathcal{A}$ is locally small,

(2) for each $X \in \mathcal{A}$ there is a $Y \in \mathcal{A}$ such that $X \subset Y$, and

(3) $(\mathcal{A}, \subset)$ is well-founded,

then TWO has a winning 2-tactic in $\mathrm{MG}(\mathcal{A}, J)$.

PROOF. The proof is analogous to that of Corollary 17; now we refer to the proof of Theorem 4, we observe that $\omega_1 \leq \mathfrak{b}$, and invoke Theorem 16. ∎

COROLLARY 19. *Let $\lambda \leq \kappa$ be infinite cardinal numbers such that:*

(1) λ has countable cofinality,

(2) $\lambda^+ \not\to (\omega\text{-path})^2_{\omega/<\omega}$, and

(3) $[\kappa]^{\leq\lambda}$ has the irredundancy property.

Then there is a cofinal family $\mathcal{A} \subset [\kappa]^{\lambda}$ such that TWO has a winning 2-tactic in $\mathrm{MG}(\mathcal{A}, [\kappa]^{<\lambda})$.

PROOF. Let $\mathcal{A}$ be a well-founded cofinal family in $[\kappa]^{\lambda}$ which is irredundant. Since there is a rank-function from $\mathcal{A}$ to λ^+ it follows from hypothesis 2 that $(\mathcal{A}, \subset) \not\to (\omega\text{-path})^2_{\omega/<\omega}$. By Corollary 8, $\mathcal{A}$ has a coherent decomposition. By Theorem 16, TWO has a winning 2-tactic in the game $\mathrm{MG}(\mathcal{A}, [\kappa]^{<\lambda})$. ∎

The next theorem shows that under certain circumstances there is for each n a free ideal J_n and a cofinal family $\mathcal{A}_n \subset \langle J_n \rangle$ such that TWO does not have a winning n-tactic, but does have a winning $n+1$-tactic in $\mathrm{MG}(\mathcal{A}_n, J_n)$. We think that Theorem 20 indicates some relevance of the games as considered here for Telgarsky's Conjecture (see 3.4).

THEOREM 20. *Let λ be an infinite cardinal number and let $2 \leq n < \omega$. If there is a linearly ordered set $(L_n, <_n)$ such that:*

(1) $\mathrm{cof}(L_n, <_n) > \omega$,

(2) $(L_n, <_n) \longrightarrow (\omega\text{-path})^n_{\lambda/<\omega}$, *but*

(3) $(L_n, <_n) \not\longrightarrow (\omega\text{-path})^{n+1}_{\lambda/<\omega}$,

then there is a free ideal J_n and a cofinal family $\mathcal{A}_n \subset \langle J_n \rangle$ such that TWO does not have a winning n-tactic, but does have a winning $n+1$-tactic in $\mathrm{MG}(\mathcal{A}_n, J_n)$.

REMARK. It follows from Propositions 3 and 4 of [S2] that if there is a linearly ordered set which stisfies these hypotheses for $n = 2$, then there there is for each integer $n > 1$ a linearly ordered set which satisfies these hypotheses.

PROOF. Let λ, n and $(L_n, <_n)$ be as in the hypotheses, fixed for the rest of the proof. We may assume that the underlying set, L_n, is disjoint from $\mathcal{P}\big(\mathcal{P}(\lambda)\big) \cup \mathcal{P}(\lambda) \cup \lambda$.

Define a free ideal J_n as follows: The underlying set on which J_n lives, say S_n, is $[\lambda]^{<\aleph_0} \cup L_n$. For each $\alpha \in \lambda$ let X_α be the set $\{Z \in [\lambda]^{<\aleph_0} : \alpha \notin Z\}$. Let $\mathcal{T}$ be $\{X_\alpha : \alpha \in \lambda\}$. Put a subset X of S_n in J_n if:

$X \cap [\lambda]^{<\aleph_0}$ is a subset of a union of finitely many elements of $\mathcal{T}$, and $X \cap L_n$ is bounded above.

Then the cofinality of $\langle J_n \rangle$ is $\mathrm{cof}(L_n, <_n)$. Define $\mathcal{A}_n$ so that $X \in \mathcal{A}_n$ if:

$$X \cap L_n = \{t \in L_n : t < z\} \text{ for some } z \in L_n.$$

Then $\mathcal{A}_n$ is cofinal in $\langle J_n \rangle$.

CLAIM 1. *TWO does not have a winning n-tactic in* $\mathrm{MG}(\mathcal{A}_n, J_n)$.

For let Φ be an n-tactic of TWO. For $x \in L_n$ put $V_x = [\lambda]^{<\aleph_0} \cup \{y \in L_n : y <_n x\}$. Define a partition $\Psi\colon [L_n]^n \longrightarrow [\lambda]^{<\aleph_0}$ so that

$$\Big(\Phi(V_{x_1}) \cup \Phi(V_{x_1}, V_{x_2}) \cup \cdots \cup \Phi(V_{x_1}, \ldots, V_{x_n})\Big) \cap [\lambda]^{<\aleph_0}$$

is a subset of $\bigcup\big\{X_\alpha : \alpha \in \Psi(\{x_1, \ldots, x_n\})\big\}$.

By (1) we obtain an ω-path $x_1 <_n x_2 <_n \cdots <_n x_k <_n \cdots$ and a finite set $F \subset \lambda$ such that $\Psi(x_{j+1}, \ldots, x_{j+n}) \subseteq F$ for all j. For each m we define: $O_m = [\lambda]^{<\aleph_0} \cup V_{x_m}$. Letting $(O_1, T_1, \ldots, O_k, T_k, \ldots)$ be the corresponding Φ-play, we find that TWO has lost this play since $[\lambda]^{<\aleph_0} \cap (\bigcup_{m=1}^\infty T_m) \subseteq \bigcup_{\alpha \in F} X_\alpha \neq [\lambda]^{<\aleph_0}$.

It follows that TWO does not have a winning n-tactic.

CLAIM 2. *TWO has a winning $n+1$-tactic in* $\mathrm{MG}(\mathcal{A}_n, J_n)$.

First observe that $\bigcup_{\alpha \in F} X_\alpha = [\lambda]^{<\aleph_0}$ whenever F is an infinite subset of λ.

Here is a definition of an $n+1$-tactic for TWO in this game: Let $\{t_\alpha : \alpha < \lambda\}$ enumerate $[\lambda]^{<\aleph_0}$ bijectively. Let $\Phi\colon [L_n]^{n+1} \longrightarrow \lambda$ be a coloring which witnesses that $(L_n, <_n) \not\longrightarrow (\omega\text{-path})^{n+1}_{\lambda/<\omega}$. For each X in $\mathcal{A}_n$ let ϕ_X be that element of L_n for which $X \cap L_n = \{t \in L_n : t < \phi_X\}$.

For $U_1 \subset \cdots \subset U_{n+1}$ elements of $\mathcal{A}_n$, observe that $\phi_{U_1} \leq \cdots \leq \phi_{U_{n+1}}$. For $X \subset Y$ sets in $\mathcal{A}_n$ such that $X \cap [\lambda]^{<\aleph_0} \neq Y \cap [\lambda]^{<\aleph_0}$ we set $\Psi(X, Y) = \min\{\alpha : t_\alpha \in Y \setminus X\}$.

Let $U_1 \subset \cdots \subset U_{n+1} \in \mathcal{A}_n$ be given. We define:

(1) $G(U_1, \ldots, U_j) = \emptyset$ when $j < n + 1$,

(2) $G(U_1, \ldots, U_{n+1}) = X_\alpha \cup (L_n \cap U_{n+1})$ when $\phi_{U_1} < \cdots < \phi_{U_{n+1}}$, and $\Phi(\{\phi_{U_1}, \ldots, \phi_{U_{n+1}}\}) = \alpha$,

(3) $G(U_1, \ldots, U_{n+1}) = X_\alpha \cup (L_n \cap U_{n+1})$ where α is minimal such that $t_\alpha \in U_{i+1} \setminus U_i$ for some $i \leq n$, otherwise.

We show that G is a winning $n + 1$-tactic for TWO. Thus, let

$$(O_1, T_1, \ldots, O_m, T_m, \ldots)$$

be a G-play of the game. For typographical convenience we define:

(1) $x_i = \phi_{O_i}$ for each i, and

(2) $\alpha_i = \Psi(O_i, O_{i+1})$ for each i for which this is defined.

There are two cases to consider.

Case 1. $\{i : x_i = x_{i+1}\}$ is finite.

Choose m such that $x_i < x_{i+1}$ for all $i \geq m$. Then the set

$$\left\{\Phi(\{x_{m+k+1}, \ldots, x_{m+k+n+1}\}) : k = 1, 2, \ldots\right\}$$

is an infinite subset of λ and it follows from 2. in the definition of G that this play is won by TWO.

Case 2. $\{i : x_i = x_{i+1}\}$ is infinite. Then the set $\{i : \Psi(O_i, O_{i+1}) \text{ is defined}\}$ is infinite. But then it follows from 3. in the definition of G that TWO wins this play. ∎

The hypotheses of Theorem 20 are consistent with ZFC (see Corollary 27 and Proposition 29 of [S2]). At this point it is an open problem whether the hypotheses (and for that matter the conclusion) of Theorem 20 are satisfied simply in the theory ZFC (see Problem 9 of [S2]).

For the case when $\lambda = \omega$, the example constructed in the proof of Theorem 20 shows that hypothesis 2 of Theorem 16 is to some extent necessary. This is because:

(1) $\mathcal{A}_n$ has the coherent decomposition property: For choose $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ from ω, and set $T_m = X_{\alpha_1} \cup \cdots \cup X_{\alpha_m}$ for each m. Then $[\omega]^{<\aleph_0} = \bigcup_{m=1}^{\infty} X_{\alpha_m}$, and $X_{\alpha_j} \subseteq X_{\alpha_i}$ for $j < i$. For $A \in \mathcal{A}_n$ we put $A_m = (A \cap T_m) \cup (A \cap L_n)$.

