COMBINANTAL FORMS OF A QUADRIC NET APPLIED TO IRREDUCIBLE CONCOMITANT SYSTEMS

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1. Introduction

A complete system of combinantal forms (generalised and ordinary) of a pencil of quadrics $f_\lambda \equiv \lambda_1 f_1 + \lambda_2 f_2$ can be chosen such that the coefficients of the various power products of λ_1, λ_2 in the former give a complete irreducible system of concomitants of the two quadrics f_1, f_2, and conversely. This result was proved by Todd (**1**), who used it in conjunction with Schur function analysis (**2**) (**3**) to derive the complete irreducible system of concomitants of two quaternary quadratics (**4**).

The aim of the present paper is to generalise the result contained in (**1**) by deriving a similar result applicable to a net of n-ary quadratic forms, and also to develop a method of symbolical polarisation whereby a concomitant of partial degrees such as $(m+r+s,\ n-r,\ p-s)$ in the respective coefficients of three quadrics of the net can be derived explicitly from any $(m,\ n,\ p)$ concomitant. From the method of proof it will be seen that a further generalisation is possible for an ∞^r system of n-ary quadratics.

A combination of these three methods, viz. combinantal forms, S-function analysis and symbolical polarisation has been applied to the relatively simple complete system of the 47 invariants of three quaternary quadratics, as derived by Turnbull (**5**) and also to the more complex system of linear complexes formulated by Williamson (**6**), in order to obtain the corresponding irreducible systems. Only the above-mentioned techniques for analysing such concomitant systems in general are developed at present, their applications being deferred to a subsequent account.

2. Application of the Gordon-Capelli Theorem

The three n-ary quadratics $f_1 \equiv a_x^2, f_2 \equiv b_x^2, f_3 \equiv c_x^2$, where $a_x \equiv \sum_{i=1}^{n} a_i x_i$ etc., have a complete irreducible system of concomitants which may be denoted by $\{C\}$, a member of this being a combinant of the net $f_\lambda \equiv \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ if it is unaltered except for a factor involving the parameters λ_i, when f_1, f_2, f_3 are replaced by any three quadrics of the net. Moreover, if $f_\lambda \equiv r_x^2 \rho_\lambda \equiv s_x^2 \sigma_\lambda \equiv t_x^2 \tau_\lambda$ is treated as a double form, its complete system includes, besides such combinants, members which are covariant and contravariant in λ. Such generalised combinants are of the form:

$$P(\alpha\beta\gamma)^i(\beta\gamma\xi)^r(\gamma\alpha\xi)^s(\alpha\beta\xi)^t\alpha_\lambda^{m-i-s-t}\beta_\lambda^{n-i-t-r}\gamma_\lambda^{p-i-r-s} \quad \text{..............(i)}$$

where P contains $r', r'', r''', \ldots, r^{(m)}$; $s', s'', s''', \ldots s^{(n)}$; $t', t'', t''', \ldots t^{(p)}$, each symbol occurring twice, together with the variables $x, \pi_2, \pi_3, \ldots, \pi_{n-1}$, such variables π being determinantal compounds of i cogredient sets $x, y, z, \ldots, w$ with $\pi_i \equiv \overline{xyz\ldots w}$ (7, p. 249). In the quaternary case, for example, $p = \pi_2$, $u = \pi_3$ and so x, p, u, the point, line and plane co-ordinates can appear in (i). Also

$$a^m = \rho'\rho''\rho'''\ldots\rho^{(m)}; \quad \beta^n = \sigma'\sigma''\sigma'''\ldots\sigma^{(n)}; \quad \gamma^p = \tau'\tau''\tau'''\ldots\tau^{(p)}$$

and

$$\xi_1 = (\lambda\mu)_{23}; \quad \xi_2 = (\lambda\mu)_{31}; \quad \xi_3 = (\lambda\mu)_{12},$$

λ and μ being cogredient parameters.

