12

Canonical anti-commutation relations

Throughout this chapter, $(\mathcal{Y}, \nu)$ is a Euclidean space, that is, a real vector space $\mathcal{Y}$ equipped with a positive definite form ν.

In this chapter we introduce the concept of *representations of the canonical anti-commutation relations* (CAR representations). The definition that we use is very similar to the definition of a *representation of the Clifford relations*, which will be discussed in Chap. 15. In the case of CAR representations we assume in addition that operators satisfying the Clifford relations act on a Hilbert space and are self-adjoint, whereas in the standard definition of Clifford relations the self-adjointness is not required.

CAR representations are used in quantum physics to describe fermions. Actually, CAR representations, as introduced in Def. 12.1, are appropriate for the so-called neutral fermions. Most fermions in physics are charged, and for them a slightly different formalism is used, which we introduce under the name *charged CAR representations*. Charged CAR representations can be viewed as a special case of (neutral) CAR representations, where the dual phase space $\mathcal{Y}$ is complex and a somewhat different notation is used.

CAR representations appear in quantum physics in at least two contexts. First, they describe fermionic systems. This is to us the primary meaning of the CAR, and most of our motivation and terminology is derived from it. Second, they describe spinors, that is, representations of the *Spin* and *Pin groups*. In most applications the second meaning is restricted to the finite-dimensional case. We will also discuss the second meaning (including the Spin and Pin groups over infinite-dimensional spaces).

12.1 CAR representations

12.1.1 Definition of a CAR representation

Let $\mathcal{H}$ be a Hilbert space. Recall that $B_{\mathrm{h}}(\mathcal{H})$ denotes the set of bounded self-adjoint operators on $\mathcal{H}$ and $[A, B]_+ := AB + BA$ is the anti-commutator of A and B.

Definition 12.1 *A* representation of the canonical anti-commutation relations *or a* CAR representation over $\mathcal{Y}$ in $\mathcal{H}$ *is a linear map*

$$\mathcal{Y} \ni y \mapsto \phi^{\pi}(y) \in B_{\mathrm{h}}(\mathcal{H}) \tag{12.1}$$

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

satisfying

$$[\phi^\pi(y_1), \phi^\pi(y_2)]_+ = 2y_1{\cdot}\nu y_2 1\!\!1, \quad y_1, y_2 \in \mathcal{Y}. \tag{12.2}$$

The operators $\phi^\pi(y)$ are called (fermionic) field operators*.*

Remark 12.2 *The superscript π is an example of a "name" of a given CAR representation.*

Remark 12.3 *Unfortunately, the analogy between the CAR (12.2) and the CCR (8.23) is somewhat violated by the number 2 on the r.h.s. of (12.2). The reason for this convention is the identity $\phi^\pi(y)^2 = (y{\cdot}\nu y)1\!\!1$.*

Remark 12.4 *Later on we will sometimes call (12.1)* neutral CAR representations*, to distinguish them from* charged CAR representations *introduced in Def. 12.17.*

In what follows we assume that we are given a CAR representation (12.1).

By complex linearity we can extend the definition of $\phi^\pi(y)$ to $\mathbb{C}\mathcal{Y}$:

$$\phi^\pi(y_1 + \mathrm{i}y_2) := \phi^\pi(y_1) + \mathrm{i}\phi^\pi(y_2), \quad y_1, y_2 \in \mathcal{Y}.$$

Definition 12.5 *The operators $\phi^\pi(w)$ for $w \in \mathbb{C}\mathcal{Y}$ are also called* field operators*.*

We have

$$[\phi^\pi(w_1), \phi^\pi(w_2)]_+ = 2w_1{\cdot}\nu_{\mathbb{C}} w_2 1\!\!1, \quad w_1, w_2 \in \mathbb{C}\mathcal{Y},$$

where $\nu_{\mathbb{C}}$ is the complexification of ν.

We will sometimes use a different terminology. Let I be a set. We will say that $\{\phi_i^\pi \ : \ i \in I\} \subset B_{\mathrm{h}}(\mathcal{H})$ is a CAR representation iff

$$[\phi_i^\pi, \phi_j^\pi]_+ = 2\delta_{ij}. \tag{12.3}$$

Clearly, if $\mathcal{Y} \ni y \mapsto \phi^\pi(y)$ is a CAR representation and we choose an o.n. basis $\{e_i \ : \ i \in I\}$, then $\phi_i^\pi := \phi^\pi(e_i)$ is a CAR representation in the second meaning.

Theorem 12.6 *Introduce the notation $|y|_\nu := (y{\cdot}\nu y)^{\frac{1}{2}}$. Let $y \in \mathcal{Y}$, $\dim \mathcal{Y} > 1$.*

(1) spec $\phi^\pi(y) = \{-|y|_\nu, |y|_\nu\}$, $\|\phi^\pi(y)\| = |y|_\nu$.
(2) *Let $t \in \mathbb{C}$, $y \in \mathcal{Y}$. Then* $\|t1\!\!1 + \phi^\pi(y)\| = \max\{|t + |y|_\nu|, |t - |y|_\nu|\}$.
(3) $\mathrm{e}^{\mathrm{i}\phi^\pi(y)} = \cos|y|_\nu 1\!\!1 + \mathrm{i}\frac{\sin|y|_\nu}{|y|_\nu}\phi^\pi(y)$.
(4) *Let $\mathcal{Y}^{\mathrm{cpl}}$ be the completion of $\mathcal{Y}$. Then there exists a unique extension of (12.1) to a continuous map*

$$\mathcal{Y}^{\mathrm{cpl}} \ni y \mapsto \phi^{\pi^{\mathrm{cpl}}}(y) \in B(\mathcal{H}). \tag{12.4}$$

(12.4) is a representation of CAR.

Proof Since $(\phi^\pi)^2(y) = |y|_\nu^2 1\!\!1$, we have spec $\phi^\pi(y) \subset \{-|y|_\nu, |y|_\nu\}$. If there exists $y_0 \in \mathcal{Y}$ with $y_0 \neq 0$ and $\phi^\pi(y_0) = \lambda 1\!\!1_{\mathcal{H}}$, then $\dim \mathcal{Y} = 1$. But we assumed that $\dim \mathcal{Y} > 1$. Therefore, the spectrum of $\phi^\pi(y)$ cannot consist of only one element,

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

which proves (1). Statements (2) and (3) follow from (12.2), and statement (4) follows from (1). □

Motivated by Thm. 12.6 (4), henceforth we will assume that $\mathcal{Y}$ is a real Hilbert space.

12.1.2 CAR representations over a direct sum

Constructing a CAR representation over a direct sum of two spaces is not as simple as the analogous construction for CCR relations (compare with Prop. 8.6).

Proposition 12.7 *Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be two real Hilbert spaces. Suppose that $I_1 \in B(\mathcal{H}_1)$ is such that*

$$\mathcal{Y}_1 \oplus \mathbb{R} \ni (y_1, t) \mapsto \phi^1(y_1) + tI_1 \in B(\mathcal{H}_1),$$
$$\mathcal{Y}_2 \ni y_2 \mapsto \phi^2(y_2) \in B(\mathcal{H}_2)$$

are CAR representations. Then

$$\mathcal{Y}_1 \oplus \mathcal{Y}_2 \ni (y_1, y_2) \mapsto \phi^1(y_1) \otimes \mathbb{1} + I_1 \otimes \phi^2(y_2) \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

is a CAR representation.

12.1.3 Cyclicity and irreducibility

The following concepts are essentially the same as in the case of CCR representations.

Definition 12.8 *We say that a subset $\mathcal{U} \subset \mathcal{H}$ is* cyclic *for (12.1) if*

$$\mathrm{Span}\{\phi^\pi(y) \cdots \phi^\pi(y_n)\Psi \ : \ \Psi \in \mathcal{U},\ y_1, \dots, y_n \in \mathcal{Y}\}$$

is dense in $\mathcal{H}$. We say that $\Psi_0 \in \mathcal{H}$ is cyclic *for (12.1) if $\{\Psi_0\}$ is cyclic for (12.1).*

Definition 12.9 *We say that the representation (12.1) is* irreducible *if the only closed subspaces of $\mathcal{H}$ invariant under the $\phi^\pi(y)$ for $y \in \mathcal{Y}$ are $\{0\}$ and $\mathcal{H}$.*

Proposition 12.10 (1) *A CAR representation is irreducible iff $B \in B(\mathcal{H})$ and $[\phi^\pi(y), B] = 0$ for all $y \in \mathcal{Y}$ implies that B is proportional to identity.*

(2) *In the case of an irreducible CAR representation, all non-zero vectors in $\mathcal{H}$ are cyclic.*

12.1.4 Intertwining operators

Let

$$\mathcal{Y} \ni y \mapsto \phi^1(y) \in B_{\mathrm{h}}(\mathcal{H}_1), \tag{12.5}$$
$$\mathcal{Y} \ni y \mapsto \phi^2(y) \in B_{\mathrm{h}}(\mathcal{H}_2) \tag{12.6}$$

be CAR representations over the same Euclidean space $\mathcal{Y}$.

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

Definition 12.11 *We say that an operator* $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ intertwines (12.5) and (12.6) *iff*

$$A\phi^1(y) = \phi^2(y)A, \quad y \in \mathcal{Y}.$$

We say that it anti-intertwines (12.5) and (12.6) *iff*

$$A\phi^1(y) = -\phi^2(y)A, \quad y \in \mathcal{Y}.$$

We say that (12.5) and (12.6) are unitarily equivalent, *resp.* anti-equivalent *if there exists a unitary* $U \in U(\mathcal{H}_1, \mathcal{H}_2)$ *intertwining, resp. anti-intertwining (12.5) and (12.6).*

Proposition 12.12 *If the representations (8.11) and (8.12) are irreducible, then the set of operators (anti-)intertwining them equals either* $\{0\}$ *or* $\{\lambda U \ : \lambda \in \mathbb{C}\}$ *for some* $U \in U(\mathcal{H})$.

Proof The proof is an obvious modification of the proof of the analogous fact about CCR representations and about C^*-algebras; see Thm. 8.13. □

12.1.5 Volume element

Consider a CAR representation (12.1). Let $\mathcal{X}$ be a finite-dimensional oriented subspace of $\mathcal{Y}$. Let $(e_1, \dots, e_n)$ be an o.n. basis of $\mathcal{X}$ compatible with the orientation.

