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ON NEIGHBOURHOODS IN THE ENHANCED POWER GRAPH ASSOCIATED WITH A FINITE GROUP

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Abstract

We investigate neighbourhood sizes in the enhanced power graph (also known as the cyclic graph) associated with a finite group. In particular, we characterise finite p-groups with the smallest maximum size for neighbourhoods of a nontrivial element in its enhanced power graph.

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1. Introduction

All groups considered in this paper are finite unless otherwise stated. To study the structure of a group, one can look at the invariants of some graphs whose vertices are the elements of the group and whose edges reveal some properties of the group itself. More precisely, if G is a group and $\mathcal{B}$ is a class of groups, the $\mathcal{B}$-graph associated with G, denoted by $\Gamma_{\mathcal{B}}(G)$, is a simple and undirected graph whose vertices are the elements of G, and there is an edge between two elements x and y of G if the subgroup generated by x and y is a $\mathcal{B}$-group.

Several features of a finite group can be detected by analysing the invariants of its $\mathcal{B}$-graph. We refer to [5] for a survey on this topic and to [10, 11] for related work. Recent papers deal with the investigation of the (closed) neighbourhood $\mathcal{I}_{\mathcal{B}}(x)$ of a vertex x in $\Gamma_{\mathcal{B}}(G)$, that is, the set of all y in G such that x and y generate a $\mathcal{B}$-group. When $\mathcal{B}$ is the class of abelian groups, then $\mathcal{I}_{\mathcal{B}}(x)$ coincides with the centraliser of x in G, thus $\mathcal{I}_{\mathcal{B}}(x)$ is a subgroup. However, in general this is not the case when $\mathcal{B}$ is distinct from the class of abelian groups. Nevertheless, even though $\mathcal{I}_{\mathcal{B}}(x)$ is not a subgroup of G in general, it can happen that the characteristics of a single neighbourhood in

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a $\mathcal{B}$-graph could affect the structure of the whole group G. For instance, when $\mathcal{B}$ coincides with the class $\mathcal{S}$ of soluble groups, it has been shown that the combinatorial properties, as well as arithmetic ones, of $\mathcal{I}_{\mathcal{B}}(x)$ may force the whole group to be abelian or nilpotent (see [2, 3] for more details).

Here we start by considering the class C of all cyclic groups. Cameron in [5] calls the graph $\Gamma_C(G)$ the *enhanced power graph*. The term enhanced power graph appears to have originated in [1]. However, this graph was first studied in [12] under the name *cyclic graph*. Further investigations under this name occurred in [13]. Recently, this graph has been investigated in [6–8] and there are still several open questions, as described in [15].

Our interest in $\Gamma_C(G)$ chiefly concerns the cardinality of $\mathcal{I}_C(x)$, discussing the possible values that can occur for $|\mathcal{I}_C(x)|$ when x belongs to a p-group G. Denote by n_G the maximum of the sizes of all $\mathcal{I}_C(x)$ for $x \in G \setminus \{1\}$. Then clearly

$$\exp(G) \leq n_G \leq |G|,$$

where $\exp(G)$ denotes the exponent of the group G. Whenever G has a nontrivial universal vertex, that is, a nontrivial element adjacent to any element of G, $n_G = |G|$. These groups have been characterised in the soluble case in [8]. Our first goal is to characterise p-groups G with $n_G = \exp(G)$. Indeed we prove the following result.

THEOREM 1.1. *Let G be a finite p-group. Then $n_G = \exp(G)$ if and only if G is cyclic, or $\exp(G) = p$, or G is a dihedral 2-group.*

It is worth mentioning that a problem connected to closed neighbourhoods has been addressed in [14]. Going further, one may ask what is the second value that can occur for n_G, and the answer is given by the following proposition.

PROPOSITION 1.2. *Let G be a p-group and assume $n_G > \exp(G)$. Then we have $n_G \geq p^{\alpha+1} - p^{\alpha} + p^{\alpha-1}$.*

We point out that the bound in Theorem 1.2 is sharp in some sense. Indeed, for $G = C_{p^2} \times C_p$ we have $n_G = p^3 - p^2 + p$, where C_k denotes the cyclic group of order k.

2. The cyclic graph

In this section we will deal with the enhanced power graph of a group, or what we like to call the cyclic graph of a group. Recall that the cyclic graph of a group G, denoted by $\Delta(G)$, is the graph whose vertex set is $G \setminus \{1\}$, and two distinct elements x, y of G are adjacent if and only if $\langle x, y\rangle$ is cyclic. When x and y are adjacent we will write $x \sim y$. We denote by n_G the maximum of the sizes of all $\mathcal{I}_C(x)$ for $x \in G \setminus \{1\}$. We begin with the following useful lemma.