It is now proposed to prove that $\{C\}$ can be replaced by an equivalent complete system $\{C^1\}$, each of whose members is either a combinant or the coefficient of a term involving $\xi_1^d\xi_2^e\xi_3^f\lambda_1^u\lambda_2^v\lambda_3^w$ in a system of generalised combinants represented by (i). This latter system together with the ordinary combinants will be denoted by $\{G^1\}$. Unlike the result given in (1) for a pencil of quadrics there is not necessarily a one-to-one correspondence between members of $\{C^1\}$ and the coefficients of $\{G^1\}$, since some of the latter may be equivalent (i.e. they may differ only by reducible terms).

The proof is as follows: any concomitant of the three quadrics of respective degrees m, n, p in their coefficients may be represented by $P\alpha_1^m\beta_2^n\gamma_3^p$. By the Gordon-Capelli Theorem for $m, n \geqq p$ (with similar expressions for $n, p \geqq m$ and $p, m \geqq n$) (8):

$$\alpha_\lambda^m\beta_\mu^n\gamma_\nu^p = \sum_{i=0}^{p}\sum_{j=0}^{p-i}(\lambda\mu\nu)^i(\alpha\beta\gamma)^i\left(\nu\frac{\partial}{\partial\mu}\right)^j\left(\nu\frac{\partial}{\partial\lambda}\right)^{p-i-j}\phi_{i,j}(\lambda,\mu) \quad \text{(ii)}$$

where

$$\phi_{i,j}(\lambda,\mu) = \sum_\sigma k_\sigma \alpha_\lambda^{r'_\sigma}\alpha_\mu^{s'_\sigma}\beta_\lambda^{r''_\sigma}\beta_\mu^{s''_\sigma}\gamma_\lambda^{r'''_\sigma}\gamma_\mu^{s'''_\sigma} \quad \text{(iii)}$$

and

$$r'_\sigma + s'_\sigma + i = m; \quad r''_\sigma + s''_\sigma + i = n; \quad r'''_\sigma + s'''_\sigma + i = p$$

while

$$r'_\sigma + r''_\sigma + r'''_\sigma = m + p - 2i - j = \delta, \text{ say}$$

and

$$s'_\sigma + s''_\sigma + s'''_\sigma = n - i + j = \varepsilon.$$

Each term of (iii), the numerical coefficient k_σ being neglected, can therefore be written in the form:

$$d_\lambda^\delta e_\mu^\varepsilon = \sum_{\omega=0}^{\theta} k_\omega (de/\lambda\mu)^\omega\left(\mu\frac{\partial}{\partial\lambda}\right)^{\varepsilon-\omega}(d_\lambda^{\delta-\omega}e_\lambda^{\varepsilon-\omega}) \quad \text{(iv)}$$

by an extension of Gordon's Series for ternary variables λ, μ (7, p. 255), θ being the lower value of δ, ε.

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If $\xi = (\lambda\mu)$, $\eta = (\mu\nu)$, $\zeta = (\nu\lambda)$, where $\xi_i = (\lambda\mu)_{jk}$ etc., ($i, j, k = 1, 2, 3$ in cyclic order), it follows that

$$\left(\nu\frac{\partial}{\partial\lambda}\right)^r (de\,|\,\lambda\mu)^\omega = (-1)^r \frac{\omega!}{(\omega-r)!}(de\,|\,\lambda\mu)^{\omega-r}(de\,|\,\mu\nu)^r$$

$$= (-1)^r \frac{\omega!}{(\omega-r)!}(de\xi)^{\omega-r}(de\eta)^r$$

$$= (-1)^r \left(\eta\frac{\partial}{\partial\xi}\right)^r (de\xi)^\omega.$$

Similarly,

$$\left(\nu\frac{\partial}{\partial\mu}\right)^s (de\,|\,\lambda\mu)^\omega = (-1)^s \left(\zeta\frac{\partial}{\partial\xi}\right)^s (de\xi)^\omega.$$