Definition 12.13 *The* volume element of the subspace $\mathcal{X}$ in the representation (12.1) *is defined by*

$$Q^\pi_{\mathcal{X}} := \phi^\pi(e_1) \cdots \phi^\pi(e_n).$$

In what follows we drop the superscript π. Note that $Q_{\mathcal{X}}$ does not depend on the choice of an oriented o.n. basis. Changing the orientation amounts to changing $Q_{\mathcal{X}}$ into $-Q_{\mathcal{X}}$. We have

$$Q^2_{\mathcal{X}} = (-1)^{n(n-1)/2}\mathbb{1}, \qquad Q^*_{\mathcal{X}} = Q^{-1}_{\mathcal{X}} = (-1)^{n(n-1)/2}Q_{\mathcal{X}}.$$

Thus $Q_{\mathcal{X}}$ is self-adjoint iff $n = 0, 1 \pmod 4$; otherwise it is anti-self-adjoint.

Define $u_{\mathcal{X}} \in O(\mathcal{Y})$ by

$$u_{\mathcal{X}} = (-\mathbb{1}) \oplus \mathbb{1},$$

where we use the decomposition $\mathcal{Y} = \mathcal{X} \oplus \mathcal{X}^\perp$. Clearly,

$$Q_{\mathcal{X}}\phi(y)Q^{-1}_{\mathcal{X}} = (-1)^n\phi(u_{\mathcal{X}}y), \quad y \in \mathcal{Y}.$$

12.1.6 CAR over Kähler spaces

In this subsection we fix a CAR representation (12.1). We use the notation and results of Subsects. 1.3.6, 1.3.8 and 1.3.9.

The following proposition shows that choosing a sufficiently large subspace of anti-commuting field operators is equivalent to fixing a Kähler structure in $(\mathcal{Y}, \nu)$.

Proposition 12.14 *Suppose that $\mathcal{Z} \subset \mathbb{C}\mathcal{Y}$ is a subspace such that*

(1) $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$;
(2) $z_1, z_2 \in \mathcal{Z}$ *implies* $[\phi^\pi(z_1), \phi^\pi(z_2)]_+ = 0$ *(or equivalently, $\mathcal{Z}$ is isotropic for $\nu_{\mathbb{C}}$).*

Then there exists a unique Kähler anti-involution j *on* $(\mathcal{Y}, \nu)$ *such that*

$$\mathcal{Z} = \{y - \mathrm{i}\mathrm{j}y \ : \ y \in \mathcal{Y}\}. \tag{12.7}$$

Proof (1) implies that there exists a linear map $\mathrm{j} \in L(\mathcal{Y})$ such that $\mathcal{Z}$ is given by (12.7). (2) implies

$$\begin{aligned} 0 &= (y_1 + \mathrm{i}\mathrm{j}y_1)\cdot\nu_{\mathbb{C}}(y_2 + \mathrm{i}\mathrm{j}y_2) \\ &= y_1\cdot\nu y_2 - (\mathrm{j}y_1)\cdot\nu(\mathrm{j}y_2) + \mathrm{i}\left((\mathrm{j}y_1)\cdot\nu y_2 + y_1\cdot\nu\mathrm{j}y_2\right). \end{aligned}$$

Hence,

$$y_1\cdot\nu y_2 - (\mathrm{j}y_1)\cdot\nu(\mathrm{j}y_2) = 0, \quad (\mathrm{j}y_1)\cdot\nu y_2 + y_1\cdot\nu\mathrm{j}y_2 = 0,$$

which shows that j is orthogonal and anti-symmetric, hence is a Kähler anti-involution. □

Motivated in part by the above proposition, let us fix j, a Kähler anti-involution on $(\mathcal{Y}, \nu)$. Recall that the space $\mathcal{Z}$ given by (12.7) is called the holomorphic subspace of $\mathbb{C}\mathcal{Y}$.

Definition 12.15 *We define the* j-creation *and* j-annihilation operators*:*

$$a^{\pi*}(z) := \phi^\pi(z), \quad a^\pi(z) := \phi^\pi(\overline{z}), \quad z \in \mathcal{Z}.$$

They are bounded operators, adjoint to one another.

Proposition 12.16 *One has $\phi^\pi(z, \overline{z}) = a^{\pi*}(z) + a^\pi(z)$, $z \in \mathcal{Z}$,*

$$[a^{\pi*}(z_1), a^{\pi*}(z_2)]_+ = 0, \quad [a^\pi(z_1), a^\pi(z_2)]_+ = 0,$$

$$[a^\pi(z_1), a^{\pi*}(z_2)]_+ = (z_1|z_2)\mathbb{1}, \quad z_1, z_2 \in \mathcal{Z}.$$

The Kähler structure appears naturally in the context of Fock representations. It also arises when the Euclidean space $(\mathcal{Y}, \nu)$ is equipped with a charge 1 $U(1)$ symmetry, as in Subsect. 1.3.11. We discuss the latter application in the following subsection.

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

12.1.7 Charged CAR representations

CAR representations, as defined in Def. 12.1, provide a natural framework for the description of neutral fermions. Therefore, sometimes we will call them *neutral CAR representations*. In the context of charged fermions (much more common than neutral fermions) physicists prefer to use another formalism described in the following definition.

Definition 12.17 *Suppose that $(\mathcal{Y}, (\cdot|\cdot))$ is a unitary space and $\mathcal{H}$ a Hilbert space. We say that an anti-linear map*

$$\mathcal{Y} \ni y \mapsto \psi^{\pi}(y) \in B(\mathcal{H})$$

is a charged CAR representation *if*

$$\begin{aligned} [\psi^{\pi *}(y_1), \psi^{\pi *}(y_2)]_+ &= [\psi^{\pi}(y_1), \psi^{\pi}(y_2)]_+ = 0, \\ [\psi^{\pi}(y_1), \psi^{\pi *}(y_2)]_+ &= (y_1|y_2)\mathbb{1}, \qquad y_1, y_2 \in \mathcal{Y}. \end{aligned}$$

Suppose that $y \mapsto \psi^{\pi}(y)$ is a charged CAR representation. Set

$$\begin{aligned} \phi^{\pi}(y) &:= \big(\psi^{\pi}(y) + \psi^{\pi *}(y)\big), \\ y_1 \cdot \nu y_2 &:= \mathrm{Re}(y_1|y_2). \end{aligned}$$

Then $\mathcal{Y} \ni y \mapsto \phi^{\pi}(y) \in B_{\mathrm{h}}(\mathcal{H})$ is a neutral CAR representation over the Euclidean space $(\mathcal{Y}, \nu)$. In addition, $\mathcal{Y}$ is equipped with a charge 1 symmetry $U(1) \ni \theta \mapsto \mathrm{e}^{\mathrm{i}\theta} \in O(\mathcal{Y})$.

Conversely, charged CAR representations arise when we have a (neutral) CAR representation and the underlying Euclidean space is equipped with a charge 1 $U(1)$ symmetry. Let us make this precise. Suppose that $(\mathcal{Y}, \nu)$ is a Euclidean space and

$$\mathcal{Y} \ni y \mapsto \phi^{\pi}(y) \in B_{\mathrm{h}}(\mathcal{H})$$

is a neutral CAR representation. Suppose that

$$U(1) \ni \theta \mapsto u_{\theta} = \cos\theta \mathbb{1} + \sin\theta \mathrm{j}_{\mathrm{ch}} \in O(\mathcal{Y})$$

is a charge 1 symmetry. We know that j_{ch} is a Kähler anti-involution. Following the standard procedure described in the previous subsection, we introduce the holomorphic subspace for j_{ch}, that is,

$$\mathcal{Z}_{\mathrm{ch}} := \{y - \mathrm{i}\mathrm{j}_{\mathrm{ch}} y \ : \ y \in \mathcal{Y}\} \subset \mathbb{C}\mathcal{Y}.$$

We define creation and annihilation operators associated with j_{ch}. We have a natural identification of the space $\mathcal{Z}_{\mathrm{ch}}$ with $\mathcal{Y}$:

$$\mathcal{Y} \ni y \mapsto z = \frac{1}{2}(\mathbb{1} - \mathrm{i}\mathrm{j}_{\mathrm{ch}})y \in \mathcal{Z}_{\mathrm{ch}}. \tag{12.8}$$

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

We use the identification (12.8) to introduce charged fields parametrized by elements of $\mathcal{Y}$:

$$\psi^{\pi *}(y) := \phi^{\pi}(z), \qquad \psi^{\pi}(y) := \phi^{\pi}(\overline{z}).$$

Then we obtain a charged CAR representation over $\mathcal{Y}^{\mathbb{C}}$ with the complex structure given by j_{ch} and the scalar product

$$(y_1|y_2) := y_1 \cdot \nu y_2 - \mathrm{i} y_1 \cdot \nu \mathrm{j}_{\mathrm{ch}} y_2, \quad y_1, y_2 \in \mathcal{Y}. \tag{12.9}$$

12.1.8 Bogoliubov rotations

Consider a CAR representation (12.1). To simplify notation, we drop π, that is, we consider a CAR representation

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\mathcal{H}). \tag{12.10}$$

Let $r \in O(\mathcal{Y})$. Clearly,

$$\mathcal{Y} \ni y \mapsto \phi^{r}(y) := \phi(ry) \in U(\mathcal{H}) \tag{12.11}$$

is also a CAR representation.

Definition 12.18 *We say that the representation (12.11) is* the Bogoliubov rotation *or* transformation of the representation (12.10) by $r^{\#}$.

Proposition 12.19 (1) *If* $r_1, r_2 \in O(\mathcal{Y})$*, then* $(\phi^{r_1})^{r_2}(y) = \phi^{r_2 r_1}(y)$*.*
(2) *The set of* $r \in O(\mathcal{Y})$ *such that (12.11) is unitarily equivalent to (12.10) is a subgroup of* $O(\mathcal{Y})$*.*
(3) *(12.11) is irreducible iff (12.10) is.*

12.2 CAR representations in finite dimensions

Throughout the section we assume that $(\mathcal{Y}, \nu)$ is a finite-dimensional Euclidean space.