LEMMA 2.1. *Let p be a prime and let G be a p-group. Then there exists an element $z \in G$ of order p such that $|\mathcal{I}_C(z)| = n_G$.*

PROOF. Observe that there exists an element $x \in G$ such that $|\mathcal{I}_C(x)| = n_G$. If $o(x) = p$, then we are done. Therefore, we assume that $o(x) = p^k$, where k is an integer so that $k \geq 2$. Take $z = x^{p^{k-1}}$, and observe that x and z belong to the same connected component Υ in $\Delta(G)$, and that z is the only element of order p in Υ. By [6, Lemma 2.2], $z \sim y$ for any element $y \in \Upsilon$, and so $|\mathcal{I}_C(z)| \geq |\mathcal{I}_C(x)| = n_G$, which implies $|\mathcal{I}_C(z)| = n_G$. □

By Lemma 2.1 and [6, Lemma 2.2], one can easily see that $n_G = |\Upsilon| - 1$, where Υ is a connected component of $\Delta(G)$ containing a vertex of degree n_G.

2.1. Abelian *p*-groups. In this subsection, we focus on Abelian p-groups. In the next lemma, we compute n_G when G is a nontrivial cyclic group.

LEMMA 2.2. *If G is a nontrivial cyclic group, then $n_G = |G|$.*

PROOF. Let $x \in G$ such that $G = \langle x \rangle$. Since $o(x) = |G|$ and $G \setminus \langle x \rangle = \emptyset$, we conclude that $n_G = |G|$. □

We next compute n_G when G is a p-group having exponent p.

LEMMA 2.3. *Let p be a prime and let G be a p-group of exponent p. Then $n_G = p$.*

PROOF. If G is a cyclic group of order p, then the result follows from Lemma 2.2. Assume that G is not cyclic, and consider an element $x \in G$ such that $|\mathcal{I}_C(x)| = n_G$. As $o(x) = p$, we have $n_G \geq p$.

Now observe that if $y \in G \setminus \langle x \rangle$, then $\langle x, y \rangle$ is not cyclic. Indeed, arguing by contradiction, let $z \in G$ be such that $\langle x, y \rangle = \langle z \rangle$. Since G has exponent p, there exist $i, j \in \{1, \ldots, p-1\}$ such that $x = z^i$ and $y = z^j$. Therefore, from $(i, p) = 1$ it follows that $\langle x \rangle = \langle z^i \rangle = \langle z \rangle$ and $y \in \langle x \rangle$, a contradiction. Hence, we conclude that $n_G = p$. □

We now show that if G is a noncyclic abelian group whose exponent is larger than p, then n_G is larger than the exponent of G.

LEMMA 2.4. *Let p be a prime and let G be a noncyclic abelian p-group of exponent $exp(G) = p^\alpha$, where $\alpha \geq 2$. Then $n_G \geq p^{\alpha+1} - p^\alpha + p^{\alpha-1}$. As a consequence, $n_G > \exp(G)$.*

PROOF. As G is abelian, we may assume

$$G = C_{p^{\alpha_1}} \times \cdots \times C_{p^{\alpha_r}},$$

where $r \geq 2$, $1 \leq \alpha_1 \leq \cdots \leq \alpha_r = \alpha$ and $C_{p^{\alpha_i}} = \langle x_i \rangle$ is a cyclic group of order p^{α_i}.

If $\alpha_{r-1} = 1$, then the vertex $x_r^{p^{\alpha-1}}$ is adjacent to $p^\alpha - 2$ nontrivial elements of $\langle x_r \rangle$ and to any element of the form $x_{r-1}^i x_r^k$, where $i = 1, \ldots, p-1$ and k is a positive integer less than p^α and coprime with p. Hence, there are precisely $p^\alpha - p^{\alpha-1}$ choices for k, which implies

$$|I_C(x)| \geq p^\alpha + (p-1)(p^\alpha - p^{\alpha-1}) = p^{\alpha+1} - p^\alpha + p^{\alpha-1}.$$

If $\alpha_{r-1} > 1$, then one can consider the subgroup $\langle x_r^{p^{\alpha_{r-1}-1}}, x_r \rangle$, arguing as in the previous case. □

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We now collect these lemmas in a proposition where we note that, for an abelian p-group G, n_G equals the exponent of G if and only if G is cyclic or elementary abelian.