By application of Leibniz's Theorem for differentiation of products,

$$\left(\nu\frac{\partial}{\partial\mu}\right)^j \left(\nu\frac{\partial}{\partial\lambda}\right)^{p-i-j} (de\,|\,\lambda\mu)^\omega \left(\mu\frac{\partial}{\partial\lambda}\right)^{\varepsilon-\omega} (d_\lambda^{\delta-\omega}e^{\varepsilon-\omega})$$

$$= \sum_{r=0}^{p-i-j}\sum_{s=0}^{j} k_{rs}\left[\left(\zeta\frac{\partial}{\partial\xi}\right)^{j-s}\left(\eta\frac{\partial}{\partial\xi}\right)^{p-i-j-r}(de\xi)^\omega\right]$$

$$\left[\left(\nu\frac{\partial}{\partial\mu}\right)^s\left(\nu\frac{\partial}{\partial\lambda}\right)^r\left(\mu\frac{\partial}{\partial\lambda}\right)^{\varepsilon-\omega}(d^{\delta-\omega}e^{\varepsilon-\omega})\right] \quad \text{......(v)}$$

and since $(de\xi)^\omega d_\lambda^{\delta-w}e^{\varepsilon-w}$ is of the form

$$(\beta\gamma\xi)^f(\gamma\alpha\xi)^g(\alpha\beta\gamma)^h\alpha_\lambda^\mu\beta_\lambda^\nu\gamma_\lambda^w,$$

where $f+g+h = \omega$, $u+v+w = \delta+\varepsilon-2\omega = m+n+p-3i-2\omega$, it is seen from (iii), (iv) and (v) that a typical term of (ii) is

$$k(\lambda\mu\nu)^i(\alpha\beta\gamma)^i\Delta_{i,\,j,\,\omega}^{r,\,s}[(\beta\gamma\xi)^f(\gamma\alpha\xi)^g(\alpha\beta\xi)^h\alpha_\lambda^\mu\beta_\lambda^\nu\gamma_\lambda^\omega] \quad \text{...............(vi)}$$

where

$$\Delta_{i,\,j,\,\omega}^{r,\,s} = \left(\zeta\frac{\partial}{\partial\xi}\right)^{j-s}\left(\eta\frac{\partial}{\partial\xi}\right)^{p-i-j-r}\left(\nu\frac{\partial}{\partial\mu}\right)^s\left(\nu\frac{\partial}{\partial\lambda}\right)^r\left(\mu\frac{\partial}{\partial\lambda}\right)^{\varepsilon-\omega}$$

and k is a numerical coefficient.

Now $P\alpha_1^m\beta_2^n\gamma_3^p$, which is the coefficient of $\lambda_1^m\mu_2^n\nu_3^p$ in $P\alpha_\lambda^m\beta_\mu^n\gamma_\nu^p$ is thus a linear combination of coefficients of products

$$\xi_1^{p-i-j-r}\xi_2^{j-s}\xi_3^{\omega-p+i+r+s}\lambda_1^{\delta-\omega-r}\lambda_2^{\varepsilon-\omega-s}\lambda_2^{r+s}$$

i.e., of

$$\xi_1^{p-i-j-r}\xi_2^{j-s}\xi_3^{\omega-p+i+r+s}\lambda_1^{m+p-2i-j-\omega-r}\lambda_2^{n-i+j-\omega-s}\lambda_3^{r+s}$$

in a combinantal form of type (i), viz. $P(\alpha\beta\gamma)^i(\beta\gamma\xi)^f(\gamma\alpha\xi)^g(\alpha\beta\xi)^h\alpha_\lambda^\mu\beta_\lambda^\nu\gamma_\lambda^\omega$, where $i = 0$ to p, $j = 0$ to $p-i$, $\omega = 0$ to $n-i+j$, $r = 0$ to $p-i-j$, $s = 0$ to j.

Thus, any concomitant of f_1, f_2, f_3 can be expressed as a linear combination of the coefficients in such combinantal forms (ordinary combinants being included).