In this section we discuss CAR representations in the finite-dimensional case. In the literature the material of this section is usually described as a part of the theory of spinors and Clifford algebras.

12.2.1 Volume element

Suppose that $\mathcal{Y}$ is oriented and we are given a CAR representation (12.1). Let $(e_1, \dots, e_n)$ be an o.n. basis of $\mathcal{Y}$ compatible with the orientation. The following definition is a special case of Def. 12.13.

Definition 12.20 *The operator*

$$Q^{\pi} := \phi^{\pi}(e_1) \cdots \phi^{\pi}(e_n)$$

will be called the volume element in the representation (12.1)

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

In what follows we drop the superscript π. Let us summarize the properties of Q, which follow from Subsect. 12.1.5:

Theorem 12.21 (1) *Q depends only on the orientation of $\mathcal{Y}$ and changes sign under the change of the orientation.*

(2) *Q is unitary. It is self-adjoint for $n \equiv 0, 1 \pmod 4$, otherwise anti-self-adjoint. Moreover, $Q^2 = (-\mathbb{1})^{n(n-1)/2}$.*

(3) *$Q\phi(y) = (-1)^{n-1}\phi(y)Q$, $y \in \mathcal{Y}$.*

(4) *If $n = 2m$, then $Q^2 = (-\mathbb{1})^m$, Q anti-commutes with $\phi(y)$, $y \in \mathcal{Y}$, and*

$$\mathbb{R}^{2m+1} \ni (y,t) \mapsto \phi(y) \pm t\mathrm{i}^m Q$$

are two representations of the CAR.

(5) *If $n = 2m+1$, then $Q^2 = (-\mathbb{1})^m$, Q commutes with $\phi(y)$, $y \in \mathcal{Y}$, and*

$$\mathcal{H} = \mathrm{Ker}(Q - \mathrm{i}^m\,\mathbb{1}) \oplus \mathrm{Ker}(Q + \mathrm{i}^m\,\mathbb{1})$$

gives a decomposition of $\mathcal{H}$ into a direct sum of subspaces invariant for the CAR representation.

Definition 12.22 *Let* $\dim\mathcal{Y} = 2m+1$. *We will say that a CAR representation is* compatible with the orientation *if $Q = \mathrm{i}^m\,\mathbb{1}$.*

12.2.2 Pauli matrices

Consider the space $\mathbb{C}^2$. Occasionally we will need its canonical basis, whose elements will be denoted $|\uparrow)$, $|\downarrow)$.

Definition 12.23 Pauli matrices *are defined as*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\sigma_i^2 = 1$, $\sigma_i^* = \sigma_i$, $i = 1,2,3$, and

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \mathrm{i}\sigma_3,$$

$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = \mathrm{i}\sigma_1,$$

$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = \mathrm{i}\sigma_2.$$

Moreover, $B(\mathbb{C}^2)$ is generated by $\{\sigma_1, \sigma_2\}$. Clearly, $\{\sigma_1, \sigma_2, \sigma_3\}$ is a CAR representation over $\mathbb{R}^3$.

Lemma 12.24 *Let $\{\phi_1, \phi_2\}$ be a CAR representation over $\mathbb{R}^2$ in a Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$ and a unitary operator $U : \mathbb{C}^2 \otimes \mathcal{K} \to \mathcal{H}$ such that*

$$\sigma_1 \otimes \mathbb{1}_{\mathcal{K}}\, U = U\phi_1, \qquad \sigma_2 \otimes \mathbb{1}_{\mathcal{K}}\, U = U\phi_2.$$

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

Proof Set $I := \mathrm{i}\phi_1\phi_2$. Clearly, $I = I^*$ and $I^2 = \mathbb{1}$. Note that $I \neq \mathbb{1}$, since $I = \mathbb{1}$ would contradict the fact that ϕ_2 is self-adjoint. Hence, $\operatorname{spec} I = \{1, -1\}$. Let $\mathcal{K} := \operatorname{Ker}(I - \mathbb{1})$. We unitarily identify $\mathcal{H}$ with $\mathcal{K} \oplus \mathcal{K}$ by the map

$$\Psi \mapsto U\Psi := \Big(\frac{1}{2}(\phi_1 - \phi_1 I)\Psi, \frac{1}{2}(\mathbb{1} + I)\Psi\Big).$$

Then $U\phi_1 = \sigma_1 \otimes \mathbb{1}_{\mathcal{K}}\, U$, $U\phi_2 = \sigma_2 \otimes \mathbb{1}_{\mathcal{K}}\, U$. □

12.2.3 Jordan–Wigner representation

In this subsection we introduce certain basic CAR representations over a finite-dimensional space. We start with the case of an even dimension, which is simpler.

In the algebra $B(\otimes^m \mathbb{C}^2)$ we introduce the operators

$$\sigma_i^{(j)} := \mathbb{1}^{\otimes(j-1)} \otimes \sigma_i \otimes \mathbb{1}^{\otimes(m-j)}, \quad i = 1, 2, 3, \quad j = 1, \dots, m.$$

Note that $\sigma_3^{(j)} = \mathrm{i}\sigma_1^{(j)}\sigma_2^{(j)}$. Moreover, $B(\otimes^m \mathbb{C}^2)$ is generated by

$$\{\sigma_i^{(j)} \ : \ j = 1, \dots, m,\ i = 1, 2\}.$$

We also set $I_0 := \mathbb{1}$, $I_j := \sigma_3^{(1)} \cdots \sigma_3^{(j)}$ for $j = 1, \dots, m$. If we set

$$\phi_{2j-1}^{\mathrm{JW}} := I_{j-1}\sigma_1^{(j)}, \quad \phi_{2j}^{\mathrm{JW}} := I_{j-1}\sigma_2^{(j)}, \ j = 1, \dots, m, \tag{12.12}$$

it is easy to see that

$$(\phi_1^{\mathrm{JW}}, \dots, \phi_{2m}^{\mathrm{JW}}) \tag{12.13}$$

is an irreducible CAR representation over $\mathbb{R}^{2m}$ in the Hilbert space $\otimes^m \mathbb{C}^2$. Note that $Q = I_m$.

Definition 12.25 *The CAR representation (12.13) will be called the* Jordan–Wigner representation over $\mathbb{R}^{2m}$*.*

In the odd-dimensional case with $n = 2m + 1$, there exist two inequivalent irreducible CAR representations, both in $\otimes^m \mathbb{C}^2$. They are obtained by adding the operator $\pm I_m$ to (12.12). In other words,

$$(\phi_1^{\mathrm{JW}}, \dots, \phi_{2m}^{\mathrm{JW}}, I_m), \tag{12.14}$$

$$(\phi_1^{\mathrm{JW}}, \dots, \phi_{2m}^{\mathrm{JW}}, -I_m) \tag{12.15}$$

are irreducible CAR representations over $\mathbb{R}^{2m+1}$. We have $Q = \mathrm{i}^m \mathbb{1}$ in the case of (12.14) and $Q = -\mathrm{i}^m \mathbb{1}$ in the case of (12.15). Thus the representation (12.14) is compatible with the natural orientation of $\mathbb{R}^{2m+1}$.

Another useful, but reducible, CAR representation over $\mathbb{R}^{2m+1}$ acts on the space $\otimes^{m+1}\mathbb{C}^2$. It is given by

$$(\phi_1^{\mathrm{JW}}, \dots, \phi_{2m}^{\mathrm{JW}}, \phi_{2m+1}^{\mathrm{JW}}). \tag{12.16}$$

It decomposes into the sum of two irreducible sub-representations, one equivalent to (12.14) and the other equivalent to (12.15). We have $Q = \mathrm{i}^m \mathbb{1}^{\otimes m} \otimes \sigma_1$. The commutant of (12.16) is spanned by $\mathbb{1}^{\otimes(m+1)}$ and Q.

Definition 12.26 *The CAR representation (12.16) will be called the* Jordan–Wigner representation over $\mathbb{R}^{2m+1}$.

12.2.4 Unitary equivalence of the CAR in finite dimensions

The following two theorems can be viewed as the fermionic analog of the Stone–von Neumann theorem. Again, we first deal with the even-dimensional case.

Theorem 12.27 *Let $(\phi_1, \phi_2, \dots, \phi_{2m})$ be a CAR representation over $\mathbb{R}^{2m}$ in a Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$ and a unitary operator $U : \otimes^m \mathbb{C}^2 \otimes \mathcal{K} \to \mathcal{H}$ such that*

$$U\phi_j^{\mathrm{JW}} \otimes \mathbb{1}_{\mathcal{K}} = \phi_j U, \quad j = 1, \dots, 2m.$$

The representation is irreducible iff $\mathcal{K} = \mathbb{C}$.

In particular, every irreducible CAR representation over an even-dimensional space is unitarily equivalent to the corresponding Jordan–Wigner representation.