PROPOSITION 2.5. *Let p be a prime and let G be an abelian p-group. Then $n_G = \exp(G)$ if and only if G is either cyclic or elementary abelian.*

PROOF. If G is either cyclic or elementary abelian, then the result follows from Lemmas 2.2 and 2.3. Conversely, assume that $n_G = \exp(G)$. If G is neither cyclic nor elementary abelian, then, applying Lemma 2.4, we have $n_G > \exp(G)$, a contradiction. □

2.2. Nonabelian *p*-groups. We now shift our focus to nonabelian p-groups. When p is a prime, we take α to be an integer greater than 1 when p is odd and an integer greater than 2 when $p = 2$. We denote by $M_{p^{\alpha+1}}$ the group

$$M_{p^{\alpha+1}} = \langle x, y \mid x^{p^\alpha} = y^p = 1,\ x^y = x^{p^{\alpha-1}+1}\rangle.$$

Going further, we respectively denote by $D_{2^{\alpha+1}}$, $S_{p^{\alpha+1}}$ and $Q_{2^{\alpha+1}}$ the dihedral, semidihedral and generalised quaternion groups given by the following presentations:

$$D_{2^{\alpha+1}} = \langle x, y \mid x^{2^\alpha} = y^2 = 1,\ x^y = x^{-1}\rangle,$$
$$S_{p^{\alpha+1}} = \langle x, y \mid x^{p^\alpha} = y^p = 1,\ x^y = x^{p^{\alpha-1}-1}\rangle,$$
$$Q_{2^{\alpha+1}} = \langle x, y \mid x^{2^{\alpha-1}} = y^2,\ y^4 = 1,\ x^y = x^{-1}\rangle.$$

The characterisation of nonabelian p-groups with a cyclic maximal subgroup is well known (see [9]).

THEOREM 2.6. *Let p be a prime and let G be a nonabelian p-group of order $p^{\alpha+1}$ with a cyclic subgroup of order p^α.*

(i) *If p is odd then G is isomorphic to $M_{p^{\alpha+1}}$.*
(ii) *If $p = 2$ and $\alpha = 2$, then G is isomorphic to either D_8 or Q_8.*
(iii) *If $p = 2$ and $\alpha > 3$, then G is isomorphic to either $M_{2^{\alpha+1}}$, $D_{2^{\alpha+1}}$, $Q_{2^{\alpha+1}}$ or $S_{2^{\alpha+1}}$.*

We compute n_G for nonabelian p-groups with a maximal cyclic subgroup of index p.

PROPOSITION 2.7. *Let p be a prime and let G be a p-group of order $p^{\alpha+1}$. Assume that G has a maximal cyclic subgroup of order p^α. Then $n_G = \exp(G)$ if and only if either G is cyclic, or $\exp(G) = p$, or $G \simeq D_{2^{\alpha+1}}$.*

PROOF. If G is cyclic or $\exp(G) = p$, then $n_G = \exp(G)$ by Lemmas 2.3 and 2.2. Moreover, if $G \simeq D_{2^{\alpha+1}}$, then G has only one cyclic subgroup of order 2^α while all the other cyclic subgroups have order 2, which implies $n_G = \exp(G)$.

Now assume that $n_G = \exp(G)$. If G is abelian then G is either cyclic or elementary abelian by Proposition 2.5. Now assume that G is neither abelian nor of exponent p. From Theorem 2.6 we have to analyse two cases. First assume that G is isomorphic to $M_{p^{\alpha+1}}$. Then $(yx)^p = x^{(p(p-1)/2)p^{\alpha-1}+p}$, which yields a contradiction. Indeed, when p is odd, we have $(yx)^p = x^p$ and $|\mathcal{I}_C(x^p)| > \exp(G)$ as x^p is connected to every element of

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$\langle x\rangle$ and to every element of $\langle yx\rangle$. If $p = 2$, then $(yx)^2 = x^{2^{\alpha-1}+2}$ and $\mathcal{I}_C(x^{2^{\alpha-1}+2})$ contains more than 2^α elements.

Finally, assume that $p = 2$ and G isomorphic to $S_{2^{\alpha+1}}$. Then $(yx)^2 = x^{2^{\alpha-1}}$ and $|\mathcal{I}_C(yx)| > \exp(G)$. □

We are now in a position to prove Theorem 1.1.