(2) $(\mathcal{A}_m, \subset) \longrightarrow (\omega\text{-path})^m_{\omega/<\omega}$, but

(3) $(\mathcal{A}_m, \subset) \not\longrightarrow (\omega\text{-path})^{m+1}_{\omega/<\omega}$.

4.2. *The game* MG(J). MG(J) denotes the version of MG($\mathcal{A}, J$) where $\langle J \rangle = \mathcal{A}$. In Problem 1 of [S1] it was asked whether there is for each k a free ideal J_k such that TWO does not have a winning k-tactic in MG(J_k), but does have a winning $k + 1$-tactic in MG(J_k). This problem is still open. In [S1], Corollary 10, it was proven that TWO does not have a winning 2-tactic in the game MG($\mathcal{NWD}_{\mathbb{R}}$), but that TWO has a winning 3-tactic in MG($\mathcal{NWD}_{\mathbb{R}}$) if for example the Continuum Hypothesis is assumed. We now extend these results in two directions:

(1) We solve Problem 3 of that paper affirmatively.

(2) We identify circumstances under which TWO does not have a winning k-tactic in $\mathrm{MG}(\mathcal{NWD}_{\mathbb{R}})$ for any k; combining this with a consistency result of Todorčević (given in the appendix), it follows that it is also consistent that there is no k for which TWO has a winning k-tactic in $\mathrm{MG}(\mathcal{NWD}_{\mathbb{R}})$.

It follows that the existence of a winning k-tactic for TWO in $\mathrm{MG}(\mathcal{NWD}_{\mathbb{R}})$ is not decided by the axioms of traditional set theory. One might now wonder if it is consistent that for example TWO does not have a winning 3-tactic in $\mathrm{MG}(\mathcal{NWD}_{\mathbb{R}})$, but does have a winning 4-tactic? This is not possible since a theorem of [S3] implies that either TWO has a winning 3-tactic, or else there is no k such that TWO has a winning k-tactic in $\mathrm{MG}(\mathcal{NWD}_{\mathbb{R}})$.

PROPOSITION 21. *The theory* "ZFC $+\neg$ CH + *TWO has a winning* 3-*tactic in* $\mathrm{MG}(J_{\mathbb{R}})$" *is consistent.*

PROOF. Start with a model in which $\langle\mathcal{NWD}_{\mathbb{R}}\rangle$ has a cofinal chain and in which $\big(\mathcal{P}(\mathfrak{c}),\subset\big)\not\to(\omega\,\mathrm{path})^3_{\omega/<\omega}$. Let C denote this cofinal chain. By Theorem 15 we may assume that C has a coherent decomposition and that it satisfies the partition relation $(C,\subset)\not\to(\omega\text{-path})^2_{\omega/<\omega}$. It follows that:

(1) $(\langle\mathcal{NWD}_{\mathbb{R}}\rangle,\subset)\not\to(\omega\text{-path})^3_{\omega/<\omega}$, and

(2) The family $\langle\mathcal{NWD}_{\mathbb{R}}\rangle$ has a coherent decomposition.

Theorem 16 implies that TWO has a winning 3-tactic in $\mathrm{MG}(\langle\mathcal{NWD}_{\mathbb{R}}\rangle,\mathcal{NWD}_{\mathbb{R}})$. This completes the proof of the proposition. ∎

The hypotheses used in the proof of this theorem hold for example in a model constructed by Woodin ([W], pp. 31–47). Also, see see [S2], top of p. 60.

Our proof of Proposition 21 shows more generally that if J is a free ideal on a set of cardinality at most $\mathfrak{c}$, and if $\langle J\rangle$ has a cofinal chain and the coherent decomposition property, and if the negative partition relation $\big(\mathcal{P}(\mathfrak{c}),\subset\big)\not\to(\omega\,\mathrm{path})^3_{\omega/<\omega}$ holds, TWO has a winning 3-tactic in $\mathrm{MG}(J)$. This generalizes Theorem 8(a) of [S1].

Next we give hypotheses under which there is no k for which TWO has a winning k-tactic in $\mathrm{MG}(J_{\mathbb{R}})$. In the appendix we give a proof that these hypotheses are consistent with ZFC. This consistency result is due to Todorčević.

THEOREM 22. *Assume that* $\mathrm{cof}(J_{\mathbb{R}},\subset)=\lambda$ *and that the partition relation* $\big(\mathcal{P}(\mathfrak{c}),\subset\big)\to(\omega\text{-path})^3_{\lambda/<\omega}$ *holds. Then there is no k for which TWO has a winning k-tactic in* $\mathrm{MG}(\mathcal{NWD}_{\mathbb{R}})$.

PROOF. Let k as well as a k-tactic F for TWO be given. Let X be a nowhere dense subset of cardinality $\mathfrak{c}$ of $\mathbb{R}\setminus\mathbb{Q}$. Let $\mathcal{A}=\{A_\alpha:\alpha<\lambda\}$ be a bijectively enumerated cofinal subfamily of $\mathcal{NWD}_{\mathbb{R}}$.

Define a partition $\Phi\colon[\mathcal{P}(X)]^k\longrightarrow\lambda$ so that

$$\Phi(\{X_1,\ldots,X_k\})=\beta$$

where β is minimal such that

$$F(\mathbb{Q} \cup X_1) \cup \cdots \cup F(\mathbb{Q} \cup X_1, \ldots, \mathbb{Q} \cup X_k) \subset A_\beta.$$

Since $\big(\mathcal{P}(\mathfrak{c}), \subset\big) \longrightarrow (\omega\text{-path})^3_{\lambda/<\omega}$, it follows that $\big(\mathcal{P}(\mathfrak{c}), \subset\big) \longrightarrow (\omega\text{-path})^k_{\lambda/<\omega}$ (see [S2], Proposition 36). Accordingly, choose a finite set $G \subset \lambda$ and an increasing ω-sequence $X_1 \subset X_2 \subset \cdots$ of subsets of X such that $\Phi(\{X_{j+1}, \ldots, X_{j+k}\}) \in G$ for all j. Put $O_n = X_n \cup \mathbb{Q}$ for all n. Let B be the nowhere dense set $\bigcup\{A_\alpha : \alpha \in G\}$. Also define $T_j = F(O_1, \ldots, O_j)$ for $j \leq k$, and $T_{j+k} = F(O_{j+1}, \ldots, O_{j+k})$ for all j. Then

$$(O_1, T_1, O_2, T_2, \ldots)$$

is an F-play of $\mathrm{MG}(\mathcal{N}\mathcal{W}\mathcal{D}_{\mathbb{R}})$ for which $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} O_n$ and $\bigcup_{n=1}^{\infty} T_n \subseteq B$. Since B is nowhere dense, $\mathbb{Q} \setminus B \neq \emptyset$. It follows that TWO has lost this play. ∎

We now consider games of the form $\mathrm{MG}([\kappa]^{<\lambda})$. In Proposition 15 of [S1] it was shown that if TWO has a winning k-tactic in this game for some k, then TWO in fact has a winning 3-tactic. It is not known if "3" is optimal (this is Problem 7 of [S1]). It also follows from [S1], Proposition 5, that if $\lambda \longrightarrow (\omega\text{-path})^2_{\omega/<\omega}$, then TWO does not have a winning k-tactic in this game for any k. We now present slightly sharper results.

THEOREM 23. *Let λ be an uncountable cardinal number of countable cofinality. Let $k > 1$ be an integer. The following statements are equivalent:*

(1) Player TWO has a winning k-tactic in the game $\mathrm{MG}([\lambda^+]^{<\lambda})$.

(2) $([\lambda^+]^{\leq\lambda}, \subset) \not\rightarrow (\omega\text{-path})^k_{\omega/<\omega}$.

(3) $\lambda^+ \not\rightarrow (\omega\text{-path})^2_{\omega/<\omega}$ *and* $\big(\mathcal{P}(\lambda), \subset\big) \not\rightarrow (\omega\text{-path})^k_{\omega/<\omega}$.

PROOF. By Theorem 1 and Proposition 15 of [S1] we may assume that $k \in \{2, 3\}$. Let $\lambda_1 < \cdots < \lambda_n < \cdots$ be a sequence of cardinal numbers converging to λ.

1. $\Rightarrow$ 2. Let F be a winning k-tactic for TWO in $\mathrm{MG}([\lambda^+]^{<\lambda})$. Put $S = \lambda^+ \setminus \lambda$. Define a coloring Φ: $[[S]^{\leq\lambda}]^k \longrightarrow \omega$ so that

$$\Phi(X_1, \ldots, X_k) = \min\{n : |F(\lambda \cup X_1, \ldots, \lambda \cup X_k)| \leq \lambda_n\}.$$

Since F is a winning k-tactic for TWO, Φ is a coloring which witnesses the partition relation in 2.

2. $\Rightarrow$ 1. The cofinal chain $\{\alpha : \alpha < \lambda^+\}$ of $[\lambda^+]^{\leq\lambda}$ has a coherent decomposition whence this entire family of sets has a coherent decomposition. The partition property in 2 implies that $[\lambda^+]^{\leq\lambda}$ satisfies the hypotheses of Theorem 16; thus TWO has a winning k-tactic in $\mathrm{MG}([\lambda^+]^{<\lambda})$.

The equivalence of 2. and 3. is also easy to establish. ∎

COROLLARY 24. *Let λ be an uncountable cardinal number of countable cofinality. Assume that there is a strict order preserving map from* $([\lambda^+]^\lambda, \subset)$ *into* $({}^\omega\omega, \ll)$. *Then TWO has a winning* 2*-tactic in* $\mathrm{MG}([\lambda^+]^{<\lambda})$.