Proof Set

$$\tilde{I}_0 := \mathbb{1}, \qquad \tilde{I}_j := \mathrm{i}^j \phi_1 \cdots \phi_j, \quad j = 1, \dots, n,$$
$$\tilde{\sigma}_1^{(j)} := \tilde{I}_{j-1}\phi_{2j-1}, \quad \tilde{\sigma}_2^{(j)} := \tilde{I}_{j-1}\phi_{2j}, \quad j = 1, \dots, m.$$

From the CAR we get

$$\begin{aligned} &\tilde{I}_j^* = \tilde{I}_j, \quad \tilde{I}_j^2 = \mathbb{1}, \\ &\phi_k \tilde{I}_j = -\tilde{I}_j \phi_k, \ \ k \leq 2j, \ \ \phi_k \tilde{I}_j = \tilde{I}_j \phi_k, \ \ k > 2j. \end{aligned} \tag{12.17}$$

This implies that

$$\begin{aligned} &\tilde{\sigma}_1^{(j)} \tilde{\sigma}_2^{(j)} = \phi_{2j-1}\phi_{2j}, \quad \tilde{I}_j = \mathrm{i}^j \tilde{\sigma}_1^{(1)} \tilde{\sigma}_2^{(1)} \cdots \tilde{\sigma}_1^{(j)} \tilde{\sigma}_2^{(j)}, \\ &\phi_{2j-1} = \tilde{I}_{j-1}\tilde{\sigma}_1^{j}, \quad \phi_{2j} = \tilde{I}_{j-1}\tilde{\sigma}_2^{(j)}. \end{aligned} \tag{12.18}$$

We observe that the pairs $\{\tilde{\sigma}_1^{(j)}, \tilde{\sigma}_2^{(j)}\}$ satisfy the CAR over $\mathbb{R}^2$ and commute with each other. Applying Lemma 12.24 inductively we see that there exists a Hilbert space $\mathcal{K}$ and a unitary map $U : \otimes^n \mathbb{C}^2 \otimes \mathcal{K} \to \mathcal{H}$ such that

$$U\sigma_1^{(j)} \otimes \mathbb{1}_{\mathcal{K}} = \tilde{\sigma}_1^{(j)} U, \quad U\sigma_2^{(j)} \otimes \mathbb{1}_{\mathcal{K}} = \tilde{\sigma}_2^{(j)} U, \quad j = 1, \cdots, m.$$

From (12.18) we get that $UI_j \otimes \mathbb{1}_{\mathcal{K}} = \tilde{I}_j U$, and hence

$$UI_{j-1}\sigma_1^{(j)} \otimes \mathbb{1}_{\mathcal{K}} = \phi_{2j-1}U, \quad UI_{j-1}\sigma_2^{(j)} \otimes \mathbb{1}_{\mathcal{K}} = \phi_{2j}U, \quad j = 1, \cdots, m.$$

This completes the proof of the theorem. □

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

Theorem 12.28 *Let $(\phi_1, \phi_2, \dots, \phi_{2m+1})$ be a representation of CAR over $\mathbb{R}^{2m+1}$ in a Hilbert space $\mathcal{H}$. Then there exist Hilbert spaces $\mathcal{K}_-$ and $\mathcal{K}_+$ and a unitary operator $U : \otimes^m \mathbb{C}^2 \otimes (\mathcal{K}_+ \oplus \mathcal{K}_-) \to \mathcal{H}$ such that*

$$\begin{aligned} U\phi_j^{\mathrm{JW}} \otimes 1\!\!1_{\mathcal{K}_+ \oplus \mathcal{K}_-} &= \phi_j U, \quad j = 1, \dots, 2m; \\ U I_m \otimes (1\!\!1_{\mathcal{K}_+} \oplus -1\!\!1_{\mathcal{K}_-}) &= \phi_{2m+1} U. \end{aligned}$$

The representation is irreducible iff $\mathcal{K}_+ \oplus \mathcal{K}_- = \mathbb{C}$.

In particular, every irreducible CAR representation over an odd-dimensional oriented space compatible with its orientation is unitarily equivalent to (12.14).

Proof Let $\tilde{I}_j$, $\tilde{\sigma}_1^{(j)}$, $\tilde{\sigma}_2^{(j)}$ be as in the proof of Thm. 12.27. Let $U_1 : \otimes^m \mathbb{C}^2 \otimes \mathcal{K} \to \mathcal{H}$ be a unitary operator as in Thm. 12.27 for the CAR representation $(\phi_1, \dots, \phi_{2m})$ over $\mathbb{R}^{2m}$. From (12.17) and the CAR we get

$$[\phi_j, \tilde{I}_m \phi_{2m+1}] = 0, \quad j = 1, \dots, 2m. \tag{12.19}$$

Since $B(\otimes^m \mathbb{C}^2)$ is generated by $\{\sigma_i^{(j)}, j = 1, \dots, m,\ i = 1, 2\}$, we see that

$$U_1^* \tilde{I}_m \phi_{2m+1} U_1 = 1\!\!1 \otimes A,\ A \in B(\mathcal{K}).$$

Again using the CAR, we get $(\tilde{I}_m \phi_{2m+1})^2 = 1\!\!1$. Hence, $A^2 = 1\!\!1$.

If $A = \pm 1\!\!1_{\mathcal{K}}$, we get $\tilde{I}_m \phi_{2m+1} = \pm 1\!\!1_{\mathcal{H}}$. Hence, $\phi_{2m+1} = \pm \tilde{I}_m$. In this case the CAR representation is one of the two constructed in Subsect. 12.2.3. In the general case we have $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, $A = 1\!\!1_{\mathcal{K}_+} \oplus -1\!\!1_{\mathcal{K}_-}$, and hence

$$U I_m \otimes (1\!\!1_{\mathcal{K}_+} \oplus -1\!\!1_{\mathcal{K}_-}) = \phi_{2m+1} U.$$

The other identities follow from Thm. 12.27. □

Corollary 12.29 (1) *Suppose that $\mathcal{Y}$ is an even-dimensional Euclidean space. Let $\mathcal{Y} \ni y \mapsto \phi^1(y) \in B_{\mathrm{h}}(\mathcal{H})$ and $\mathcal{Y} \ni y \mapsto \phi^2(y) \in B_{\mathrm{h}}(\mathcal{H})$ be two irreducible representations of the CAR. Then they are unitarily equivalent.*

(2) *The same is true if $\mathcal{Y}$ is odd-dimensional and oriented, and both representations are compatible with its orientation.*

12.3 CAR algebras: finite dimensions

As in the previous section, we assume that $(\mathcal{Y}, \nu)$ is a finite-dimensional Euclidean space.

In this section we discuss $*$-algebras generated by the CAR in finite dimension. As pure $*$-algebras they are not very interesting – they are full matrix algebras over a 2^m-dimensional space in the case of even dimension, and the direct sum of two such algebras in the case of odd dimension. They become interesting when we consider them together with the linear subspace of distinguished elements $\phi(y),\ y \in \mathcal{Y}$.

12.3.1 CAR algebra

Let $(\mathcal{Y}, \nu)$ be a finite-dimensional Euclidean space.

Definition 12.30 $\mathrm{CAR}(\mathcal{Y})$ *is the complex unital $*$-algebra generated by elements $\phi(y)$, $y \in \mathcal{Y}$, with relations*

$$\phi(\lambda y) = \lambda\phi(y),\ \lambda \in \mathbb{R}, \qquad \phi(y_1 + y_2) = \phi(y_1) + \phi(y_2),$$
$$\phi^*(y) = \phi(y), \qquad \phi(y_1)\phi(y_2) + \phi(y_2)\phi(y_1) = 2y_1\cdot\nu y_2 \mathbb{1}.$$

The following theorem is a simple algebraic fact:

Proposition 12.31 *If*

$$\mathcal{Y} \ni y \mapsto \phi^\pi(y) \in B(\mathcal{H})$$

is a CAR representation, then there exists a unique $*$-homomorphism

$$\pi : \mathrm{CAR}(\mathcal{Y}) \to B(\mathcal{H})$$

such that $\pi(\mathbb{1}) = \mathbb{1}_\mathcal{H}$ and $\pi(\phi(y)) = \phi^\pi(y)$, $y \in \mathcal{Y}$.

Definition 12.32 *Applying Prop. 12.31 to the Jordan–Wigner representations (12.13) or (12.16) we obtain $*$-homomorphisms*

$$\pi^{\mathrm{JW}} : \mathrm{CAR}(\mathbb{R}^{2m}) \to B(\otimes^m \mathbb{C}^2),$$
$$\pi^{\mathrm{JW}} : \mathrm{CAR}(\mathbb{R}^{2m+1}) \to B(\otimes^{m+1} \mathbb{C}^2).$$

Proposition 12.33 (1) *The $*$-homomorphisms π^{JW} for $n = 2m$ are bijective and $\mathrm{CAR}(\mathbb{R}^{2m})$ is $*$-isomorphic to $B(\otimes^m \mathbb{C}^2)$.*

(2) *The $*$-homomorphisms π^{JW} for $n = 2m + 1$ are injective and $\mathrm{CAR}(\mathbb{R}^{2m+1})$ is $*$-isomorphic to $B(\otimes^m \mathbb{C}^2) \oplus B(\otimes^m \mathbb{C}^2)$.*

Proof Choose an o.n. basis $(e_1, \dots, e_n)$ in $\mathcal{Y}$. For an ordered subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, set $\phi_{\mathrm{i}_1,\dots,i_k} := \mathrm{i}^{k(k-1)/2}\phi_{i_1} \cdots \phi_{i_k}$. It is easy to prove that the elements $\phi_{\mathrm{i}_1,\dots,i_k}$ are self-adjoint and are a basis of $\mathrm{CAR}(\mathbb{R}^n)$. Their commutation relations are determined by the CAR.

Following the construction of the Jordan–Wigner representations we see that $B(\otimes^m \mathbb{C}^2)$, if $n = 2m$, and $B(\otimes^m \mathbb{C}^2) \oplus B(\otimes^m \mathbb{C}^2)$, if $n = 2m + 1$, have self-adjoint bases satisfying the same relations. $\square$

The Jordan–Wigner representation determines a unique C^*-norm on $\mathrm{CAR}(\mathcal{Y})$. Henceforth we will treat $\mathrm{CAR}(\mathcal{Y})$ as a C^*-algebra.

If $\mathcal{Y}_1 \subset \mathcal{Y}_2$ are two finite-dimensional spaces, then $\mathrm{CAR}(\mathcal{Y}_1)$ is isometrically embedded in $\mathrm{CAR}(\mathcal{Y}_2)$.

12.3.2 Parity

CAR algebras have a natural $\mathbb{Z}_2$-grading. Therefore, they are examples of super-algebras. Consistently with the terminology of super-algebras, we introduce the following definition:

Definition 12.34 *The map* $\alpha(\phi(y)) := -\phi(y)$ *extends to a unique* $*$*-isomorphism* α *of* $\mathrm{CAR}(\mathcal{Y})$*. For* $j = 0, 1$*, we set*

$$\mathrm{CAR}_j(\mathcal{Y}) := \big\{B \in \mathrm{CAR}(\mathcal{Y}) \ : \ \alpha(B) = (-1)^j B\big\}.$$

Elements of $\mathrm{CAR}_0(\mathcal{Y})$*, resp.* $\mathrm{CAR}_1(\mathcal{Y})$ *are called* even*, resp.* odd*.*

Suppose that $\mathcal{Y}$ is oriented. The following definition is closely related to Def. 12.13.

Definition 12.35 *The* volume element of the algebra $\mathrm{CAR}(\mathcal{Y})$ *is defined by*

$$Q := \phi(e_1)\cdots\phi(e_n), \tag{12.20}$$

where $(e_1, \dots, e_n)$ *is any o.n. basis of* $\mathcal{Y}$ *compatible with its orientation.*

Note that Q is never proportional to $1\!\!1$ as an element of $\mathrm{CAR}(\mathcal{Y})$.