3. Combinantal Forms and Symbolical Polarisation

The net f_λ may be written in non-symbolical form as follows:

$$f_\lambda \equiv \sum_{(j)} (a_{(j)}x^{(j)}\lambda_1 + b_{(j)}x^{(j)}\lambda_2 + c_{(j)}x^{(j)}\lambda_3) \quad \text{....................(i)}$$

where $(j) = j_1 j_2 j_3 \cdots j_n$, $x^{(j)} = x_1^{j_1} x_2^{j_2} x_3^{j_2} \ldots x_n^{j_n}$ and $\sum_{r=1}^{n} j_n = 2$.

(i) represents a mixed form and so any generalised combinant of the net possesses six annihilators (**9**, pp. 410-412):

$$\Omega_{rs} + \xi_s \frac{\partial}{\partial \xi_r} - \lambda_r \frac{\partial}{\partial \lambda_s} \; (r,\, s = 1,\, 2,\, 3 \text{ and } r \neq s),$$

where, for example,

$$\Omega_{13} = \sum_{(j)} c_{(j)} \frac{\partial}{\partial a_{(j)}},\; \Omega_{31} = \sum_{(j)} a_{(j)} \frac{\partial}{\partial c_{(j)}}.$$

The coefficients of power products $\xi_1^f \xi_2^g \xi_3^h \lambda_1^u \lambda_2^v \lambda_3^w$ in any such combinantal form are derived from a leading term $S\lambda_3^{\tilde{\omega}} \xi_3^{\tilde{\omega}^1}$ by expressing the form as the expansion of

$$\left\{\xi_3^{\tilde{\omega}^1} e^{-\xi_1/\xi_3\left(\Omega_{13} - \lambda_1 \frac{\partial}{\partial \lambda_3}\right) - \xi_2/\xi_3\left(\Omega_{23} - \lambda_2 \frac{\partial}{\partial \lambda_3}\right)}\right\}\left\{\lambda_3^{\tilde{\omega}} e^{\lambda_1/\lambda_3 \Omega_{31} + \lambda_2/\lambda_3 \Omega_{32}}\right\} S$$

$$= \left\{\xi_3^{\tilde{\omega}^1} \sum_{r=0}^{\tilde{\omega}^1} (-1)^r \frac{1}{r!} \Theta^r\right\}\left\{\lambda_3^{\tilde{\omega}} \sum_{s=0}^{\tilde{\omega}} \frac{1}{s!} \Phi^s\right\} S \quad \text{..................(ii)}$$

where

$$\Theta = \frac{1}{\xi_3}\left\{(\xi_1\Omega_{13} + \xi_2\Omega_{23}) - (\xi_1\lambda_1 + \xi_2\lambda_2)\frac{\partial}{\partial \lambda_3}\right\}$$

and

$$\Phi = \frac{1}{\lambda_3}\{\lambda_1\Omega_{31} + \lambda_2\Omega_{32}\}.$$

Thus, if any one coefficient of a power product of ξ's and λ's in (ii) is irreducible, so are all the coefficients and consequently a set of combinantal forms $\{G^1\}$ can be chosen whose coefficients form a complete irreducible system $\{C^1\}$ of concomitants of f_1, f_2, f_3 with the possibility of repetitions or equivalences among the coefficients.

For expansion of (ii) shows that the coefficients of terms such as

$$\xi_1^{a-r}\xi_2^b\xi_3^{c+r}\lambda_1^{d-r}\lambda_2^e\lambda_3^{f+r} \quad \text{and} \quad \xi_1^{a+r}\xi_2^{b-r}\xi_3^c\lambda_1^{d+r}\xi_2^{e-r}\lambda_3^f,$$

where $a+b+c = \tilde{\omega}^1$ and $d+e+f = \tilde{\omega}$, are all of the same partial degrees in the coefficients of f_1, f_2, f_3 irrespective of the value of r, involving as they do the respective combinations of expressions, $\Omega_{13}^{a-r}\Omega_{23}^b\Omega_{31}^{d-r}\Omega_{32}^e S$ and $\Omega_{13}^{a+r}\Omega_{23}^{b-r}\Omega_{31}^{d+r}\Omega_{32}^{e-r} S$. It may therefore happen that some of the coefficients are equivalent as far as irreducibility is concerned. Thus, when concomitants of the same partial degrees occur as coefficients in combinantal forms of $\{G^1\}$ it is necessary to test for equivalence.