PROOF. The hypothesis implies that both λ^+ and $\big(\mathcal{P}(\lambda), \subset\big)$ embed in $({}^{\omega}\omega, \ll)$ for any $\lambda < \mathfrak{c}$. It then follows from Corollary 13 of [S2] that the partition relations in 3. of Theorem 23 hold for $k = 2$ for each $\lambda < \mathfrak{c}$. ∎

4.3. *The game* SMG(J). For a free ideal J on an infinite set S, the game SMG(J) (read "strongly monotonic game on J") is defined so that an ω-sequence $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ is a play if for each n,

(1) $O_n \in \langle J\rangle$ is player ONE's move in inning n,
(2) $T_n \in J$ is player TWO's move in inning n, and
(3) $O_n \cup T_n \subseteq O_{n+1}$.

Player TWO wins this play if $\bigcup_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty} T_n$.

Throughout this section we assume that $\langle J\rangle$ is a proper ideal on S.

THEOREM 25. *Let $J \subset \mathcal{P}(S)$ be a free ideal and let $\mathcal{A}$ be a cofinal subfamily of $\langle J\rangle$ such that:*

(1) TWO has a winning k-tactic in MG($\mathcal{A}, J$),
(2) there are functions $\Phi_1 \colon \langle J\rangle \longrightarrow J$ and $\Phi_2 \colon \langle J\rangle \longrightarrow \mathcal{A}$ such that:
  (a) $A \subset \Phi_2(A)$ for each $A \in \langle J\rangle$, and
  (b) $\Phi_2(A) \subset \Phi_2(B)$ whenever $A \cup \Phi_1(A) \subseteq B \in \langle J\rangle$.

Then TWO has a winning 2*-tactic in* SMG(J).

PROOF. Let $\mathcal{A}$, Φ_1 and Φ_2 be as in the hypotheses. For each A in $\langle J\rangle$ define $(A_1, \ldots, A_k)$ so that $A_1 = \Phi_2(A)$ and $A_{j+1} = \Phi_2(A_j)$ for each $j < k$. Also define: $\Psi(A) = \Phi_1(A) \cup \Phi_1(A_1) \cup \cdots \cup \Phi_1(A_k)$.

Let F be a winning k-tactic for TWO in MG($\mathcal{A}, J$). Define a k-tactic, G, for TWO as follows. Let $A \subset B$ be given.

CASE 1. $G(A) = F(A_1) \cup \cdots \cup F(A_1, \ldots, A_k) \cup \Psi(A)$.

CASE 2. If $A_k \subset B_1$, we let $G(A, B)$ be the set

$$F(A_2, \ldots, A_k, B_1) \cup F(A_3, \ldots, A_k, B_1, B_2) \cup \cdots \cup F(B_1, \ldots, B_k) \cup \Psi_1(B).$$

CASE 3. Otherwise we put $G(A, B) = G(B)$.

Then G is a winning 2-tactic for TWO in SMG(J). For let

$$(O_1, T_1, \ldots, O_n, T_n, \ldots)$$

be a play of SMG(J) during which TWO followed the 2-tactic G. For each j we put $M_j^1 = \Phi_2(O_j), \ldots, M_j^k = \Phi_2(M_j^{k-1})$. An inductive computation shows that

- $(M_1^1, M_1^2, \ldots, M_1^k, M_2^1, M_2^2, \ldots, M_2^k, \ldots)$ is a sequence of legal moves for *ONE* in the game MG($\mathcal{A}, J$), and that
- (1) $F(M_1^1) \cup \cdots \cup F(M_1^1, \ldots, M_1^k) \subseteq T_1$, and
  (2) $F(M_j^1, \ldots, M_j^k) \cup F(M_j^2, \ldots, M_j^k, M_{j+1}^1) \cup \cdots \cup F(M_j^k, M_{j+1}^1, \ldots, M_{j+1}^{k-1}) \subseteq T_{j+1}$ for each j.

Since F is a winning k-tactic for TWO in the game MG($\mathcal{A}, J$), and since $\bigcup_{n=1}^{\infty} O_n \subseteq \bigcup_{n=1}^{\infty} M_n^1$, TWO won the given play of SMG(J). ∎

The next corollary solves Problems 10 and 11 of [S1]. The notation $\mathbb{N} \cdot X$ used in its proof denotes the set $\{n \cdot x : n \in \mathbb{N} \text{ and } x \in X\}$.

COROLLARY 26. *Player TWO has a winning* 2*-tactic in the game* SMG($J_{\mathbb{R}}$).

PROOF. Fix, by Corollary 17, a cofinal family $\mathcal{A} \subset \langle \mathcal{NWD}_{\mathbb{R}} \rangle$ such that TWO has a winning 2-tactic in MG($\mathcal{A}, \mathcal{NWD}_{\mathbb{R}}$).

We define $\Phi_1 : \langle \mathcal{NWD}_{\mathbb{R}} \rangle \longrightarrow \mathcal{NWD}_{\mathbb{R}}$ and $\Phi_2 : \langle \mathcal{NWD}_{\mathbb{R}} \rangle \longrightarrow \mathcal{A}$ as follows:

Fix $X \in \langle \mathcal{NWD}_{\mathbb{R}} \rangle$, and choose a sequence $(X_0, X_1, \ldots, X_n, \ldots)$ such that:

(1) $X_0 = X$,

(2) $X_{n+1} \in \mathcal{A}$ and $\mathbb{N} \cdot X_n \subseteq X_{n+1}$

for each n. Put $\Phi_2(X) = \bigcup_{n=1}^{\infty} X_n$.

Fix $X \in \langle \mathcal{NWD}_{\mathbb{R}} \rangle$ and let $\Phi_1(X)$ be a nowhere dense set for which $\Phi_2(X) \subset \mathbb{N} \cdot \Phi_1(X)$.

Then $\mathcal{A}$, Φ_1 and Φ_2 are as required by Theorem 25. ∎

COROLLARY 27. *For each of the ideals J_n constructed in the proof of Theorem 20, TWO has a winning* 2*-tactic in* SMG(J_n).

PROOF. Let $\mathcal{A}_n$ be as in the proof of Theorem 20. For each $X \in \langle J_n \rangle$ we let $\Phi_2(X)$ be an element of $\mathcal{A}_n$ which contains it, and we let $\Phi_1(X) = \{a_X\}$ where $a_X \in L_n \setminus \Phi_2(X)$. Then $\mathcal{A}_n$, Φ_1 and Φ_2 are as required by Theorem 25. ∎

Before turning to another application of Theorem 25 we give examples of free ideals J which show that TWO does not always have a winning k-tactic in the game SMG(J) for some k. These examples are also relevant to the material of the next section. The symbol $M(\omega, 2)$ denotes the smallest ordinal α for which the partition relation $\alpha \longrightarrow (\omega\text{-path})^2_{\omega/<\omega}$ holds. $M(\omega, 2)$ is a regular uncountable cardinal less than or equal to $\mathfrak{c}^+$. It in fact satisfies the partition relation $M(\omega, 2) \longrightarrow (\omega\text{-path})^n_{\omega/<\omega}$ for all n. Let κ be an initial ordinal number. It is consistent that $M(\omega, 2)$ is equal to $\aleph_2$ while $\mathfrak{c}$ is larger than κ (this is yet another result of Todorčevič).

THEOREM 28. *Let λ be a cardinal number of countable cofinality and let κ be a cardinal number larger than λ. If $M(\omega, 2) \leq \lambda^+$, then there is no k such that player TWO has a winning k-tactic in* SMG($[\kappa]^{<\lambda}$).

PROOF. Let F be a k-tactic for TWO.

Player ONE's counter-strategy will be to play judiciously chosen subsets from κ. We first single out those sets from which ONE will make moves.

Choose sets $S_0 \subset S_1 \subset \cdots \subset S_\alpha \subset \cdots \in [\kappa]^\lambda$ for $\alpha < \lambda^+$ such that:

(1) $\lambda \subset S_0$,

(2) $\bigcup\{F(S_{i_1}, \ldots, S_{i_j}) : j \leq k, i_1 < \cdots < i_j < \alpha\} \subset S_\alpha$ for each $0 < \alpha < \lambda^+$.

Now let $\aleph_0 < \lambda_1 < \lambda_2 < \cdots < \lambda$ be an increasing sequence of regular cardinal numbers converging to λ. Define a function $\Gamma\colon [\lambda^+]^k \longrightarrow \omega$ so that

$$\Gamma(\xi_1, \ldots, \xi_k) = \min\{m : |F(S_{\xi_1}, \ldots, S_{\xi_k})| \leq \lambda_m\}.$$

Then, on account of the relation $M(\omega, 2) \leq \lambda^+$, choose an $m < \omega$ and a sequence $\alpha_{k+1} < \cdots < \alpha_{k+m} < \cdots$ from λ^+ such that $\Gamma(\alpha_{j+1}, \ldots, \alpha_{j+k}) \leq m$ for all j.

Consider the sequence

$$\Big(S_{\alpha_1}, F(S_{\alpha_1}), \ldots, S_{\alpha_k}, F(S_{\alpha_1}, \ldots, S_{\alpha_k}), \ldots, S_{\alpha_{k+m}}, F(S_{\alpha_{1+m}}, \ldots, S_{\alpha_{k+m}}), \ldots\Big).$$

It is a play of the game $\mathrm{SMG}([\kappa]^{<\lambda})$ during which TWO used the k-tactic F. But $|\bigcup_{n=1}^{\infty} T_n| < \lambda = |\bigcup_{n=1}^{\infty} O_n|$, so that TWO lost the play. ∎

COROLLARY 29. *For* $\omega = \mathrm{cof}(\lambda) \leq \lambda < \kappa$ *cardinal numbers with* $\mathrm{cof}([\kappa]^{\leq\lambda}, \subset) = \kappa$, *the following statements are equivalent:*

(1) TWO has a winning 2*-tactic in* $\mathrm{SMG}([\kappa]^{<\lambda})$.

(2) $\lambda^+ \not\rightarrow (\omega\text{-path})^2_{\omega/<\omega}$.

PROOF. It follows from Theorem 28 that 1. implies 2.

