GROUPS WITH FINITELY MANY CONJUGACY CLASSES OF SUBGROUPS WITH LARGE SUBNORMAL DEFECT

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(Received 21 June, 1993)

Given a group G and a positive integer k, let $\nu_k(G)$ denote the number of conjugacy classes of subgroups of G which are not subnormal of defect at most k. Groups G such that $\nu_k(G) < \infty$ for some k are considered in Section 2 of [**1**], and Theorem 2.4 of that paper states that an infinite group G for which $\nu_k(G) < \infty$ (for some k) is nilpotent provided only that all chief factors of G are locally (soluble or finite). Now it is easy to see that a group G whose chief factors are of this type is *locally graded*, that is, every nontrivial, finitely generated subgroup F of G has a nontrivial finite image (since there is a chief factor H/K of G such that F is contained in H but not in K). On the other hand, every (locally) free group is locally graded and so there is in general no restriction on the chief factors of such groups. The class of locally graded groups is a suitable class to consider if one wishes to do no more than exclude the occurrence of finitely generated, infinite simple groups and, in particular, Tarski p-groups. As pointed out in [**1**], Ivanov and Ol'shanskiĭ have constructed (finitely generated) infinite simple groups all of whose proper nontrivial subgroups are conjugate; clearly a group G with this property satisfies $\nu_1(G) = 1$. The purpose of this note is to provide the following generalization of the above-mentioned theorem from [**1**].

THEOREM. *Let G be an infinite, locally graded group satisfying $\nu_k(G) < \infty$, for some positive integer k. Then G is nilpotent.*

Most of the proof is based on that in [**1**]. Indeed, the main objective will be to show that our group G satisfies the stated hypothesis on chief factors. (For the standard results on infinite groups which are used in our proof, the reader is referred to [**2**].) The following elementary results are a little stronger than required but are easy to prove.

LEMMA 1. *Let G be a group such that all rank* 1 *abelian subgroups of G fall into finitely many conjugacy classes. Then G has finite exponent.*

Proof. If G is periodic then the result is clear, since the order of any cyclic subgroup is bounded. So assume, for a contradiction, that G contains an element x of infinite order. Let p be an arbitrary prime. The subgroups $\langle x\rangle$, $\langle x^p\rangle$, $\langle x^{p^2}\rangle, \ldots$ fall into finitely many conjugacy classes and so there exist integers m, n with $m < n$ such that $\langle x^{p^m}\rangle$ is conjugate to $\langle x^{p^n}\rangle$. Let $y = x^{p^m}$, $t = n - m$. Then there exists g in G such that $(y^{p^t})^g = y^{\pm 1}$ and we have $\langle y\rangle < \langle y^g\rangle < \langle y^{g^2}\rangle < \ldots$. Clearly the union Y of these subgroups is isomorphic to the additive group of p-adic rationals. Since p was arbitrary, we see that G contains a copy of the p-adic rationals for all primes p. But these subgroups are of course locally cyclic (that is, of rank one) and pairwise nonisomorphic. Thus we have a contradiction and the result follows.

LEMMA 2. *Let G be a group whose nonsubnormal, rank* 1 *abelian subgroups fall into finitely many conjugacy classes. Let B denote the Baer radical of G and suppose that the element x of G has finite order* >1 *mod B. Then x has finite order.*

Glasgow Math. J. **37** (1995) 69–71.

https://doi.org/10.1017/S0017089500030408 Published online by Cambridge University Press

Proof. Suppose x has order $r>1$ mod B and let p be a prime not dividing r. Then none of the subgroups $\langle x\rangle$, $\langle x^p\rangle$, $\langle x^{p^2}\rangle, \ldots$ is subnormal in G and so, as in the proof of Lemma 1, if x has infinite order then there is an integer m such that x^{p^m} is contained in a subgroup P of G which is isomorphic to the additive group of p-adic rationals. But P is not subnormal in G and, since there are infinitely many such primes p, we obtain the required contradiction.

Our final requirement is as follows.

LEMMA 3. *Let G be a locally graded group of finite exponent. Then G is locally finite.*

Proof. Assuming the result false, we may suppose that G is finitely generated and infinite. By Zel'manov's solution to the restricted Burnside problem ([**4**], [**5**]), every finite image of G has bounded order and so the finite residual R of G has finite index. But then R is finitely generated and nontrivial and therefore contains a proper G-invariant subgroup of finite index. This contradiction completes the proof.

We note that the use of such a deep result as that of Zel'manov is not very satisfactory in the context of our discussion. It seems reasonable to hope that it could be avoided here, given that the group to which we shall be applying Lemma 3 has the additional hypothesis on conjugacy classes.

Proof of the theorem. Let G be as stated and suppose that $\nu_k(G)=m$. Let B denote the Baer radical of G and suppose that B is not nilpotent. Then, by the well-known theorem of Roseblade [**3**], there is a finitely generated subgroup F of B whose subnormal defect l in G is at least $k+m+1$. For each $i=0,1,\ldots,l$, let F_i denote the ith term of the normal closure series of F in G (thus, in particular, $F_0=G$ and $F=F_l<F_{l-1}$). Then the subgroups $F_{k+1},\ldots,F_l$ are more than m in number and so there exist i, j with $k+1\leq i<j\leq l$ such that $F_i^g=F_j$, for some g in G. But the subnormal defects in G of F_i and F_j are i and j respectively, and a contradiction ensues. Thus B is nilpotent. Now all cyclic subgroups of G/B fall into finitely many conjugacy classes and so G/B has only finitely many normal subgroups. In particular, G has a maximal normal soluble subgroup S, say. As above, we see that the Baer radical of G/S is nilpotent and hence trivial, and we deduce from Lemma 1 that G/S has finite exponent. Next, suppose that x is an element of $G\backslash S$ and assume that x has infinite order. Then there exist normal subgroups U, V of G, with $U<V$, such that V/U is the Baer radical of G/U and x has finite order >1 mod V but infinite order mod U. Using bars to denote factor groups mod U, we have that $\langle\bar{x}\rangle$ is not contained in the Baer radical $\bar{V}$ of $\bar{G}$ but has finite order mod $\bar{V}$. Lemma 2 now applies to give a contradiction. It follows that every element of $G\backslash S$ has finite order.

We now proceed to show that G/S is locally graded. Suppose, for a contradiction once more, that F is a finitely generated subgroup of G such that FS/S is nontrivial but has no nontrivial finite images. Since F contains an element of $G\backslash S$ we see that F has a finite generating set consisting of elements of finite order. If X is any subgroup of G with such a generating set then of course X/X' is finite and X' is finitely generated. Since no term of the derived series of F is contained in S, repeated application of the above allows us to deduce that $F/F^{(d)}$ is finite and $F^{(d)}$ is finitely generated, where d is the derived length of S. Since G is locally graded, there is a normal subgroup N of F, properly contained in $F^{(d)}$, such that F/N is finite. By hypothesis, we have $F=N(F\cap S)$. This gives the contradiction $F^{(d)}\leq N$ and so G/S is locally graded. By Lemma 3, G/S is therefore locally finite. Since S is soluble, all chief factors of G are abelian or locally finite. The result now follows by Theorem 2.4 of [**1**].

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https://doi.org/10.1017/S0017089500030408 Published online by Cambridge University Press