Recourse is now had to the principle of symbolical polarisation derived as follows:

Let $C(a_{ij})$ be a concomitant of degree m in the coefficients of $f_1 \equiv \sum_{i,j=1}^{n} a_{ij}x_i x_j$, a typical term of C being $Ea_{ij}^r a_{kl}^s a_{pq}^t \ldots$, where $r+s+t+\ldots = m$; $i, j, k, l, \ldots = 1, 2, 3, \ldots, n$ with possible repetitions, but with $ij \neq kl \neq pq \ldots$; and E containing $b_{ij}, c_{ij}, \ldots$ and a numerical coefficient.

In terms of Clebsch-Aronhold symbols

$$Ea_{ij}^r a_{kl}^s a_{pq}^t \ldots = E \prod_{\alpha, \beta, \gamma} a_i^{(\alpha)} a_j^{(\alpha)} a_k^{(\beta)} a_l^{(\beta)} a_p^{(\gamma)} a_q^{(\gamma)} \ldots = X,$$

say, where $\alpha = 1$ to r, $\beta = (r+1)$ to $(r+s)$, $\gamma = (r+s+1)$ to $(r+s+t)$, ..., Thus, m pairs of equivalent symbols are introduced.

Now

$$\left(b_{ij}\frac{\partial}{\partial a_{ij}}\right)X = Era_{ij}^{r-1} b_{ij} a_{kl}^s a_{pq}^t \ldots$$
$$= \sum_{\alpha=1}^{r}\left(b_i^{(\alpha)}\frac{\partial}{\partial a_i^{(\alpha)}}\right)\left(b_j^{(\alpha)}\frac{\partial}{\partial a_j^{(\alpha)}}\right)X.$$

Also, since $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, Ω_{12} can be written as

$$\Omega_{12} = \sum_{i,j=1}^{n}\left(b_{ij}\frac{\partial}{\partial a_{ij}}\right) = \sum_{\omega=1}^{m}\tfrac{1}{2}\left(b^{(\omega)}\frac{\partial}{\partial a^{(\omega)}}\right)^2,$$

where

$$\left(b^{(\omega)}\frac{\partial}{\partial a^{(\omega)}}\right) = \sum_{r=1}^{n}\left(b_r^{(\omega)}\frac{\partial}{\partial a_r^{(\omega)}}\right)$$

and so

$$\sum_{i,j=1}^{n}\left(b_{ij}\frac{\partial}{\partial a_{ij}}\right)X = \sum_{\omega=1}^{m}\tfrac{1}{2}\left(b^{(\omega)}\frac{\partial}{\partial a^{(\omega)}}\right)^2 X.$$

Extension of this result to $C(a_{ij})$ gives

$$\Omega_{12}C(a_{ij}) = \sum_{\omega=1}^{m}\tfrac{1}{2}\left(b^{(\omega)}\frac{\partial}{\partial a^{(\omega)}}\right)^2 C(a^{(1)}, a^{(1)};\ a^{(2)}, a^{(2)};\ \ldots;\ a^{(m)}, a^{(m)}).$$

For any set of paired symbols $a^{(\omega)}, a^{(\omega)}$ let $C = a_\delta^{(\omega)} a_\varepsilon^{(\omega)} D$.

Thus $\tfrac{1}{2}\left(b^{(\omega)}\frac{\partial}{\partial a^{(\omega)}}\right)^2 C = b_\delta^{(\omega)} b_\varepsilon^{(\omega)} D$ and so the effect of operating with $\tfrac{1}{2}\left(b^{(\omega)}\frac{\partial}{\partial a^{(\omega)}}\right)^2$ is to replace the double $a^{(\omega)}$ by a double $b^{(\omega)}$ in the symbolical form of the concomitant C.