That 2. implies 1. By the cofinality hypothesis and by 2. we find, according to Corollary 19, a well-founded cofinal family $\mathcal{A}$ such that TWO has a winning 2-tactic in $\mathrm{MG}(\mathcal{A}, [\kappa]^{<\lambda})$. We may assume that there is an enumeration $\{A_\alpha : \alpha < \kappa\}$ of $\mathcal{A}$ for which $\alpha \in A_\alpha$ for each α. Define Φ_1 and Φ_2 as follows:

For $X \in [\kappa]^{\leq\lambda}$ define a sequence $(X_0, \ldots, X_m, \ldots)$ such that:

(1) $X_0 = X$, and

(2) $X_{n+1} = \bigcup_{\alpha \in X_n} A_\alpha$

for each n.

Choose $\Phi_2(X) \in \mathcal{A}$ such that $\bigcup_{n<\omega} X_n \subseteq \Phi_2(X)$.

Pick $z_X \in \big(\kappa \setminus \Phi_2(X)\big)$ and pick ρ_X minimal such that $\rho_X \notin \Phi_2(X)$, and $\Phi_2(X) \subset A_{\rho_X}$. Put $\Phi_1(X) = \{z_X, \rho_X\}$.

Then $\mathcal{A}$, Φ_1 and Φ_2 are as required by Theorem 25. ∎

Results related to Corollary 29 will be discussed after Theorem 33.

We finally mention that it is still unknown whether there is for each m a free ideal J_m such that TWO does not have a winning m-tactic, but does have a winning $m+1$-tactic in $\mathrm{SMG}(J_m)$. This is Problem 9 of [S1]. In this connection it is worth noting the following relationship between winning k-tactics in $\mathrm{MG}(J)$ and winning m-tactics in $\mathrm{SMG}(J)$. The proof uses ideas as in the proof of Theorem 25.

THEOREM 30. *If TWO has a winning k-tactic in* $\mathrm{MG}(J)$, *then TWO has a winning* 2*-tactic in* $\mathrm{SMG}(J)$.

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4.4. *The game* VSG(J). For a free ideal J on an infinite set S, the game VSG(J) (read "very strong game on J") is defined so that an ω-sequence $\big(O_1, (T_1, S_1), \ldots, O_n, (T_n, S_n), \ldots\big)$ is a play if for each n,

(1) $O_n \in \langle J \rangle$ is player ONE's move in inning n,

(2) $(T_n, S_n) \in J \times \langle J \rangle$ is player TWO's move in inning n, and

(3) $O_n \cup T_n \cup S_n \subseteq O_{n+1}$.

Player TWO wins this play if $\bigcup_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty} T_n$.

We assume for this section that $\langle J \rangle$ is also a proper ideal on S. Given a cofinal family $\mathcal{A} \subset \langle J \rangle$, we may assume whenever convenient that ONE is playing from $\mathcal{A}$ in the game VSG(J). It is clear that if TWO has a winning k-tactic in SMG(J), then TWO has a winning k-tactic is VSG(J).

PROBLEM 1. Let J be an ideal on a set S and let k be a positive integer. Is it true that if TWO has a winning k-tactic in VSG(J), then TWO has a winning k-tactic in SMG(J)?

In the next theorem we find a partial answer.

THEOREM 31. *Let J be a free ideal on a set S and let k be a positive integer. If* $\mathrm{add}(\langle J \rangle, \subset) = \mathrm{cof}(\langle J \rangle, \subset)$, *then the following statements are equivalent:*

(1) TWO has a winning 2*-tactic in* SMG(J).

(2) TWO has a winning k-tactic in SMG(J).

(3) TWO has a winning k-tactic in VSG(J).

PROOF. That 1. and 2. are equivalent: This is Theorem 19 of [S1]. That 2. implies 3.: Let F be a winning k-tactic for TWO in SMG(J). Define G so that

$$G(A_1, \ldots, A_j) = \big(F(A_1, \ldots, A_j), A_j \cup F(A_1, \ldots, A_j)\big)$$

for $j \leq k$. Then G is a winning k-tactic for TWO in VSG(J). That 3. implies 2.: Let G be a winning k-tactic for TWO in VSG(J). Then choose a sequence $\big(M_\xi : \xi < \mathrm{cof}(\langle J \rangle, \subset)\big)$ such that:

(1) $M_\xi \subset M_\nu$ for $\xi < \nu < \mathrm{cof}(\langle J \rangle, \subset)$ and

(2) $\{M_\xi : \xi < \mathrm{cof}(\langle J \rangle, \subset)\}$ is cofinal in $\langle J \rangle$.

Now $\mathrm{cof}(\langle J \rangle, \subset)$ is a regular uncountable cardinal number. We may thus further assume that the sequence $\big(M_\xi : \xi < \mathrm{cof}(\langle J \rangle, \subset)\big)$ has been chosen such that if $(U, T) = G(M_{\xi_1}, \ldots, M_{\xi_j})$, then $U \cup T \subset M_\eta$ for all $\xi_j < \eta < \mathrm{cof}(\langle J \rangle, \subset)$.

For each $X \in \langle J \rangle$ define $\alpha(X) = \min\{\xi : X \subset M_\xi\}$. For each ξ choose $z_\xi \in S \setminus M_\xi$. We now define a k-tactic, F, for TWO in SMG(J).

Let $X_1 \subset \cdots \subset X_j \in \langle J \rangle$ for a $j \leq k$ be given.

CASE 1. $\alpha(X_1) < \cdots < \alpha(X_j)$. Let $(U, T) = G(M_{\alpha(X_1)}, \ldots, M_{\alpha(X_j)})$ and define $F(X_1, \ldots, X_j) = U \bigcup \{z_{\alpha(X_j)+1}\}$.

CASE 2. Otherwise, set $F(X_1, \ldots, X_j) = \{z_{\alpha(X_j)+1}\}$. Then F is a winning k-tactic for TWO in SMG(J). ∎

There is the following analogue of Theorem 25 for the very strong game:

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PROPOSITION 32. *Let J be a free ideal on a set S. If there is a cofinal family $\mathcal{A} \subset \langle J\rangle$ such that TWO has a winning k-tactic in* $\mathrm{MG}(\mathcal{A}, J)$, *then TWO has a winning* 2*-tactic in* $\mathrm{VSG}(J)$.

PROOF. Let $\mathcal{A} \subset \langle J\rangle$ be a cofinal family such that TWO has a winning k-tactic in $\mathrm{MG}(\mathcal{A}, J)$. We will define a winning 2-tactic for TWO for the game $\mathrm{VSG}(J)$. To this end, choose a winning k-tactic, F, for TWO for the game $\mathrm{MG}(\mathcal{A}, J)$. For each $X \in \langle J\rangle$ choose a set $A_1(X) \subset \cdots \subset A_k(X)$ from $\mathcal{A}$ such that $X \subset A_1(X)$, and choose $\Psi(X)$ from $\mathcal{A}$ such that $A_k(X) \subset \Psi(X)$.

Let $X \subset Y$ be sets from $\langle J\rangle$.

CASE 1. $G(X) = \Big(F\big(A_1(X)\big) \cup \cdots \cup F\big(A_1(X), \ldots, A_k(X)\big), \Psi(X)\Big)$.

CASE 2. Define $G(X, Y)$ so that:

(1) $G(X, Y) = (F(A_2(X), \ldots, A_k(X), A_1(Y) \cup \cdots \cup F\big(A_1(Y), \ldots, A_k(Y)\big), \Psi(Y))$ if $\Psi(X) \subset Y$, and

(2) $G(X, Y) = G(Y)$ otherwise.

Then G is a winning 2-tactic for TWO in $\mathrm{VSG}(J)$. For let $\big(O_1, (T_1, S_1), O_2, (T_2, S_2), \ldots\big)$ be a play of $\mathrm{VSG}(J)$ such that $(T_1, S_1) = G(O_1)$ and $(T_{n+1}, S_{n+1}) = G(O_n, O_{n+1})$ for all n. Then $S_n = \Psi(O_n)$ and $A_k(O_n) \subset A_1(O_{n+1})$ for each n. An inductive computation, using this information, shows that TWO won this play of $\mathrm{VSG}(J)$. ■

Combining Theorem 31 and Theorem 28 we see that TWO does not always have a winning k-tactic in games of the form $\mathrm{VSG}(J)$. Combining Theorem 31 and Corollary 29 we obtain another game-theoretic characterization of the partition relation $\lambda^+ \longrightarrow (\omega\text{-path})^2_{\omega/<\omega}$ when λ is an uncountable cardinal of countable cofinality.

Analogous to the case of the ideal of countable subsets of an infinite set, there is for each uncountable cardinal number λ which is of countable cofinality, a proper class of cardinals κ for which the ideal $[\kappa]^{\leq\lambda}$ has the irredundancy property. It is also a consequence of MA $+\mathfrak{c} > \lambda$ that the partition relation $\lambda^+ \not\longrightarrow (\omega\text{-path})^2_{\omega/<\omega}$ holds. Accordingly it is consistent that there is a proper class of cardinals κ such that TWO has a winning 2-tactic in the game $\mathrm{VSG}([\kappa]^{\leq\lambda})$. The following problem (to be compared with the upcoming Conjecture 1) is open.

PROBLEM 2. Let λ be an uncountable cardinal of countable cofinality. Is it true that if TWO has a winning 2-tactic in the game $\mathrm{VSG}([\lambda^+]^{<\lambda})$, then TWO has a winning 2-tactic in $\mathrm{VSG}([\kappa]^{<\lambda})$ for all $\kappa > \lambda$?

Our next theorem (Theorem 33) applies to abstract free ideals whose σ-completions have small principal bursting number. It is not clear to us whether "3" occurring in Theorem 33 is optimal. One of its applications is that ZFC + GCH implies that TWO has a winning 3-tactic in $\mathrm{VSG}([\kappa]^{<\aleph_0})$ for all κ. It is *very* likely that the "3" appearing in this application is not optimal, as will be discussed later.