Proposition 12.36 (1) *Let* $A \in \mathrm{CAR}_0(\mathcal{Y})$ *commute with* $\phi(y)$*,* $y \in \mathcal{Y}$ *(and hence with all* $\mathrm{CAR}(\mathcal{Y})$*). Then* A *is proportional to* $1\!\!1$*.*
(2) *Let a non-zero* $A \in \mathrm{CAR}_1(\mathcal{Y})$ *commute with* $\phi(y)$*,* $y \in \mathcal{Y}$ *(and hence with all* $\mathrm{CAR}(\mathcal{Y})$*). Then* $\dim\mathcal{Y}$ *is odd, and* A *is proportional to* Q*.*
(3) *Let a non-zero* $A \in \mathrm{CAR}(\mathcal{Y})$ *anti-commute with* $\phi(y)$*,* $y \in \mathcal{Y}$ *(and hence with all* $\mathrm{CAR}_1(\mathcal{Y})$*). Then* $\dim\mathcal{Y}$ *is even, and* A *is proportional to* Q*.*

12.3.3 Complex conjugation and transposition

Definition 12.37 *The map* $\mathrm{c}(\phi(w)) := \phi(\overline{w})$*,* $w \in \mathbb{C}\mathcal{Y}$*, extends to a unique anti-linear* $*$*-isomorphism* c *of* $\mathrm{CAR}(\mathcal{Y})$*. We introduce the* Clifford algebra over $(\mathcal{Y}, \nu)$ *as the real sub-algebra*

$$\mathrm{Cliff}(\mathcal{Y}) := \big\{B \in \mathrm{CAR}(\mathcal{Y}) \ : \ \mathrm{c}(B) = B\big\}. \tag{12.21}$$

We also introduce the transposition $A^{\#} := \mathrm{c}(A^*)$*, which is a linear anti-automorphism.*

$\mathrm{Cliff}(\mathcal{Y})$ is a real $*$-algebra with a basis

$$\phi_{i_1}\cdots\phi_{i_k}, \quad \{i_1, \dots, i_k\} \subset \{1, \dots, n\}. \tag{12.22}$$

In Chap. 15 we will introduce a more general notion of Clifford algebras, defined for an arbitrary symmetric form on a vector space. The algebra $\mathrm{Cliff}(\mathcal{Y})$ defined in Def. 12.37 corresponds to the special case of a real space equipped with a Euclidean scalar product.

12.3.4 Bogoliubov automorphisms

Proposition 12.38 *If $r \in O(\mathcal{Y})$, then the map $\hat{r}\,(\phi(y)) := \phi(ry)$ extends to a unique $*$-automorphism $\hat{r}$ of* $\mathrm{CAR}(\mathcal{Y})$. *We have $\widehat{r_1 r_2} = \hat{r}_1 \hat{r}_2$.*

Definition 12.39 *$\hat{r}$ is called the* Bogoliubov automorphism associated with r*.*

12.4 Anti-symmetric quantization and real-wave CAR representation

In this section we introduce a natural parametrization of operators in a CAR algebra by anti-symmetric polynomials. This parametrization, which we call the *anti-symmetric quantization*, can be viewed as the fermionic analog of the Weyl–Wigner quantization.

We also define a representation given by the GNS construction from the tracial state. This representation has some analogy to the real-wave CCR representation considered in Sect. 9.3; therefore we will call it the *real-wave CAR representation*. In this section we describe the real-wave CAR representation only in the case of a finite number of degrees of freedom. We will extend it to the case of an infinite dimension in Subsect. 12.5.3, and then we will continue its study using the formalism of Fock spaces in Subsect. 13.2.1.

In this section, $(\mathcal{Y}, \nu)$ is a finite-dimensional Euclidean space.

12.4.1 Anti-symmetric quantization

Definition 12.40 *Let $y_1, \dots, y_n \in \mathbb{C}\mathcal{Y}$. We can treat these as elements of $\mathbb{C}\mathrm{Pol}^1_{\mathrm{a}}(\mathcal{Y}^{\#})$ and take their product $y_1 \cdots y_n \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$. We define*

$$\mathrm{Op}(y_1 \cdots y_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, \phi(y_{\sigma(1)}) \cdots \phi(y_{\sigma(n)}) \in \mathrm{CAR}(\mathcal{Y}). \tag{12.23}$$

The map extends uniquely to a linear bijective map

$$\mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#}) \ni b \mapsto \mathrm{Op}(b) \in \mathrm{CAR}(\mathcal{Y}), \tag{12.24}$$

called the anti-symmetric quantization*.*

The above definition should be compared with Def. 8.65, where the Weyl–Wigner quantization was introduced.

Definition 12.41 *The inverse of (12.24) will be called the* anti-symmetric symbol*. The anti-symmetric symbol of an operator $B \in \mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ will be denoted $\mathrm{s}_B \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$.*

As usual, N denotes the number operator, which in the context of $\mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$ is perhaps better called the *degree operator*. Recall that in Chap. 3 we introduced

$$\Lambda := (-1)^{N(N-1)/2}, \quad I := (-1)^{N};$$

see (3.9) and (3.29). We will use the functional notation for elements of $\mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$; see Subsect. 7.1.1. The generic variable in $\mathcal{Y}^{\#}$ will be denoted v. We equip $\mathcal{Y}$ with a volume form compatible with the scalar product ν. We will use the corresponding Berezin integral on $\mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$, defined in Subsect. 7.1.4.

Proposition 12.42 (1) $\mathrm{Op}(b)^* = \mathrm{Op}(\Lambda\overline{b})$.

(2) *Let $\mathcal{Z}$ be an isotropic subspace of $\mathbb{C}\mathcal{Y}$ for $\nu_{\mathbb{C}}$. Let $f_1, \dots, f_n \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Z}^{\#}) \subset \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$. Then*

$$\mathrm{Op}(f_1)\cdots\mathrm{Op}(f_n) = \mathrm{Op}(f_1\cdots f_n).$$

(3) *If $b, b_1, b_2 \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$ and $\mathrm{Op}(b) = \mathrm{Op}(b_1)\mathrm{Op}(b_2)$, then*

$$b(v) = \exp\left(\nabla_{v_2}\cdot\nu\nabla_{v_1}\right) b_1(v_1)b_2(v_2)\big|_{v_1=v_2=v} \tag{12.25}$$

$$= \int\limits_{\mathcal{Y}^{\#}}\int\limits_{\mathcal{Y}^{\#}} \mathrm{e}^{(v-v_1)\cdot\nu^{-1}(v-v_2)} b_1(v_1)b_2(v_2)\mathrm{d}v_2\mathrm{d}v_1. \tag{12.26}$$

(4) *If $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$, $y \in \mathbb{C}\mathcal{Y}$, then*

$$\frac{1}{2}\left(\phi(y)\mathrm{Op}(b) + \mathrm{Op}(Ib)\phi(y)\right) = \mathrm{Op}(y\cdot b), \tag{12.27}$$

$$\frac{1}{2}\left(\phi(y)\mathrm{Op}(b) - \mathrm{Op}(Ib)\phi(y)\right) = \mathrm{Op}((\nu y)\cdot\nabla_v b). \tag{12.28}$$

Proof Statements (1) and (2) are immediate. Let us prove (12.25). Let us fix an o.n. basis $(e_1, \dots, e_d)$ of $\mathcal{Y}$ such that $\Xi = e^n \wedge \cdots \wedge e^1$. We use the Berezin calculus introduced in Subsect. 7.1.5. We rename the variable v_1 as v and the variable v_2 as w. We will write $v_i = e_i \cdot v$, $w_i = e_i \cdot w$. Without loss of generality, we can assume that $b_1(v) = \prod\limits_{i\in I} v_i$, $b_2(w) = \prod\limits_{i\in J} w_i$ for $I, J \subset \{1, \dots, d\}$. We have

$$\exp\left(\sum_{i=1}^{d} \nabla_{w_i}\cdot\nabla_{v_i}\right) = \sum_{K\subset\{i,\dots,d\}}\prod_{i\in K}\nabla_{w_i}\cdot\nabla_{v_i}. \tag{12.29}$$

The only term in (12.29) giving a non-zero contribution to

$$\exp\left(\sum_{i=1}^{d} \nabla_{w_i}\cdot\nabla_{v_i}\right) b_1(v)\cdot b_2(w)\big|_{w=v}$$

is $K = I \cap J$. Without loss of generality we can further assume that $1 \le p \le n \le m$ and

$$b_1(v) = v_1\cdots v_p\cdot v_{p+1}\cdots v_n, \quad b_2(w) = w_n\cdots w_{p+1}\cdot w_{n+1}\cdots w_m.$$

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

Then

$$\prod_{i=p+1}^{n} \nabla_{w_i} \cdot \nabla_{v_i} b_1(v) \cdot b_2(w)\big|_{w=v} = v_1 \cdots v_p \cdot v_{n+1} \cdots v_m =: b(v).$$

We have

$$\begin{aligned}\mathrm{Op}(b_1) &= \phi(e_1) \cdots \phi(e_p) \cdot \phi(e_{p+1}) \cdots \phi(e_n),\\ \mathrm{Op}(b_2) &= \phi(e_n) \cdots \phi(e_{p+1}) \cdot \phi(e_{n+1}) \cdots \phi(e_m),\end{aligned}$$

and

$$\mathrm{Op}(b_1)\mathrm{Op}(b_2) = \phi(e_1) \cdots \phi(e_p) \cdot \phi(e_{n+1}) \cdots \phi(e_m) = \mathrm{Op}(b),$$

using the CAR. This proves (12.25).

To obtain (12.26), we apply Prop. 7.19 to the even-dimensional space $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}$. Let $x = (y_1, y_2)$, $\xi = (v_1, v_2)$ be the generic variables in $\mathcal{X}$ and $\mathcal{X}^{\#}$. Let $\zeta = \begin{bmatrix} 0 & \nu^{-1} \\ -\nu^{-1} & 0 \end{bmatrix} \in L_{\mathrm{a}}(\mathcal{X}^{\#}, \mathcal{X})$, so that $\frac{1}{2} x \cdot \zeta^{-1} x = y_2 \cdot \nu y_1$ and $\frac{1}{2}\xi \cdot \zeta\xi = v_1 \cdot \nu^{-1} v_2$. The Pfaffian of ζ w.r.t. $\mathrm{d}v_2 \wedge \mathrm{d}v_1$ is equal to 1, which by Prop. 7.19 proves the second identity of (3).