Such symbolical polarising processes can be applied to any coefficient of a combinantal form of $\{G^1\}$ *to obtain other coefficients, and can on occasion be* *used to establish irreducibility when S-function analysis fails to give a clear-cut answer due to the existence of syzygies among concomitants.*

Finally, the system $\{G^1\}$ having been determined, any concomitant F of f_1, f_2, f_3 (i.e. a rational integral function of the forms of $\{C^1\}$) is a real combinant if, and only if, it is an invariant of the set of forms of $\{G^1\}$ regarded as functions of the ξ's and λ's. But any invariant of the complete system of generalised combinants of the net is a combinant of the quadrics and so $\{G^1\}$ is this complete system.

4. Application of Schur Functions

In accordance with the general methods developed by Littlewood (**2**) and applied by Todd (**3**) (**4**) to a pair of quaternary quadrics similar results will now be formulated for application to three quaternary quadrics. A concomitant of partial degrees (m, n, p) in the coefficients of f_1, f_2, f_3 and of orders n_1, n_2, n_3 in the variables x, p, u is considered. Then n_1, n_2, n_3 are the numbers of a_x, (abp), $(abcu)$ type factors respectively, while n_4 is taken to be the number of $(abcd)$ type brackets. Such a concomitant corresponds to a partition $(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of $2N$, where

$$\left.\begin{aligned} N = m+n+p \quad \text{and} \quad \lambda_1+\lambda_2+\lambda_3+\lambda_4 &= 2N \\ \lambda_1-\lambda_2 &= n_1 \\ \lambda_2-\lambda_3 &= n_2 \\ \lambda_3-\lambda_4 &= n_3 \\ \lambda_4 &= n_4 \end{aligned}\right\} \qquad \text{(i)}$$

To a partition (λ) corresponds an S-function of type $\{\lambda\}$, and the total number of linearly independent (m, n, p) concomitants of orders n_1, n_2, n_3 in x, p, u is equal to the coefficient of $\{\lambda\}$ in the expression:

$$[\{2\}\otimes\{m\}][\{2\}\otimes\{n\}][\{2\}\otimes\{p\}] \qquad \text{(ii)}$$

where $\{2\}\otimes\{r\}$ is the sum of the S-functions corresponding to the partitions of $2r$ into four or fewer parts. The calculation involved in (ii) is performed by the construction of regular Young Tableaux (**3**).

For the invariants of three quadrics $n_1 = n_2 = n_3 = 0$ and so from (i)

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = n_4 = \frac{N}{2} \quad \text{and} \quad (\lambda) = (n_4^4). \qquad \text{(iii)}$$

Similarly, for the linear complexes $n_1 = n_3 = 0$, $n_2 = 1$ and so

$$\lambda_1 = \lambda_2 = \lambda_3+1 = \lambda_4+1 = n_4+1.$$

Thus, the linear complexes have partitions of the type $((n_4+1)^2, n_4^2)$, where $2n_4+1 = N$, and the total degree is therefore odd with $n_4 = 2, 3, 4, \ldots$.

The total number v of concomitants of any degree and type can thus be determined by S-function analysis. Also the number v^1 of such which are reducible can also be calculated from a knowledge of the irreducible invariants and relevant concomitants of lower degree. The number of irreducible concomitants of the given type is therefore $v-(v^1-\sigma)$, where σ is the number of linearly independent syzygies connecting the v^1 products so obtained. A complete system of concomitants of given type provides an upper bound to the number of irreducibles of this type, while $v-v^1$ gives a lower bound. If these numbers are equal, that is when $\sigma = 0$, then all the members of the complete system are irreducible.

If the results obtained in §§ **2** and **3** for combinantal forms are used in conjunction with the above procedure the task of obtaining an irreducible system can be considerably shortened, since in such a form the coefficients are either all reducible or all irreducible. Moreover, *symbolical polarisation can serve as a useful ancillary method in determining irreducibility in cases where* $\sigma \neq 0$ *and syzygies do exist.*

Such specific applications are given in a subsequent paper.

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