THEOREM 33. *Let J be a free ideal on a set S such that*

(1) $\mathrm{bu}(\langle J\rangle, \subset) = \aleph_n$ *for some finite n,*

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(2) *there is an* (ω_k, ω_k)*-pseudo-Lusin set in* $(\langle J\rangle, \subset)$ *for each* $k \in \{1, \ldots, n\}$,
(3) $\mathrm{cof}(\langle J\rangle, \subset) = \lambda$, *and*
(4) $([\lambda]^{<\aleph_0}, \subset)$ *has the coherent decomposition property.*
Then player TWO has a winning $n+1$*-tactic in* $\mathrm{VSG}(J)$.

PROOF. We present a proof for the special (and more transparent) case where $n = 2$. Thus, let J be a free ideal (on a set S) such that
(1) $\mathrm{bu}(\langle J\rangle, \subset) = \aleph_2$,
(2) $\mathrm{add}(\langle J\rangle, \subset) = \aleph_1$,
(3) $\mathrm{cof}(\langle J\rangle, \subset) = \lambda$ and
(4) $[\lambda]^{<\aleph_0}$ has the coherent decomposition property.
Let $\mathcal{A}$ be a well-founded cofinal family of cardinality λ, such that $|\{B \in \mathcal{A} : B \subseteq A\}| \leq \aleph_1$ for each $A \in \mathcal{A}$.

For each $A \in \mathcal{A}$ fix $\nu_A \leq \omega_1$ and a bijective enumeration $\{J_\xi(A) : \xi < \nu_A\}$ of the set $\{X \in \mathcal{A} : X \subseteq A\}$.

Choose a sequence $(C_\xi : \xi < \omega_1)$ from $\langle J\rangle$ such that:
(1) $C_\xi \subset C_\nu$ for $\xi < \nu$ and
(2) $\bigcup_{\xi<\omega_1} C_\xi \notin \langle J\rangle$.

For $A \in \mathcal{A}$ define $\xi_A = \min\{\xi < \omega_1 : C_\xi \not\subseteq A\}$.

For $A \subset B$ elements from $\mathcal{A}$, define a set $\tau(A, B)$ such that $(S_1, \ldots, S_n)$ is in $\tau(A, B)$ if:
(1) $2 \leq n < \omega$,
(2) $S_1 = B$ and $S_2 = A$,
(3) $S_{j+1} \in \{J_\xi(S_j) : \xi < \nu_{S_j}$ and $C_\xi \subset S_{j-1}\}$ for $2 \leq j < n$.
For $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_m)$ in $\tau(A, B)$ define $(S_1, \ldots, S_n) < (T_1, \ldots, T_m)$ if $n < m$ and $(S_1, \ldots, S_n) = (T_1, \ldots, T_n)$. Then $\big(\tau(A, B), <\big)$ is a tree. Each branch of this tree is finite since $(\mathcal{A}, \subset)$ is well-founded. Indeed, $\tau(A, B)$ is a countable set.

Define $F(A, B)$ to be the set of $X \in \mathcal{A}$ such that $X \in \{S_1, \ldots, S_m\}$ for some $(S_1, \ldots, S_m) \in \tau(A, B)$. Then $F(A, B)$ is a countable set. Notice that if $C \subset A \subset B$ are elements of $\mathcal{A}$ such that $C \in \{J_\xi(A) : \xi \leq \nu_A$ and $C_\xi \subset B\}$, then $F(C, A) \subset F(A, B)$.

Let $\mathcal{B} \subset [\mathcal{A}]^{\aleph_0}$ be cofinal, well-founded and with the coherent decomposition property. For each $B \in \mathcal{B}$ choose a decomposition $B = \bigcup_{n=1}^{\infty} B^n$ where each B^n is finite, and these decompositions satisfy the coherent decomposition requirement. Using Proposition 15 of [S2] we also fix a function

$$\mathcal{K} \colon [\mathcal{B}]^2 \longrightarrow \omega$$

which witnesses that $(\mathcal{B}, \subset) \not\to (\omega\text{-path})^2_{\omega/<\omega}$.

Define $\Phi_1 \colon [\mathcal{A}]^2 \longrightarrow \mathcal{B}$ such that

$$\bigcup\Big\{F(X, Y) : \big(\exists (S_1, \ldots, S_n) \in \tau(A, B)\big)(X \subset Y \text{ and } X, Y \in \{S_1, \ldots, S_n\})\Big\}$$

is a subset of $\Phi_1(A, B)$. Also define $\Phi_2 \colon [\mathcal{A}]^2 \longrightarrow \mathcal{A}$ such that

$$C_\xi \cup C_{\xi_B} \cup \Big(\bigcup \Phi_1(A, B)\Big) \subset \Phi_2(A, B)$$

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where now $A = J_\xi(B)$.

Note that if A, B and C are elements of $\mathcal{A}$ such that $A \subset B \subset \Phi_2(A,B) \subset C$, then $\Phi_1(A,B) \subset \Phi_1(B,C)$.

Finally, choose for each $A \in \mathcal{A}$ a $\Phi_3(A) \in \mathcal{A}$ such that $A \cup C_{\xi_A} \subseteq \Phi_3(A)$.

Choose for each $A \in \mathcal{A}$ a sequence of sets $A^0 \subseteq \cdots, A^n \subseteq \cdots$ such that each A^i is in J and $A = \bigcup_{n=0}^\infty A^n$.

We now define a 3-tactic for TWO: First note that for the very strong game we may make the harmless assumption that player ONE's moves are all from the cofinal family $\mathcal{A}$. Let $A \subset B \subset C$ be sets from $\mathcal{A}$. Here are player TWO's responses $\mathcal{F}(A)$, $\mathcal{F}(A,B)$ and $\mathcal{F}(A,B,C)$:

CASE 1. $\mathcal{F}(A) = \big(\emptyset, \Phi_3(A)\big)$.

CASE 2. $\mathcal{F}(A,B) = \big(\emptyset, \Phi_2(A,B)\big)$.

CASE 3. $\mathcal{F}(A,B,C) = \big(D, \Phi_2(B,C)\big)$ if $\Phi_2(A,B) \subseteq C$, where $D = C_1^m \cup \cdots \cup C_r^m$ is given by: $m \geq \mathcal{K}\big(\{\Phi_1(A,B), \Phi_1(B,C)\}\big)$ is minimal such that $\big(\Phi_1(A,B)\big)^n \subseteq \big(\Phi_1(B,C)\big)^n$ for all $n \geq m$, and $\big(\Phi_1(B,C)\big)^m = \{C_1, \ldots, C_r\}$.

CASE 4. In all other cases define $\mathcal{F}(A,B,C) = \mathcal{F}(B,C)$.

To see that $\mathcal{F}$ is a winning 3-tactic for TWO, consider a play

$$\big(O_1, (T_1, S_1), O_2, (T_2, S_2), \ldots\big)$$

of VSG(J) for which

(1) $(T_1, S_1) = \mathcal{F}(O_1)$,
(2) $(T_2, S_2) = \mathcal{F}(O_1, O_2)$ and
(3) $(T_{n+3}, S_{n+3}) = \mathcal{F}(O_{n+1}, O_{n+2}, O_{n+3})$

for all n.

Then $T_1 = T_2 = \emptyset$, $S_1 = \Phi_3(O_1)$, $S_2 = \Phi_2(O_1, O_2)$ and $S_{n+1} = \Phi_2(O_n, O_{n+1})$ for all $n \geq 2$. From the fact that $O_n \supseteq S_{n-1}$ for all $n \geq 2$ it follows that

$$O_1 \subset O_2 \subset \Phi_2(O_1, O_2) \subseteq O_3 \subset \Phi_2(O_2, O_3) \subseteq O_4 \subset \cdots,$$

whence $\Phi_1(O_1, O_2) \subset \Phi_1(O_2, O_3) \subset \Phi_1(O_3, O_4) \subset \cdots$. For each k let m_k denote the minimal integer such that

(1) $\mathcal{K}\big(\{\Phi_1(O_k, O_{k+1}), \Phi_1(O_{k+1}, O_{k+2})\}\big) \leq m_k$ and
(2) $\big(\Phi_1(O_k, O_{k+1})\big)^n \subseteq \big(\Phi(O_{k+1}, O_{k+2})\big)^n$ for all $n \geq m_k$.

From the properties of $\mathcal{K}$ it follows that there are infinitely many k such that $m_j < m_k$ for each $j < k$. Fix i, and fix the smallest $j \geq i$ such that $O_i \in \Phi_1(O_j, O_{j+1})$. Then let t be minimal such that $O_i \in \big(\Phi_1(O_j, O_{j+1})\big)^t$. For each k such that $m_\ell < m_k$ for all $\ell < k$, and $t < m_k$, $O_i^{m_k} \subseteq T_k$. It follows that $O_i \subseteq \bigcup_{n=1}^\infty T_n$. From this it follows that TWO won this $\mathcal{F}$-play of VSG(J). ∎

COROLLARY 34 (GCH). *For every infinite cardinal number κ, TWO has a winning 3-tactic in* VSG($[\kappa]^{<\aleph_0}$).

The results of Corollaries 29 and 34 should be compared with those of Koszmider [Ko] for the game $\mathrm{MG}([\kappa]^{<\aleph_0})$. In Corollary 29 we show that there is a proper class of κ such that TWO has a winning 2-tactic in $\mathrm{SMG}([\kappa]^{<\aleph_0})$, and thus in $\mathrm{VSG}([\kappa]^{<\aleph_0})$. This class includes $\aleph_n$ for all $n < \omega$. In [Ko] it is proven that TWO has a winning 2-tactic in $\mathrm{MG}([\aleph_n]^{<\aleph_0})$ for all $n \in \omega$ ([Ko], Theorem 18). Under the additional set theoretic assumption that both $\Box_\lambda$ holds and $\lambda^{\aleph_0} = \lambda^+$ for all uncountable cardinal numbers λ which are of countable cofinality, Koszmider further proves that player TWO has a winning 2-tactic in $\mathrm{MG}([\kappa]^{<\aleph_0})$ for all κ ([Ko], Theorem 19). In light of these results it is consistent that TWO has a winning 2-tactic in the game $\mathrm{SMG}([\kappa]^{<\aleph_0})$ and thus in the game $\mathrm{VSG}([\kappa]^{<\aleph_0})$ for all κ. M. Foreman-[Fo] of Ohio State University has also proved that even in the presence of supercompact cardinals TWO may have a winning 2-tactic in $\mathrm{MG}([\kappa]^{<\aleph_0})$ for all κ.