To prove (4), we can assume without loss of generality that $b(v) = v_{i_1} \cdots v_{i_p}$ and $\langle y|v\rangle = v_j$. Then $\mathrm{Op}(b) = \phi(e_{i_1}) \cdots \phi(e_{i_p})$, $\phi(y) = \phi(e_j)$. Using the CAR we get

$$\begin{aligned}&\frac{1}{2}\left(\phi(e_j)\phi(e_{i_1}) \cdots \phi(e_{i_p}) + (-1)^p \phi(e_{i_1}) \cdots \phi(e_{i_p})\phi(e_j)\right)\\ &= \begin{cases} 0 & \text{if } j \in \{i_1, \ldots, i_p\}, \\ \phi(e_j)\phi(e_{i_1}) \cdots \phi(e_{i_p}) & \text{if } j \notin \{i_1, \ldots, i_p\}, \end{cases}\end{aligned}$$

which proves the first statement of (4). The second can be proved similarly. □

Theorem 12.43 *If $b, b_1, \ldots, b_n \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$ and*

$$\mathrm{Op}(b) = \mathrm{Op}(b_1) \cdots \mathrm{Op}(b_n),$$

then the following version of the Wick theorem for the anti-symmetric quantization is true:

$$b(v) = \exp\Big(\sum_{i>j} \nabla_{v_i} \cdot \nu \nabla_{v_j}\Big) b_1(v_1) \cdots b_n(v_n)\big|_{v=v_1=\cdots=v_n}.$$

12.4.2 Real-wave CAR representation

Definition 12.44 *For $A \in \mathrm{CAR}(\mathcal{Y})$, we define*

$$\begin{aligned}\mathrm{tr} A &= 2^{-m} \mathrm{Tr}\, \pi^{\mathrm{JW}}(A), \quad \dim \mathcal{Y} = 2m,\\ \mathrm{tr} A &= 2^{-m-1} \mathrm{Tr}\, \pi^{\mathrm{JW}}(A), \quad \dim \mathcal{Y} = 2m+1,\end{aligned}$$

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

where π^{JW} *is the Jordan–Wigner representation.* tr *is called the* canonical tracial state *on* $\mathrm{CAR}(\mathcal{Y})$*.*

Theorem 12.45 (1) tr *is a tracial state on* $\mathrm{CCR}(\mathcal{Y})$*, which means*

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad A, B \in \mathrm{CAR}(\mathcal{Y}).$$

(2) *It satisfies*

$$\mathrm{tr}(A) = \mathrm{tr}(\mathrm{c}(A)) = \mathrm{tr}(\alpha(A)) = \mathrm{tr}(\hat{r}(A)), \qquad A \in \mathrm{CAR}(\mathcal{Y}),\ r \in O(\mathcal{Y}).$$

(3) *If* $b, c \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#}) \simeq \Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})$*, then*

$$\mathrm{tr}(\mathrm{Op}(b)^*\mathrm{Op}(c)) = (b|c). \tag{12.30}$$

(4) *For* $y, y_1, y_2 \in \mathcal{X}$*, we have the expectation values*

$$\begin{aligned} \mathrm{tr}\,(\phi(y)) &= 0, \\ \mathrm{tr}\,(\phi(y_1)\phi(y_2)) &= y_1 \cdot \nu y_2. \end{aligned}$$

More generally,

$$\begin{aligned} \mathrm{tr}\,(\phi(y_1)\cdots\phi(y_{2m-1})) &= 0, \\ \mathrm{tr}\,(\phi(y_1)\cdots\phi(y_{2m})) &= \sum_{\sigma \in \mathrm{Pair}_{2m}} \mathrm{sgn}(\sigma) \prod_{j=1}^{m} y_{\sigma(2j-1)} \cdot \nu y_{\sigma(2j)}. \end{aligned}$$

Definition 12.46 *Let* $(\pi_{\mathrm{tr}}, \mathcal{H}_{\mathrm{tr}}, \Omega_{\mathrm{tr}})$ *denote the GNS representation of* $\mathrm{CAR}(\mathcal{Y})$ *w.r.t. the state* tr*. The CAR representation*

$$\mathcal{Y} \ni y \mapsto \phi_{\mathrm{tr}}(y) := \pi_{\mathrm{tr}}(\phi(y)) \in B_{\mathrm{h}}(\mathcal{H}_{\mathrm{tr}})$$

will be called the real-wave *or* tracial CAR representation.

12.4.3 Real-wave CAR representation in coordinates

Let n be an integer. We are going to describe the real-wave representation over $\mathbb{R}^n$ more explicitly.

Clearly, $\otimes^n\mathbb{C}^2$ has a natural conjugation, denoted as usual $\otimes^n\mathbb{C}^2 \ni \Psi \mapsto \overline{\Psi} \in \otimes^n\mathbb{C}^2$. For typographical reasons, it will sometimes be denoted by χ. The corresponding real subspace of $\otimes^n\mathbb{C}^2$ obviously equals $\otimes^n\mathbb{R}^2$. Linear operators preserving $\otimes^n\mathbb{R}^2$ are called real.

The conjugation of $A \in B(\otimes^n\mathbb{C}^2)$ is denoted by $\overline{A}$ or $\chi A \chi$.

Define the "vacuum vector" $\Omega := |\downarrow) \otimes \cdots \otimes |\downarrow)$. Clearly, $\Omega = \overline{\Omega}$.

Introduce the following operators on $\otimes^n\mathbb{C}^2$:

$$N := \sum_{j=1}^{n} \mathbb{1}^{\otimes j} \otimes \frac{\sigma_3 + \mathbb{1}}{2} \otimes \mathbb{1}^{\otimes(n-j)}, \qquad \Lambda := (-1)^{N(N-\mathbb{1})/2}.$$

The role of these operators will become clear in Chap. 13, where we will identify $\otimes^n \mathbb{C}^2$ with the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathbb{C}^n)$ and they will coincide with the operators defined in (3.9) and (3.29). Therefore, in particular, N is called the number operator. For further reference let us note the following identities involving Λ:

$$\begin{aligned}
\Lambda \mathbb{1}^{\otimes(j-1)} \otimes \sigma_1 \otimes \mathbb{1}^{\otimes(n-j)} \Lambda &= (-\sigma_3)^{\otimes(j-1)} \otimes \sigma_1 \otimes (-\sigma_3)^{\otimes(n-j)}, \\
\Lambda \mathbb{1}^{\otimes(j-1)} \otimes \sigma_2 \otimes \mathbb{1}^{\otimes(n-j)} \Lambda &= -(-\sigma_3)^{\otimes(j-1)} \otimes \sigma_2 \otimes (-\sigma_3)^{\otimes(n-j)}, \\
\Lambda \mathbb{1}^{\otimes(j-1)} \otimes \sigma_3 \otimes \mathbb{1}^{\otimes(n-j)} \Lambda &= \mathbb{1}^{\otimes(j-1)} \otimes \sigma_3 \otimes \mathbb{1}^{\otimes(n-j)}.
\end{aligned}$$

In order to describe the real-wave CAR representation over $\mathbb{R}^n$, introduce the following operators

$$\begin{aligned}
\phi_j^{\mathrm{l}} &= \sigma_3^{\otimes(j-1)} \otimes \sigma_1 \otimes \mathbb{1}^{\otimes(n-j)}, \\
\phi_j^{\mathrm{r}} &= \mathbb{1}^{\otimes(n-j)} \otimes \sigma_1 \otimes \sigma_3^{\otimes(j-1)} \ = \ \Lambda \phi_j^{\mathrm{l}} \Lambda.
\end{aligned}$$

Theorem 12.47 (1) *We have two mutually commuting CAR representations:*

$$\phi_1^{\mathrm{l}}, \dots, \phi_n^{\mathrm{l}}, \tag{12.31}$$

$$\phi_1^{\mathrm{r}}, \dots, \phi_n^{\mathrm{r}}. \tag{12.32}$$

That means

$$[\phi_i^{\mathrm{l}}, \phi_j^{\mathrm{l}}]_+ = 2\delta_{i,j}, \quad [\phi_i^{\mathrm{r}}, \phi_j^{\mathrm{r}}]_+ = 2\delta_{i,j}, \quad [\phi_i^{\mathrm{l}}, \phi_j^{\mathrm{r}}] = 0, \quad i,j = 1, \dots, n.$$

(2) *Let* $\pi^{\mathrm{l}} : \mathrm{CAR}(\mathbb{R}^n) \to B(\otimes^n \mathbb{C}^2)$ *be the* $*$*-homomorphism obtained by Prop. 12.31 from the CAR representation (12.31). Then*

$$\mathrm{tr}(A) = (\Omega | \pi^{\mathrm{l}}(A) \Omega), \quad A \in \mathrm{CAR}(\mathbb{R}^n),$$

and $\pi^{\mathrm{l}}(\mathrm{CAR}(\mathbb{R}^n))\,\Omega = \otimes^n \mathbb{C}^2$. *Thus* Ω *is a cyclic vector representative for the state* tr, *and hence* π^{l} *is the GNS representation of* $\mathrm{CAR}(\mathbb{R}^n)$ *for the state* tr.

(3) *Let* J *be the modular conjugation for the state* tr. *Then* $J = \Lambda\chi$ *(where* χ *denotes the complex conjugation). We have*

$$J \phi_j^{\mathrm{l}} J = \phi_j^{\mathrm{r}}, \quad j = 1, \dots, n.$$

(4) *We have*

$$\overline{\phi_i^{\mathrm{l}}} = \phi_i^{\mathrm{l}}, \quad i = 1, \dots, n.$$

Therefore,

$$\pi_{\mathrm{tr}}(\mathrm{c}(A)) = \overline{\pi_{\mathrm{tr}}(A)}, \quad A \in \mathrm{CAR}(\mathbb{R}^n).$$

Consequently, $\pi_{\mathrm{tr}}(\mathrm{Cliff}(\mathbb{R}^n))$ *consists of real elements of* $\pi_{\mathrm{tr}}(\mathrm{CAR}(\mathbb{R}^n))$.