PROOF OF THEOREM 1.1. By Lemmas 2.2 and 2.3 and Proposition 2.7, we only need to prove that if $n_G = \exp(G)$ then G is either cyclic, or $\exp(G) = p$, or G is a dihedral 2-group. Thus, let $n_G = \exp(G)$, and by way of contradiction assume neither that G is cyclic, nor $\exp(G) = p$, nor G is a dihedral group of order $2^{\exp(G)+1}$, such that G has minimal order. Hence, there exists an element $x \in G$ such that $p < o(x) = \exp(G)$. By Proposition 2.7, it follows that $p \cdot o(x) < |G|$, and thus G contains a proper subgroup H such that $x \in H$ and $|H| = p \cdot o(x)$. Then $\exp(H) = \exp(G)$ and H has a cyclic subgroup of index p. By Proposition 2.7, H is a dihedral group of order $2\exp(G)$ since H is neither cyclic nor such that $\exp(H) = p$. As a consequence G is a 2-group, and by minimality, $|G : H| = 2$. If $o(x) = 4$, then $|G| = 16$ and an easy computation using GAP shows that this is a contradiction. Hence, we may assume $o(x) > 4$. Now assume that there exists an element $a \in G \setminus H$ such that $o(a) > 4$. Then $a^2 \in H$ and $o(a^2) > 2$. This implies that $a^2 \in \langle x\rangle$ and $|\mathcal{I}_C(a^2)| > \exp(G)$. Hence, we may assume that $o(a) \le 4$ for all $a \in G \setminus H$. First assume that $G \setminus H$ contains an element a of order 2. If a does not invert x, then $(xa)^2 = xx^a$ is a nontrivial element of $\langle x\rangle$, since $\langle x\rangle$ is normal in G. As a consequence, $|\mathcal{I}_C((xa)^2)| > \exp(G)$. Now assume that $x^a = x^{-1}$. Let $b \in H$ be such that $x^b = x^{-1}$. Then $x^{ab} = x$ and ab belongs to the centraliser in G of x. Thus, $(xab)^4 = x^4 \neq 1$, and $|\mathcal{I}_C(x^4)| > \exp(G)$. Therefore, we only need to address the case in which $o(a) = 4$ for every $a \in G \setminus H$. If $a^2 \in \langle x\rangle$ for some $a \in G \setminus H$, then $|\mathcal{I}_C(a^2)| > \exp(G)$. This implies that $a^2 \in H \setminus \langle x\rangle$. As a consequence a^2 inverts x. On the other hand, the dihedral groups have no automorphisms of order 4 whose square inverts its element of maximal order (see, for instance, Theorem 34.8(a) of [4]). This final contradiction proves the theorem. □

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References

[1] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish and F. Shaveisi, 'On the structure of the power graph and the enhanced power graph of a group', *Electron. J. Combin.* **24**(3) (2017), Article no. 3.16, 18 pages.

[2] B. Akbari, C. Delizia and C. Monetta, 'On the solubilizer of an element in a finite group', *Mediterr. J. Math.* **20**(3) (2023), Article no. 135, 9 pages.

[3] B. Akbari, M. L. Lewis, J. Mirzajani and A. R. Moghaddamfar, 'The solubility graph associated with a finite group', *Internat. J. Algebra Comput.* **30**(8) (2020), 1555–1564.

[4] Y. Berkovich, *Groups of Prime Power Order, Volume 1*, DeGruyter Expositions in Mathematics, 46 (DeGruyter, Berlin, 2008).

[5] P. J. Cameron, 'Graphs defined on groups', *Int. J. Group Theory* **11**(2) (2022), 53–107.

[6] D. G. Costanzo, M. L. Lewis, S. Schmidt, E. Tsegaye and G. Udell, 'The cyclic graph (deleted enhanced graph) of a direct product', *Involve* **14**(1) (2021), 167–179.

[7] D. G. Costanzo, M. L. Lewis, S. Schmidt, E. Tsegaye and G. Udell, 'The cyclic graph of a Z-group', *Bull. Aust. Math. Soc.* **104**(2) (2021), 295–301.

[8] D. G. Costanzo, M. L. Lewis, S. Schmidt, E. Tsegaye and G. Udell, 'Characterizing finite groups whose enhanced power graphs have universal vertices', *Czechoslovak Math. J.* **74**(2) (2024), 637–645.

[9] D. Gorenstein, *Finite Groups* (Chelsea Publishing Company, New York, 1980).

[10] V. Grazian, A. Lucchini and C. Monetta, 'Group nilpotency from a graph point of view', *Int. J. Group Theory* **13**(4) (2024), 351–367.

[11] V. Grazian and C. Monetta, 'A conjecture related to the nilpotency of groups with isomorphic non-commuting graphs', *J. Algebra* **633** (2023), 389–402.

[12] D. Imperatore, 'On a graph associated with a group', In *Ischia Group Theory 2008* (eds. M. Bianchi, P. Longobardi, M. Maj and M. Scoppola) (World Scientific, Hackensack, NJ, 2009), 100–115.

[13] D. Imperatore and M. L. Lewis, 'A condition in finite solvable groups related to cyclic subgroups', *Bull. Aust. Math. Soc.* **83**(2) (2011), 267–272.

[14] J. Kumar, X. M. Parveen and S. Singh, 'Certain properties of the enhanced power graph associated with a finite group', *Acta Math. Hungar.* **169**(1) (2023), 238–251.

[15] X. Ma, A. Kelarev, Y. Lin and K. Wang, 'A survey on enhanced power graphs of finite groups', *Electron. J. Graph Theory Appl. (EJGTA)* **10**(1) (2022), 89–111.

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