All this evidence suggests:

CONJECTURE 1. *One can prove in* ZFC *that player TWO has a winning* 2*-tactic in the game* $\mathrm{SMG}([\kappa]^{<\aleph_0})$ *for each infinite cardinal number* κ.

We now give an example which shows, assuming the Continuum Hypothesis, that the hypothesis that $\mathrm{add}(\langle J\rangle, \subset) = \aleph_1$ of Theorem 33 is necessary (see Corollary 36).

THEOREM 35. *Let ω_α be the initial ordinal corresponding to $\mathfrak{c}$. Then there is a free ideal $J \subset \mathcal{P}(\omega_{\alpha+1})$ such that* $\mathrm{cof}(\langle J\rangle, \subset) = \aleph_{\alpha+1}$ *and there is no positive integer k for which TWO has a winning k-tactic in* $\mathrm{VSG}(J)$.

PROOF. Define $J \subset \mathcal{P}(\omega_{\alpha+1})$ such that $X \in J$ if, and only if, $|X| \leq \aleph_\alpha$ and $X \cap \omega$ is finite. Then $\mathrm{cof}(\langle J\rangle, \subset) = \mathrm{add}(\langle J\rangle, \subset) = \omega_{\alpha+1}$. By Theorem 31 it suffices to show that TWO doesn't have a winning 2-tactic in $\mathrm{SMG}(J)$.

Let F be a 2-tactic for TWO in $\mathrm{SMG}(J)$. For $\omega < \eta < \omega_{\alpha+1}$ put $\phi(\eta) = \sup\big(\eta \cup F(\eta)\big)$. Let $C \subseteq \omega_{\alpha+1} \setminus (\omega+1)$ be a closed unbounded set such that $\phi(\gamma) < \beta$ whenever $\gamma < \beta$ are in C.

For each $\eta \in C$ define $\phi_\eta\colon C \setminus (\eta+1) \longrightarrow \omega_{\alpha+1}$ so that $\phi_\eta(\beta) = \sup\big(\beta \cup F(\eta, \beta)\big)$ for all β. Then choose a closed, unbounded set $C_\eta \subseteq C \setminus (\alpha+1)$ such that $\phi_\eta(\beta) < \gamma$ whenever $\beta < \gamma$ are in C_η.

Let D be the diagonal intersection of $(C_\eta : \eta \in C)$; *i.e.*, $D = \{\xi \in C : \xi \in \cap\{C_\eta : \eta < \xi \text{ and } \eta \in C\}$. Then D is an unbounded subset of $\omega_{\alpha+1}$. Now observe that if $\eta_1 < \eta_2 < \eta_3$ are elements of D, then

(1) $\eta_2 \in C_{\eta_1}$,
(2) $\eta_3 \in C_{\eta_1} \cap C_{\eta_2}$, and thus
(3) $F(\eta_1) \subseteq \eta_2$ and $F(\eta_1, \eta_2) \subseteq \eta_3$.

Define $\Phi\colon [D]^2 \longrightarrow \omega$ so that

$$\Phi(\eta, \beta) = \max\Big(\omega \cap \big(F(\eta) \cup F(\eta, \beta)\big)\Big).$$

By the Erdős-Rado theorem we obtain an $n < \omega$ and an uncountable $X \subset D$ such that $\Phi(\eta, \beta) = n$ for all $\eta < \beta \in X$. Pick $\eta_1 < \eta_2 < \cdots < \eta_m < \cdots$ from X and put $O_n = \eta_n$ for each n. Put $T_1 = F(O_1)$ and $T_{n+1} = F(O_n, O_{n+1})$ for each n.

Then $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ is an F-play of SMG(J) which is lost by TWO ∎

COROLLARY 36 *Assume the Continuum Hypothesis Then there is a free ideal* $J \subset \mathcal{P}(\omega_2)$ *such that* $\mathrm{cof}(\langle J\rangle, \subset) = \aleph_2$, *and there is no positive integer k for which TWO has a winning k-tactic in* VSG(J)

PROBLEM 3 Is there for each m a free ideal J_m such that TWO does not have a winning m-tactic, but does have a winning $m+1$-tactic in VSG(J_m)?

4 5 *The Banach-Mazur game and an example of Debs* The Banach-Mazur game is defined as follows for a topological space (X, τ) Players ONE and TWO alternately choose nonempty open subsets from X, in the n-th inning player ONE first chooses O_n and TWO responds with T_n An inning is played for each positive integer The sets chosen by the players must satisfy the rule

$$O_{n+1} \subseteq T_n \subseteq O_n$$

for all n Player TWO wins the play

$$(O_1, T_1, \ldots, O_n, T_n, \ldots)$$

if the intersection of these sets is nonempty, otherwise player ONE wins Following Galvin and Telgarsky [G-T], we denote this game by BM(X, τ) In the early 1980's Debs [D] solved Problem 3 of [F-K] by giving examples of topological spaces (X, τ) for which player TWO has a winning strategy in the game BM(X, τ), but no winning 1 tactic In all but one of Debs' examples it was known (in ZFC) that TWO has a winning 2-tactic We show here that also for the remaining example player TWO has a winning 2-tactic (Corollary 41) This was previously known under the assumption of some additional hypotheses

This result eliminates this example as a candidate for providing evidence (consistent, modulo ZFC) towards the following conjecture of Telgarsky

CONJECTURE 2 (TELGARSKY, [T], P 236) *For each positive integer k there is a topological space* (X_k, τ_k) *such that TWO does not have a winning k tactic, but does have a winning* $k+1$*-tactic in the game* BM(X_k, τ_k)

The following unpublished result of Galvin is the only theorem known to us which gives general conditions under which TWO has a winning 2-tactic if TWO has a winning strategy in the Banach-Mazur game

THEOREM 37 (GALVIN, UNPUBLISHED) *Let* (X, τ) *be a topological space for which TWO has a winning strategy in the Banach-Mazur game If this space has a* π *base* $\mathcal{P}$ *with the property that*

- $|\{V \in \mathcal{P} : B \subseteq V\}| < S(B)$ *for each B in* $\mathcal{P}$,

then TWO has a winning 2*-tactic*

Here the cardinal number $S(B)$ is defined to be the minimal κ such that B does not contain a collection of κ pairwise disjoint nonempty open subsets; it is said to be the *Souslin number* of B.

This subsection is organised as follows. We first prove a theorem concerning k-tactics in the Banach-Mazur game which is analogous to Theorem 5 of [G-T]. It provides an equivalent formulation of Telgarsky's conjecture which allows player TWO slightly more information: TWO may also remember the inning number. After this we give our result on Debs' example.

4.5.1. Markov k-tactics. Whereas a k-tactic for player TWO remembers at most the latest k moves of the opponent, a strategy for TWO which remembers in addition to this information also the number of the inning in progress will be called a *Markov k-tactic.*

Note that if (X,τ) has a dense set of isolated points then player TWO has a winning 1-tactic in BM(X,τ). Thus we may assume that if at all possible, player ONE will avoid playing an open set which contains an isolated point. We may therefore restrict our attention to topological spaces without isolated points. By the following proposition we may further restrict our attention to topological spaces in which each nonempty open set contains infinitely many pairwise disjoint open subsets.

PROPOSITION 38. *Let (X,τ) be a topological space with no infinite set of pairwise disjoint open subsets. Then there is a positive integer n such that:*

$$\tau \setminus \{\emptyset\} = \tau_1 \cup \cdots \cup \tau_n$$

where each τ_i has the finite intersection property.

PROOF.

CLAIM 1. *There is a positive integer n such that every collection of pairwise disjoint nonempty open subsets is of cardinality $\leq n$. (This is a well known fact: see e.g. [C-N], Lemma 2.10, p. 31.)*

Now let n be the minimal positive integer satisfying Claim 1. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a collection of pairwise disjoint nonempty open subsets of the space. Then $\mathcal{U}$ is a maximal pairwise disjoint family.

For $1 \leq i \leq n$, let τ_i be a maximal family of nonempty open sets such that:

(1) $U_i \in \tau_i$,

(2) any two elements of τ_i have nonempty intersection.

CLAIM 2. $\tau \setminus \{\emptyset\} = \tau_1 \cup \cdots \cup \tau_n$.

PROOF OF CLAIM 2. Assume the contrary and let Y be a nonempty open set which is in none of the τ_i. Then we find for each i an X_i in τ_i which is disjoint from Y (by maximality of each τ_i). We may assume that $X_i \subseteq U_i$ for each i. But then $\{X_1, \ldots, X_n, Y\}$ is a collection of $n + 1$ pairwise disjoint nonempty open subsets of (X,τ), contradicting the choice of n.) ∎

Each τ_i has the finite intersection property. ∎

PROPOSITION 39. *Let (X, τ) be a topological space for which*

(1) Player TWO has a winning strategy in the game $\mathrm{BM}(X, \tau)$ *and*

(2) every collection of pairwise disjoint open subsets is finite.

Then TWO has a winning 1-tactic in $\mathrm{BM}(X, \tau)$.

PROOF. Write, by Proposition 38,

$$\tau \setminus \{\emptyset\} = \tau_1 \cup \cdots \cup \tau_n$$

where each τ_i has the finite intersection property, and n is minimal. Choose a pairwise disjoint collection $\{U_1, \ldots, U_n\}$ such that $U_j \in \tau_j$ for each j.

CLAIM 3. *For each j, if $S_1 \supseteq S_2 \supseteq \cdots$ is a denumerable chain from τ_j, then $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$.*

PROOF OF CLAIM 3. Assume the contrary, and fix j and a chain $S_1 \supseteq S_2 \supseteq \cdots$ in τ_j such that $\bigcap_{n=1}^{\infty} S_n = \emptyset$. We may assume that $S_{n+1} \subset S_n \subset U_j$ for all n.