(5) *Let Q be the operator defined in (12.20). We have*

$$\begin{aligned}\pi^{\mathrm{l}}(Q) = J\pi^{\mathrm{l}}(Q)J = \phi_1^{\mathrm{l}}\cdots\phi_n^{\mathrm{l}} = \phi_n^{\mathrm{r}}\cdots\phi_1^{\mathrm{r}} \\ = \begin{cases}(-1)^m \sigma_2\otimes\sigma_1\otimes\cdots\otimes\sigma_2\otimes\sigma_1, & n = 2m; \\ (-1)^m \sigma_1\otimes\sigma_2\otimes\cdots\otimes\sigma_2\otimes\sigma_1, & n = 2m+1.\end{cases}\end{aligned}$$

By Thm. 12.47 (2), the representation ϕ^{l} can be identified with the real-wave CAR representation defined in Def. 12.46.

The analysis of the real-wave CAR representation, in the case of an arbitrary dimension, will be continued in Subsect. 13.2.1, where we will use the formalism of Fock spaces.

12.5 CAR algebras: infinite dimensions

Throughout this section $(\mathcal{Y}, \nu)$ is a Euclidean space, possibly infinite-dimensional.

One aspect of the theory of CAR algebras simplifies in infinite dimensions: it is not necessary to distinguish between the even and odd cases. On the other hand, the topological aspects become more subtle. In particular, it is natural to define (at least) three different kinds of CAR algebras: the algebraic, the C^*- and the W^*-CAR algebra. (The situation is, however, simpler than in the case of CCR algebras.)

12.5.1 Algebraic CAR algebra

The definition of the *algebraic* CAR algebra is the same as that of the CAR algebra in finite dimension:

Definition 12.48 *The* algebraic CAR algebra over $\mathcal{Y}$, *denoted* $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$, *is the complex unital $*$-algebra generated by elements $\phi(y)$, $y \in \mathcal{Y}$, with relations*

$$\begin{aligned}&\phi(\lambda y) = \lambda\phi(y),\ \lambda \in \mathbb{R}, \qquad \phi(y_1 + y_2) = \phi(y_1) + \phi(y_2), \\ &\phi^*(y) = \phi(y), \qquad \phi(y_1)\phi(y_2) + \phi(y_2)\phi(y_1) = 2y_1\cdot\nu y_2\,\mathbb{1}.\end{aligned}$$

Clearly, Prop. 12.31 extends to infinite dimension, with $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ replacing $\mathrm{CAR}(\mathcal{Y})$. The parity α, the complex conjugation c and the transposition $^{\#}$ naturally extend to $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$. If $r \in O(\mathcal{Y})$, we can introduce a unique $*$-automorphism $\hat{r}$ of $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$, called the *Bogoliubov automorphism*, satisfying $\hat{r}(\phi(y)) = \phi(ry)$ as in Def. 12.39.

Definition 12.49 *Let $j = 0, 1$. We introduce*

$$\begin{aligned}\mathrm{CAR}_j^{\mathrm{alg}}(\mathcal{Y}) &:= \big\{B \in \mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y}) \ :\ \alpha(B) = (-1)^j B\big\}, \\ \mathrm{Cliff}^{\mathrm{alg}}(\mathcal{Y}) &:= \big\{B \in \mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y}) \ :\ \mathrm{c}(B) = B\big\}.\end{aligned}$$

Note that if $\mathcal{Y}$ is infinite-dimensional, $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ does not contain an operator analogous to the operator Q defined in (12.20). (The same remark applies to $\mathrm{CAR}^{C^*}(\mathcal{Y})$ and $\mathrm{CAR}^{W^*}(\mathcal{Y})$ defined later on.)

12.5.2 C^*-CAR algebra

Proposition 12.50 *There exists a unique C^*-norm on* $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$.

Proof We already know that this is true if $\mathcal{Y}$ is finite-dimensional.

If $\mathcal{Y}$ has an infinite dimension, then $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ is the union of $\mathrm{CAR}(\mathcal{Y}_1)$ for finite-dimensional subspaces of $\mathcal{Y}$. So $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ is equipped with a unique C^*-norm. □

Definition 12.51 *The* CAR C^*-algebra over $\mathcal{Y}$ *is defined as*

$$\mathrm{CAR}^{C^*}(\mathcal{Y}) := \left(\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})\right)^{\mathrm{cpl}},$$

where the completion is w.r.t. the C^*-norm defined above. $\mathrm{CAR}^{C^*}(\mathcal{Y})$ *is a C^*-algebra.*

The relationship between CAR representations and the algebra $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is given by the following theorem:

Proposition 12.52 *If*

$$\mathcal{Y} \ni y \mapsto \phi^{\pi}(y) \in B_{\mathrm{h}}(\mathcal{H})$$

is a CAR representation, then there exists a unique $*$-homomorphism of C^*-algebras

$$\pi : \mathrm{CAR}^{C^*}(\mathcal{Y}) \to B(\mathcal{H})$$

such that $\pi(\phi(y)) = \phi^{\pi}(y)$, $y \in \mathcal{Y}$.

Proof We already know that this is true if we replace $\mathrm{CAR}^{C^*}(\mathcal{Y})$ with $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$. π extends to $\mathrm{CAR}^{C^*}(\mathcal{Y})$ by continuity.

To see the uniqueness, note that every $*$-homomorphism between C^*-algebras is continuous. □

Clearly, $\mathrm{CAR}^{C^*}(\mathcal{Y})$ coincides with $\mathrm{CAR}^{C^*}(\mathcal{Y}^{\mathrm{cpl}})$. Hence, it is enough to restrict to complete $\mathcal{Y}$.

Proposition 12.53 *The parity α, the complex conjugation* c *and the transposition $^{\#}$ are isometric, and hence extend by continuity from* $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ *to* $\mathrm{CAR}^{C^*}(\mathcal{Y})$. *For $r \in O(\mathcal{Y})$, the same is true concerning the Bogoliubov automorphism $\hat{r}$.*

Proof Let $A \in \mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$. Then

$$\operatorname{spec} A = \operatorname{spec} \alpha(A) = \overline{\operatorname{spec} \mathrm{c}(A)} = \overline{\operatorname{spec} A^{\#}} = \operatorname{spec} \hat{r}(A).$$

Therefore, α, c, $^{\#}$ and $\hat{r}$ do not change the spectral radius of A. Hence, they are isometric. □

Definition 12.54 *We define* $\mathrm{CAR}_i^{C^*}(\mathcal{Y})$, $i = 0, 1$, *and* $\mathrm{Cliff}^{C^*}(\mathcal{Y})$ *as in Def. 12.49.*

Theorem 12.55 *If $\mathcal{Y}$ is an infinite-dimensional real Hilbert space, then* $\mathrm{CAR}^{C^*}(\mathcal{Y})$ *is simple. If in addition $\mathcal{Y}$ is separable, then* $\mathrm{CAR}^{C^*}(\mathcal{Y})$ *is isomorphic to* $\mathrm{UHF}(2^\infty)$ *defined in Subsect. 6.2.9.*

Proof Choose an o.n. basis $(e_{1+}, e_{1-}, e_{2+}, e_{2-}, \dots)$ of $\mathcal{Y}$. Let $\mathcal{Y}_n$ be the space spanned by the first n vectors of this basis. We have a commuting diagram,

$$\begin{array}{ccc} \mathrm{CAR}(\mathcal{Y}_{2m}) & \subset & \mathrm{CAR}(\mathcal{Y}_{2m+2}) \\ \downarrow & & \downarrow \\ B(\otimes^m \mathbb{C}^2) & \rightarrow & B(\otimes^{m+1}\mathbb{C}^2), \end{array}$$

where the vertical arrows are $*$-isomorphisms and the lower horizontal arrow is $A \mapsto A \otimes 1\!\!1_{\mathbb{C}^2}$. Clearly, $\bigcup_{m=1}^\infty \mathcal{Y}_{2m}$ is dense in $\mathcal{Y}$. Hence,

$$\begin{aligned} \mathrm{CAR}^{C^*}(\mathcal{Y}) &= \left(\bigcup_{m=1}^{\infty} \mathrm{CAR}(\mathcal{Y}_{2m}) \right)^{\mathrm{cpl}} \\ &\simeq \left(\bigcup_{m=1}^{\infty} B(\otimes^m \mathbb{C}^2) \right)^{\mathrm{cpl}} = \mathrm{UHF}(2^\infty). \end{aligned}$$

□

12.5.3 W^*-CAR algebra

In Thm. 12.47 we defined the state tr on $\mathrm{CAR}(\mathcal{Y})$ for any finite-dimensional $\mathcal{Y}$. For an arbitrary $\mathcal{Y}$ this gives rise to a state on $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$, and hence on $\mathrm{CAR}^{C^*}(\mathcal{Y})$, also denoted tr. We can perform the GNS representation using the state tr and obtain the triple $(\mathcal{H}_{\mathrm{tr}}, \pi_{\mathrm{tr}}, \Omega_{\mathrm{tr}})$, where

$$\pi_{\mathrm{tr}} : \mathrm{CAR}^{C^*}(\mathcal{Y}) \to B(\mathcal{H}_{\mathrm{tr}})$$

is a faithful $*$-representation, $\Omega_{\mathrm{tr}} \in \mathcal{H}_{\mathrm{tr}}$ is a vector cyclic for $\pi_{\mathrm{tr}}(\mathcal{H}_{\mathrm{tr}})$ and

$$\mathrm{tr}(A) = (\Omega_{\mathrm{tr}} | \pi_{\mathrm{tr}}(A)\Omega_{\mathrm{tr}}).$$

Definition 12.56 *We define the W^*-CAR algebra of $\mathcal{Y}$ as*

$$\mathrm{CAR}^{W^*}(\mathcal{Y}) := \Big(\pi_{\mathrm{tr}}\big(\mathrm{CAR}^{C^*}(\mathcal{Y})\big) \Big)''.$$

Since π_{tr} is faithful, it defines an isomorphism of $\mathrm{CAR}^{C^*}(\mathcal{Y})$ onto $\pi_{\mathrm{tr}}(\mathrm{CAR}^{C^*}(\mathcal{Y}))$. Therefore, in what follows these two algebras will be identified. Thus $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is a σ-weakly dense sub-algebra of $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

We have a normal state

$$(\Omega_{\mathrm{tr}}|A\Omega_{\mathrm{tr}}), \quad A \in \mathrm{CAR}^{W^*}(\mathcal{Y}). \tag{12.33}$$

On $\mathrm{CAR}^{C^*}(\mathcal{Y})$ it coincides with tr. In what follows, we will write $\mathrm{tr}A$ also for (12.33).

