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E-ASSOCIATIVE RINGS

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ABSTRACT. A ring R is E-associative if $\varphi(xy) = \varphi(x)y$ for all endomorphisms φ of the additive group of R, and all $x, y \in R$. Unital E-associative rings are E-rings. The structure of the torsion ideal of an E-associative ring is described completely. The E-associative rings with completely decomposable torsion free additive groups are also classified. Conditions under which E-associative rings are E-rings, and other miscellaneous results are obtained.

Introduction. Rings considered in this article are not necessarily associative, and need not possess a unity. All groups (except S_n) are abelian with addition the group operation. A ring R is associative if and only if $a_\ell(xy) = a_\ell(x)y$ for all $a, x, y \in R$, where the mapping a_ℓ is left multiplication by a. If the ring R satisfies a condition stronger than associativity, namely that $\varphi(xy) = \varphi(x)y$ for all $x, y \in R$, and all endomorphisms φ of the additive group of G, then R will be said to be E-associative (endomorphism associative). Unital E-associative rings are called E-rings, and have been studied fairly extensively.

The class of E-associative rings is considerably larger than the class of E-rings. The main goal of this note is to describe E-associative rings. The torsion part of an E-associative ring is described completely, and so a classification of torsion E-associative rings is obtained. A description of E-associative rings with a completely decomposable torsion free additive group is also obtained. It will be shown that a unital ring R is an E-ring if and only if R satisfies any one of an infinite set of ring properties which will be defined. E-associativity is one of these properties.

For R a ring, and $a \in R$, right multiplication by a will be denoted by a_r, and the pure subgroup of R^+ generated by a will be denoted by $(a)_*$. Let $X \subseteq R$. Then the right annihilator of X is $r.\operatorname{ann}(X) = \{a \in R \mid Xa = 0\}$.

The reader is referred to [2] and [3] for definitions of terms and facts concerning abelian groups.

General results.

LEMMA 1. *Let R be an E-associative ring, and let A be a direct summand of R^+. Then A is a right ideal in R, and A is E-associative.*

PROOF. Let $R^+ = A \oplus B$, and let π_B be the natural projection of R^+ onto B along A. For all $a \in A$, and $x \in R$, $\pi_B(ax) = \pi_B(a) \cdot x = 0$, and so $ax \in A$. Let $a, b \in A$, and let $\varphi \in E(A)$. Clearly φ can be extended to an endomorphism of R^+, and so $\varphi(ab) = \varphi(a)b$, *i.e.*, A is E-associative.

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LEMMA 2. *Let R be an E-associative ring. If there exists $a \in R$ such that* $r.\operatorname{ann}(a) = 0$, *then R is commutative.*

PROOF. Let $x, y \in R$. Then $axy = y_r(ax) = y_r(a)x = ayx$. Therefore $a(xy - yx) = 0$, and so $xy = yx$.

It is easy to verify the following:

LEMMA 3. *Let $\{R_i \mid i \in I\}$ be a collection of E-associative rings. If* $\operatorname{Hom}(R_i^+, R_j^+) = 0$ *for all $i \neq j$, then $R = \bigoplus_{i \in I} R_i$, and $S = \prod_{i \in I} R_i$ are E-associative rings.*

An immediate consequence of Lemma 3 is:

COROLLARY 4. *Let $\{R_p \mid p$ a prime$\}$ be E-associative p-rings. Then $R = \prod_{p \text{ prime}} R_p$ is E-associative.*

EXAMPLE 5. Z/pZ is a field and so is E-associative for every prime p. Therefore $\prod_{p \text{ prime}} Z/pZ$ is E-associative, and is in fact an E-ring. The ring $\bigoplus_{p \text{ prime}} Z/pZ$ is E-associative, but is not an E-ring.

LEMMA 6. *Let R be an E-associative ring with $R^+ = (a) \oplus B$.*

(1) If a is torsion free, then there exists an integer m such that $xa = mx$ for all $x \in R$.

(2) If $|a| = n$ then there exists an integer m, with $0 \leq m \leq n - 1$, such that $xa = mx$ for all $x \in R$ satisifying $|x| \mid n$.

PROOF. Since (a) is a right ideal in R by Lemma 1, there exists an integer m such that $a^2 = ma$. If $|a| = n$, then $0 \leq m \leq n - 1$. Suppose that either a is torsion free and $x \in R$, or that $|a| = n$, and $x \in R$, with $|x| \mid n$. There exists $\varphi \in E(R^+)$ satisfying $\varphi(a) = x$. Therefore $xa = \varphi(a)a = \varphi(a^2) = m\varphi(a) = mx$.

LEMMA 7. *Let $R = \bigoplus_{i<\omega} A_i$. If $R_n = \bigoplus_{i=1}^n A_i$ is E-associative for every positive integer n, then R is E-associative.*

PROOF. Suppose that R_n is E-associative for every positive integer n; let $\varphi \in E(R^+)$ and let $a, b \in R$. There exists a positive integer n such that a, b, ab, $\varphi(a)$, and $\varphi(ab)$ all belong to R_n. Let π be the natural projection of R^+ onto R_n^+ along $\bigoplus_{i>n} A_i$. The restriction of $\psi = \pi\varphi$ to R_n^+ belongs to $E(R_n^+)$, and so $\psi(ab) = \psi(a)b$. Since $\varphi(ab)$ and $\varphi(a)$ belong to R_n, it follows that $\psi(ab) = \varphi(ab)$, and $\psi(a) = \varphi(a)$, i.e., $\varphi(ab) = \varphi(a)b$.