Let F be a winning perfect information strategy for TWO in $\mathrm{BM}(X, \tau)$. Consider the play

$$(O_1, T_1, \ldots, O_m, T_m, \ldots)$$

which is defined so that

(1) $O_1 = S_1$,

(2) $T_m = F(O_1, \ldots, O_m)$ for all m and

(3) $O_{m+1} = T_m \cap S_{m+1}$.

Note that each response by player TWO using F is a member of τ_j, whence each O_m is a legal move by ONE. But TWO lost, contradicting the assumption that F was a winning strategy. This completes the proof of Claim 3. ■

We now define a winning 1-tactic, G, for TWO. Let U be a nonempty open subset of X. Choose the minimal j such that $U_j \cap U \neq \emptyset$ and put $G(U) = U_j \cap U$. Claim 3 implies that this is a winning 1-tactic for TWO. ■

THEOREM 40. *Let k be a positive integer. If player TWO has a winning Markov k-tactic in the Banach Mazur game on some topological space, then TWO has a winning k-tactic in the Banach Mazur game on that space.*

PROOF. For $k = 1$, see Theorem 5 of [G-T]. So, assume $k > 1$, and let (X, τ) be a topological space such that TWO has a winning Markov k-tactic in the game $\mathrm{BM}(X, \tau)$. We may assume that every nonempty open subset of X contains infinitely many pairwise disjoint open subsets (player ONE may safely avoid playing open subsets not having this property).

Let F be a winning Markov k-tactic for TWO. For each nonempty open set U, let $\{J_m(U) : 0 < m < \omega\}$ bijectively enumerate a collection of infinitely many pairwise disjoint nonempty open subsets of U.

Define a k-tactic G for TWO as follows. Let $U_1 \supseteq \cdots \supseteq U_j$ be nonempty open sets, where $1 \leq j \leq k$.

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CASE 1. $j = 1$: Put $G(U_1) = F\big(J_2(U_1), 1\big)$.

CASE 2. $j > 1$ and $U_{i+1} \subseteq J_{l+i+1}(U_i)$ for $1 \leq i < j$, for some l. Put $G(U_1, \ldots, U_j) = F\big(J_{l+2}(U_1), \ldots, J_{l+j+1}(U_j), l+j\big)$.

CASE 3. In all other cases define $G(U_1, \ldots, U_j) = G(U_j)$.

To see that G is a winning k-tactic for TWO, consider a play

$$(O_1, T_1, \ldots, O_m, T_m, \ldots)$$

such that

- $T_j = G(O_1, \ldots, O_j)$ for $j \leq k$ and
- $T_{n+k} = G(O_{n+1}, \ldots, O_{n+k})$ for all n.

From the definition of G and the rules of the Banach-Mazur game it follows that T_1 is defined by Case 1 and T_m for $m > 1$ by Case 2. In particular, writing S_n for $J_{n+1}(O_n)$ we find that:

(1) $T_j = F(S_1, \ldots, S_j, j)$ for $j \leq k$ and
(2) $T_{n+k} = F(S_{n+1}, \ldots, S_{n+k}, n+k)$

for all n. Indeed,

$$O_1 \supseteq S_1 \supseteq T_1 \supseteq O_2 \supseteq S_2 \supseteq \cdots.$$

Since F is a winning Markov k-tactic, it follows that $\bigcap_{n=1}^{\infty} O_n \neq \emptyset$. ∎

4.5.2. Debs' example. Let σ be the topology of the real line whose elements are of the form $U \setminus M$ where U is open and M is meager in the usual topology. The symbol $\mathrm{BM}(\mathbb{R}, \sigma)$ denotes the Banach-Mazur game, played on the topological space $(\mathbb{R}, \sigma)$. It is known that TWO has a winning strategy but does not have a winning 1-tactic in $\mathrm{BM}(\mathbb{R}, \sigma)$.

COROLLARY 41. *Player TWO has a winning* 2*-tactic in the game* $\mathrm{BM}(\mathbb{R}, \sigma)$.

PROOF. Theorem 22 of [S1] and Corollary 26. ∎

5. **Appendix: consistency of the hypotheses of Theorem 22.** We start with a ground model V and let $\mathbf{P} \in V$ be a forcing notion of cardinality $\leq \mathfrak{c}$. For a cardinal κ, denote by $\mathbf{P}_\kappa$ the product of κ copies of $\mathbf{P}$ taken side-by-side with countable supports.

LEMMA 42. *Let λ be an uncountable cardinal. Suppose:*

(1) $\kappa \geq \kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \omega_2$ are cardinal numbers such that

- *κ is a regular cardinal,*
- $\kappa \longrightarrow (\kappa_1)^2_\lambda$,
- $\kappa_1 \longrightarrow (\kappa_2)^3_{\mathfrak{c}}$,
- $\kappa_2 \longrightarrow (\kappa_3)^2_\lambda$ *and*
- $\kappa_3 \longrightarrow (\omega_2)^3_{\mathfrak{c}}$.

(2) Forcing with $\mathbf{P}$ *adds a real to the ground model.*

Then $\mathfrak{c} \longrightarrow (\omega\text{- path})^2_{\lambda/<\omega}$ *holds in the forcing extension* $V^{\mathbf{P}_\kappa}$.

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PROOF. Let $\lambda, \kappa, \kappa_1, \kappa_2, \kappa_3, \mathbf{P}$ be as in the assumptions. Our argument closely follows Section 2 of [To1].

For sets A, B the symbol A/B denotes $\{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha < \beta\}$.

Note that $V^{\mathbf{P}_\kappa}$ satisfies $\mathfrak{c} = \kappa$; we prove that $\kappa \longrightarrow (\omega\text{-path})^2_{\lambda/<\omega}$ holds in $V^{\mathbf{P}_\kappa}$.

Let $[\kappa]^2 = \bigcup_{i<\lambda} \dot{K}_i$ be a given partition in $V^{\mathbf{P}_\kappa}$. Let $\dot{U}$ be a $\mathbf{P}_\kappa$-name for a member of $[\kappa]^\kappa$. Pick $A \in [\kappa]^\kappa$ and for each $\alpha \in A$, a $q_\alpha \in \mathbf{P}_\kappa$ such that $q_\alpha \Vdash \alpha \in \dot{U}$ and such that the q_α's form a Δ-system. Define $H\colon [A]^2 \longrightarrow (\lambda + 1)$ so that $H(\{\alpha, \beta\}) = i$ if i is the minimal j such that $p \Vdash \{\alpha, \beta\} \in \dot{K}_j$ for some $p \leq q_\alpha, q_\beta$ if such j exists (*i.e.*, if q_α and q_β are compatible), and $H(\{\alpha, \beta\}) = \lambda$ if q_α is incompatible with q_β.

By our choice of κ, the partition relation $\kappa \longrightarrow (\kappa_1)^2_\lambda$ holds. Therefore, choose $A_1 \subset [A]^{\kappa_1}$ and $i \leq \lambda$ such that $H''[A_1]^2 = \{i\}$. Since $\mathbf{P}_\kappa$ satisfies the $\mathfrak{c}^+$-c.c., we have $i < \lambda$.

Let $\langle p_{\alpha,\beta} : \{\alpha, \beta\} \in [A_1]^2\rangle$ be a fixed sequence of conditions such that $p_{\alpha,\beta} \leq q_\alpha, q_\beta$ and $p_{\alpha,\beta} \Vdash \{\alpha, \beta\} \in \dot{K}_i$. For $\alpha < \beta < \gamma$ in A_1 we define $H_0(\{\alpha, \beta, \gamma\})$ to be a pair (c, d), where c codes $p_{\alpha,\beta}$ and $p_{\alpha,\gamma}$ as structures as well as relations between the ordinals of $\mathrm{dom}(p_{\alpha,\beta})$ and $\mathrm{dom}(p_{\alpha,\gamma})$, and d does the same for $p_{\alpha,\gamma}$ and $p_{\beta,\gamma}$. Since there are only $\mathfrak{c}$ such pairs, and since $\kappa_1 \longrightarrow (\kappa_2)^3_{\mathfrak{c}}$ holds, choose $A_2 \in [A_1]^{\kappa_2}$ and (c, d) such that $H_0''[A_2]^3 = \{(c, d)\}$. For convenience, assume that A_2 has order type κ_2. It follows that for each $\alpha \in A_2$ the sequence $\langle p_{\alpha,\beta} : \beta \in A_2 \setminus (\alpha + 1)\rangle$ forms a Δ-system with root p^0_α ($\leq q_\alpha$), and that for each $\gamma \in A_2$ the sequence $\langle p_{\beta,\gamma} : \beta \in A_2 \cap \gamma\rangle$ forms a Δ-system with root p^1_γ ($\leq q_\gamma$). Moreover, the p^0_α's and p^1_γ's form Δ-systems with roots p^0 and p^1 respectively. To see the latter, note that we may shrink A_2 to a cofinal subset A_3 so that the relevant p^0_α's and p^1_α's do in fact form a Δ-system. Now consider $\alpha, \beta, \gamma \in A_3$, and $\alpha', \beta' \in A_2$. Comparing $H_0(\{\alpha, \beta, \gamma\})$, $H_0(\{\alpha, \beta', \gamma\})$ and $H_0(\{\alpha', \beta', \gamma\})$, one sees that the sequence $\langle p^0_\alpha : \alpha \in A_2\rangle$ forms a Δ-system. A similar argument works for the p^1_γ's.

Also, p^0 is compatible with p^1. We call $\langle p_{\alpha,\beta} : \{\alpha, \beta\} \in B/B\rangle$ a *double* Δ-*system* with root $p^0 \cup p^1$.

There is no reason why for a given α the conditions p^0_α and p^1_α should be compatible: if these were always compatible, our argument would yield a consistency proof of $\mathfrak{c} \longrightarrow (\omega_1)^2_\lambda$, which is false in ZFC.