Thm. 12.45 extends with obvious adjustments:

Theorem 12.57 (1) tr *is a tracial state on* $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

(2) *The conjugation* c *and the parity* α *extend to* σ*-weakly continuous involutions on* $\mathrm{CAR}^{W^*}(\mathcal{Y})$ *preserving* tr. *For* $r \in O(\mathcal{Y})$*, the same is true for the Bogoliubov automorphism* $\hat{r}$.

(3) *The identities of Thm. 12.45 (3) and (4) are true.*

Definition 12.58 *We define* $\mathrm{CAR}_i^{W^*}(\mathcal{Y})$, $i = 0, 1$, *and* $\mathrm{Cliff}^{W^*}(\mathcal{Y})$ *as in Def. 12.49.*

Theorem 12.59 *If* $\mathcal{Y}$ *is an infinite-dimensional separable Hilbert space, then* $\mathrm{CAR}^{W^*}(\mathcal{Y})$ *is isomorphic to* HF *(the unique hyper-finite type* II_1 *factor described in Subsect. 6.2.10).*

Recall from Subsect. 6.5.2 that, for any $1 \le p \le \infty$, we can define the space $L^p(\mathrm{CAR}^{W^*}(\mathcal{Y}), \mathrm{tr})$. For $p = 1$, it coincides with the space of normal functionals on $\mathrm{CAR}^{W^*}(\mathcal{Y})$. For $p = 2$, it coincides with the GNS Hilbert space for the state tr, denoted also $\mathcal{H}_{\mathrm{tr}}$. Finally, for $p = \infty$, it coincides with $\mathrm{CAR}^{W^*}(\mathcal{Y})$ itself.

For $1 \le p < \infty$, $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ is dense in $L^p(\mathrm{CAR}^{W^*}(\mathcal{Y}), \mathrm{tr})$, so that $L^p(\mathrm{CAR}^{W^*}(\mathcal{Y}), \mathrm{tr})$, can be understood as the completion of $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ in the norm $\|A\|_p := (\mathrm{tr}|A|^p)^{1/p}$.

Definition 12.60 *Similarly to the case of a finite dimension, in the general case we define the* tracial *or* real-wave CAR representation over $\mathcal{Y}$ *by*

$$\mathcal{Y} \ni y \mapsto \phi_{\mathrm{tr}}(y) := \pi_{\mathrm{tr}}(\phi(y)) \in B_{\mathrm{h}}(\mathcal{H}_{\mathrm{tr}}).$$

12.5.4 Conditional expectations between CAR algebras

Consider a closed subspace $\mathcal{Y}_1$ of $\mathcal{Y}$. Clearly, $\mathrm{CAR}^{W^*}(\mathcal{Y}_1)$ can be viewed as a W^*-sub-algebra of $\mathrm{CAR}^{W^*}(\mathcal{Y})$. Besides, $\mathrm{CAR}^{W^*}(\mathcal{Y})$ is equipped with a tracial state tr. Therefore, by Subsect. 6.5.4, there exists a unique conditional expectation

$$E_{\mathcal{Y}_1} : \mathrm{CAR}^{W^*}(\mathcal{Y}) \to \mathrm{CAR}^{W_1}(\mathcal{Y}_1)$$

such that

$$\mathrm{tr}A = \mathrm{tr}E_{\mathcal{Y}_1}(A), \quad A \in \mathrm{CAR}^{W^*}(\mathcal{Y}).$$

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

It commutes with the parity:

$$\alpha \circ E_{\mathcal{Y}_1} = E_{\mathcal{Y}_1} \circ \alpha.$$

It restricts to a conditional expectation between the corresponding CAR C^*-algebras:

$$E_{\mathcal{Y}_1} : \mathrm{CAR}^{C^*}(\mathcal{Y}) \to \mathrm{CAR}^{C^*}(\mathcal{Y}_1).$$

If $\{\mathcal{Y}_i\}_{i\in I}$ is an increasing net of closed subspaces of $\mathcal{Y}$ with $\underset{i\in I}{\cup} \mathcal{Y}_i$ dense in $\mathcal{Y}$, we have the norm convergence

$$\lim_i E_{\mathcal{Y}_i}(A) = A, \quad A \in \mathrm{CAR}^{C^*}(\mathcal{Y}), \tag{12.34}$$

and the σ-weak convergence

$$\sigma\text{--w} - \lim_i E_{\mathcal{Y}_i}(A) = A, \quad A \in \mathrm{CAR}^{W^*}(\mathcal{Y}).$$

12.5.5 Irreducibility of infinite-dimensional CAR algebras

The following proposition extends Prop. 12.36 to the infinite-dimensional case.

Proposition 12.61 (1) *Let $A \in \mathrm{CAR}_0^{C^*}(\mathcal{Y})$ commute with $\phi(y)$, $y \in \mathcal{Y}$ (and hence with all $\mathrm{CAR}^{C^*}(\mathcal{Y})$). Then A is proportional to $\mathbb{1}$.*

(2) *Let a non-zero $A \in \mathrm{CAR}_1^{C^*}(\mathcal{Y})$ commute with $\phi(y)$, $y \in \mathcal{Y}$ (and hence with all $\mathrm{CAR}^{C^*}(\mathcal{Y})$). Then $\dim \mathcal{Y}$ is finite and odd, and A is proportional to Q.*

(3) *Let a non-zero $A \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ anti-commute with $\phi(y)$, $y \in \mathcal{Y}$ (and hence with all $\mathrm{CAR}_1^{C^*}(\mathcal{Y})$). Then $\dim \mathcal{Y}$ is finite and even, and A is proportional to Q.*

Proof We pick an increasing net $\mathcal{Y}_i$, $i \in I$ of finite-dimensional subspaces of $\mathcal{Y}$ with $(\bigcup_{i\in I} \mathcal{Y}_i)^{\mathrm{cl}} = \mathcal{Y}$. Let E_i be the conditional expectation onto $\mathrm{CAR}(\mathcal{Y}_i)$. Let $A \in \mathrm{CAR}_0^{C^*}(\mathcal{Y})$ such that $A\phi(y) = \phi(y)A$ for $y \in \mathcal{Y}$. Let $A_i := E_i(A)$. Since $E_i\phi(y) = \phi(y)$, $y \in \mathcal{Y}_i$, we obtain from Prop. 6.83 that $A_i\phi(y) = \phi(y)A_i$, $y \in \mathcal{Y}_i$. By Prop. 12.36, this implies that, for all i, $A_i = \lambda_i\mathbb{1}$. We know that $\lim_i A_i = A$ by (12.34). Hence, $\lim_i \lambda_i =: \lambda$ exists and $A = \lambda\mathbb{1}$. This proves (1).

Let us now prove (2). Let us assume that there exists A with the stated properties, and that $\dim \mathcal{Y}$ is infinite. We pick an increasing net of finite-dimensional subspaces $\mathcal{Y}_i$ of odd dimensions as above, equip them with orientations and denote by Q_i the associated volume elements as in (12.20). Considering the net $A_i := E_iA$, we know by Prop. 12.36 that, for all i, $A_i = \lambda_i Q_i$. Clearly, if $i \leq j$, then $E_i(Q_j) = Q_i$, which implies that λ_i coincide and equal a certain non-zero number λ. Since $A := \lim_i A_i \neq 0$ exists, $\lim_i Q_i \neq 0$. Using now the CAR we see that if $\mathcal{Y}_i$, $\mathcal{Y}_j$ are two finite-dimensional spaces with $\mathcal{Y}_i \subsetneq \mathcal{Y}_j$, then

https://doi.org/10.1017/9781009290876.013 Published online by Cambridge University Press

$\|Q_i - Q_j\| = 1$, which is a contradiction, since $\mathcal{Y}$ is infinite-dimensional. The proof of (3) is similar. □

The following proposition is the W^*-analog of Prop. 12.61.

Proposition 12.62 (1) *Let $A \in \mathrm{CAR}_0^{W^*}(\mathcal{Y})$ commute with $\phi(y)$, $y \in \mathcal{Y}$ (and hence with all $\mathrm{CAR}^{W^*}(\mathcal{Y})$). Then A is proportional to $\mathbb{1}$.*

(2) *Let a non-zero $A \in \mathrm{CAR}_1^{W^*}(\mathcal{Y})$ commute with $\phi(y)$, $y \in \mathcal{Y}$ (and hence with all $\mathrm{CAR}^{W^*}(\mathcal{Y})$). Then $\dim \mathcal{Y}$ is finite and odd, and A is proportional to Q.*

(3) *Let a non-zero $A \in \mathrm{CAR}^{W^*}(\mathcal{Y})$ anti-commute with $\phi(y)$, $y \in \mathcal{Y}$ (and hence with all $\mathrm{CAR}_1^{W^*}(\mathcal{Y})$). Then $\dim \mathcal{Y}$ is finite and even, and A is proportional to Q.*

Proof The proof of (1) is completely analogous to Prop. 12.61. Let us explain the modifications for the proof of (2). By the same arguments as in Prop. 12.61, we obtain that $\lim_i Q_i$ exists in the σ-weak topology. Working in the GNS representation for the tracial state, we see that $\lim_i Q_i \Omega$ does not exist. The proof of (3) is similar. □

12.6 Notes

Clifford relations and Clifford algebras appeared in mathematics before quantum theory, in Clifford (1878). They will be further discussed in Chap. 15.

Canonical anti-commutation relations were introduced in the description of fermions by Jordan–Wigner (1928).

Pauli matrices were introduced by Pauli (1927) to describe spin $\frac{1}{2}$ particles.

Mathematical properties of CAR algebras were extensively studied; see e.g. the review paper by Araki (1987) and the book by Plymen–Robinson (1994).