We now save as much of the compatibility between p^0_α and p^1_α as is needed for the consistency proof of $\mathfrak{c} \longrightarrow (\omega\text{-path})^2_{\lambda/<\omega}$. Thin out A_2 to a cofinal subset A_3 such that $\mathrm{dom}(p^0_\alpha \cup p^1_\alpha) \cap \mathrm{dom}(p^0_\beta \cup p^1_\beta) = \mathrm{dom}(p^0 \cup p^1)$ for all $\{\alpha, \beta\} \in A_3/A_3$. Then in particular p^1_α and p^0_β are compatible for $\{\alpha, \beta\} \in A_3/A_3$.

Now repeat the reasoning above with A_2 in place of A, κ_2 in place of κ, κ_3 in place of κ_1, and ω_2 in place of κ_2. Also, p^1_α will now play the role of q_α, and p^0_β the role of q_β for $\{\alpha, \beta\} \in A_3/A_3$. We get a set $A_4 \subset A_3$ of order type ω_2 and some $j < \lambda$ (which may be different from i), conditions $\bar{p}_{\alpha,\beta}$ for $\{\alpha, \beta\} \in A_4/A_4$ that form a double Δ-system with root $\bar{p}^0 \cup \bar{p}^1$, and we get roots $\bar{p}^0_\alpha$ and $\bar{p}^1_\gamma$ as before. Now $\bar{p}_{\alpha,\beta} \Vdash \{\alpha, \beta\} \in \dot{K}_j$ for $\{\alpha, \beta\} \in A_4/A_4$.

Our choice of $\bar{p}_{\alpha,\beta}$ at the beginning of the second run of the argument insures that $\bar{p}^0_\alpha \leq p^1_\alpha$ and $\bar{p}^1_\gamma \leq p^0_\gamma$, and hence $\bar{p}^0 \leq p^1$ and $\bar{p}^1 \leq p^0$.

Now let $\mathbf{G}$ be a generic subset of $\mathbf{P}_\kappa$. Define:

$\dot{X} = \{\alpha \in A_4 : p^0_\alpha \in \mathbf{G}\}$,
$\dot{Y} = \{\alpha \in A_4 : p^1_\alpha \in \mathbf{G}\}$,
$\dot{W} = \{\alpha \in A_4 : \bar{p}^0_\alpha \in \mathbf{G}\}$,
$\dot{Z} = \{\alpha \in A_4 : \bar{p}^1_\alpha \in \mathbf{G}\}$.

Then $\dot{Z} \subset \dot{X}$ and $\dot{W} \subset \dot{Y}$, and all four sets are cofinal in A_4.

Now $\bar{p}^0 \cup \bar{p}^1$ forces the following facts:

(1) $\exists \delta_1 \in \omega_2 \forall \alpha \in \dot{X} \setminus \delta_1 \{\beta \in \dot{W} : \{\alpha, \beta\} \in \dot{K}_i\}$ is cofinal in A_4, and

(2) $\exists \delta_2 \in \omega_2 \forall \alpha \in \dot{Y} \setminus \delta_2 \{\beta \in \dot{Z} : \{\alpha, \beta\} \in \dot{K}_j\}$ is cofinal in A_4.

The combination of (1) and (2) suffices to construct in $V^{\mathbf{P}_\kappa}$ an ω-path of the given partition that uses only colors i and j:

Let $\delta = \max\{\delta_1, \delta_2\}$. Inductively define an increasing sequence $\langle x_n : n \in \omega\rangle$ of ordinals such that $x_{2k} \in Z$ (and hence in X), $x_{2k+1} \in W$, and $\{x_{2k}, x_{2k+1}\} \in \dot{K}_i$ (by (1)); $\{x_{2k+1}, x_{2k+2}\} \in \dot{K}_j$ (by (2)).

It remains to prove (1) and (2). We shall prove (1) only; the proof of (2) is similar, and is a special case of [To1], Section 2, property (1).

Assume that $\bar{p}^0 \cup \bar{p}^1$ does not force (1). Then we can find a condition $\bar{p}^2 \leq \bar{p}^0 \cup \bar{p}^1$ and a $\mathbf{P}_\kappa$-name $\dot{D} \in [\dot{X}]^{\omega_2}$ and for each $\beta \in \dot{D}$ a $\gamma_\beta \in A_4 \setminus (\beta + 1)$ such that $\bar{p}^2 \Vdash \{\beta, \delta\} \notin \dot{K}_i$ whenever $\delta \in \dot{W} \setminus \gamma_\beta$.

Working in V, we pick $B \in [A_4]^{\omega_2}$ such that for each $\beta \in B$ we find $r_\beta \leq p^0_\beta \cup \bar{p}^2$ such that $r_\beta \Vdash \beta \in \dot{D}$, and r_β decides the value of γ_β. We may assume that the r_β's form a Δ-system with root $\leq \bar{p}^2 \leq \bar{p}^0 \cup \bar{p}^1$, and that $\gamma_\beta < \delta$ for all $\{\beta, \delta\} \in B/B$. Since $\langle p_{\alpha,\beta} : \beta \in B \setminus (\alpha + 1)\rangle$ forms a Δ-system, we may also assume that $\mathrm{dom}(r_\beta) \cap \mathrm{dom}(p_{\beta,\delta} \setminus p^0_\beta) = \emptyset$ for all $\delta > \gamma_\beta$ in A_4.

Pick $\delta \in A_2$ such that $B \cap \delta$ is uncountable and $\mathrm{dom}(\bar{p}^0_\delta) \cap \mathrm{dom}(\bar{p}^2) = \mathrm{dom}(\bar{p}^0)$. Since $\langle p_{\beta,\delta} : \beta \in B \cap \delta\rangle$ forms a Δ-system with root p^1_δ and since $\mathrm{dom}(\bar{p}^0_\delta)$ is countable, we have $\mathrm{dom}(p_{\beta,\delta} \setminus p^1_\delta) \cap \mathrm{dom}(\bar{p}^0_\delta) \neq \emptyset$ for only countably many $\beta \in B \cap \delta$. So pick a $\beta \in B \cap \delta$ such that $\mathrm{dom}(p_{\beta,\delta} \setminus p^1_\delta) \cap \mathrm{dom}(\bar{p}^0_\delta) = \emptyset$.

Define $r \in \mathbf{P}_\kappa$ as follows:

$$\mathrm{dom}(r) = \mathrm{dom}(r_\beta) \cup \mathrm{dom}(\bar{p}^0_\delta) \cup \mathrm{dom}(p_{\beta,\delta} \setminus p^1_\delta),$$
$$r | \mathrm{dom}(r_\beta \cup \bar{p}^0_\delta) = r_\beta \cup \bar{p}^0_\delta,$$

and

$$r(\xi) = p_{\beta,\delta}(\xi) \text{ for } \xi \in \mathrm{dom}\big(p_{\beta,\delta} \setminus \mathrm{dom}(r_\beta \cup \bar{p}^0_\delta)\big).$$

Then r is a well-defined condition with the property that $r \leq r_\beta, \bar{p}^0_\delta$ and $p_{\beta,\delta}$. So r forces that $\{\beta, \delta\} \in X/W$ and that $\{\beta, \delta\} \in K_i$, which is a contradiction. ∎

If $\mathbf{P}_\kappa$ is as in the assumptions of Lemma 42, then $\mathbf{P}_\kappa$ is a $\mathfrak{c}^+$-c.c. poset. If GCH holds in the ground model and $\lambda = \omega_1$, then our proof works if $\kappa \geq \aleph_8$. One can obtain the consistency of $\mathfrak{c} \longrightarrow (\omega\text{- path})^2_{\omega_1/<\omega}$ with a smaller size of the continuum, but this is not essential for our purposes. Todorčević has for example shown that, adjoining at least ω_2 Cohen reals to a model of the Continuum Hypothesis, produces a model in which $\omega_2 \longrightarrow (\omega\text{- path})^2_{\omega/<3}$.

We have actually proved something apparently stronger than $\mathfrak{c} \longrightarrow (\omega\text{- path})^2_{\lambda/<\omega}$ in $V^{\mathbf{P}_\kappa}$, namely a relation denoted by $\mathfrak{c} \longrightarrow (\omega\text{- path})^2_{\lambda/<3}$

We do not know the answers to the following two problems concerning the ω-path partition relation

PROBLEM 4. Is it for each integer $k > 2$ consistent, for some infinite cardinal numbers κ and λ, that $\kappa \not\longrightarrow (\omega\text{- path})^2_{\lambda/<k}$, but $\kappa \longrightarrow (\omega\text{- path})^2_{\lambda/<k+1}$?

PROBLEM 5. Is it consistent, for some infinite cardinal numbers κ and λ, that for each $k < \omega$, $\kappa \not\longrightarrow (\omega\text{- path})^2_{\lambda/<k}$, but $\kappa \longrightarrow (\omega\text{- path})^2_{\lambda/<\omega}$?

THEOREM 43 (TODORČEVIĆ) *If* ZFC *is a consistent theory, then so is the theory* ZFC + $\mathrm{cof}(\langle \mathcal{NWD}_{\mathbb{R}} \rangle, \subset) = \aleph_1 + \mathfrak{c} \longrightarrow (\omega\text{- path})^2_{\omega_1/<\omega}$

PROOF. Theorem 43 is an immediate consequence of Lemma 42 It is well known that if CH holds in the ground model, and $\mathbf{P}$ is *e g* Sacks or Prikry-Silver forcing, then (b) and (c) of the lemma hold for every κ. It is also known that adding any number of Sacks or Prikry-Silver reals side-by-side with countable supports to a model of CH, one obtains a model where the collection of meager sets whose Borel codes are from the ground model, is a cofinal subfamily of $\langle \mathcal{NWD}_{\mathbb{R}} \rangle$ (see [M]) Since $|{}^{\omega}\omega \cap V| = \aleph_1$, we get $\mathrm{cof}(\langle \mathcal{NWD}_{\mathbb{R}} \rangle, \subset) = \aleph_1$ in the forcing extension ∎

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