LEMMA 8. *Let R be an E-associative ring, D the maximal divisible subgroup of R^+, and let $R^+ = B \oplus D$. Then $BD = RD_t = 0$. If B is not a torsion group then $RD = 0$.*

PROOF. $BD \subseteq B$ by Lemma 1, but $(bD)^+$ is divisible for all $b \in B$, and so $BD \subseteq B \cap D = 0$. Since $BD_t \subseteq BD = 0$, and $DD_t = 0$ by [1, 1.4.7], it follows that $RD_t = 0$. Suppose there exists $b \in B$, $b \neq 0$, and b is torsion free. Let $d, d' \in D$. There exists a homomorphism $\varphi\colon (b) \longrightarrow D$ satisfying $\varphi(b) = d$. Since D is injective in the category of abelian groups, φ can be extended to a homomorphism $\varphi\colon R^+ \longrightarrow D$. Hence $dd' = \varphi(b)d' = \varphi(bd')$. However $bd' \in BD = 0$, and so $D^2 = 0$.

LEMMA 9. *Let R be an E-associative ring, and let $a \in R$ such that $aR = Ra = R$. Then R is an E-ring.*

PROOF. There exists $e \in R$ such that $ae = a$. Clearly e is a right unity for R. Similarly there exists $f \in R$ such that $fa = a$, and f is a left unity for R. Therefore $f = fe = e$ is a unity for R, and R is an E-ring.

Lemma 2 and Lemma 9 yield:

COROLLARY 10. *Let R be an E-associative ring. If there exist $a, b \in R$ such that $aR = R$, and $r.\mathrm{ann}(b) = 0$, then R is an E-ring.*

By employing Lemma 1, and the argument used to prove Lemma 7, one can easily prove the following two results:

LEMMA 11. *Let $R = \prod_{i\in I} R_i$ be an E-associative ring. Then $T = \bigoplus_{i\in I} R_i$ is E-associative.*

The torsion case.

COROLLARY 12. *A torsion ring R is E-associative if and only if R_p is E-associative for every prime p.*

PROOF. A simple consequence of Lemmas 1 and 3.

Clearly, every zero-ring is E-associative, so the E-associative rings R of interest are those satisfying $R^2 \neq 0$. By Corollary 4, the problem of classifying E-associative torsion rings reduces to the case of E-associative p-rings, p a prime.

LEMMA 13. *Let R be an E-associative p-ring, p a prime, such that $R^2 \neq 0$. Then R^+ is reduced.*

PROOF. Suppose that $R^+ = A \oplus D$ with D divisible. It is well known that $DR = RD = 0$, [1, 1.4.7], [3, Theorem 120.5]. It therefore suffices to show that if $D \neq 0$, then $A^2 = 0$. Suppose there exist $a, b \in A$ such that $ab \neq 0$. There exists a homomorphism $\varphi\colon (ab) \longrightarrow D$, with $\varphi(ab) \neq 0$. Since D is injective in the category of abelian groups, φ can be extended to a homomorphism $\varphi\colon R^+ \longrightarrow D$. Hence $\varphi(ab) = \varphi(a)b \in DR = 0$, a contradiction.

THEOREM 14. *Let G be a p-group, p a prime. G is the additive of an E-associative ring R satisfying $R^2 \neq 0$ if and only if G is bounded.*

PROOF. Let R be an E-associative ring with $R^+ = G$, and suppose that G is not bounded. Let B be a basic subgroup of G. Lemma 13 implies that B is not bounded. Every p-ring R with B a basic subgroup of R^+ satisfies $R^2 = B^2$, [1, 1.4.6], [3, Theorem 120.1]. It therefore suffices to show that $B^2 = 0$. Let $B = \bigoplus_{i\in I}(b_i)$, and let $i \in I$. Since (b_i) is a direct summand of R^+, Lemma 6 yields that there exists an integer m_i, with $0 \leq m_i < |b_i|$ such that $xb_i = m_i x$ for all $x \in R$ with $|x| \leq |b_i|$. Let $|b_i| = p^n$, and let $j \in I$ such that $|b_j| = p^m$ with $m \geq 2n$. Since $|p^{m-n}b_j| = |b_i|$, it follows that $p^{m-n}b_j \cdot b_i = m_i p^{m-n} b_j$.

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However $m - n \geq n$, and so $p^{m-n}b_j \cdot b_i = b_j(p^{m-n}b_i) = b_j \cdot 0 = 0$. Therefore $m_i = 0$, *i.e.*,

$$(*) \qquad xb_i = 0 \text{ for all } x \in R \text{ with } |x| \leq |b_i|, \text{ and for all } i \in I.$$

Let $k \in I$ such that $|b_k| > |b_i|$. For every $j \in I$ put

$$c_j = \begin{cases} b_j, & \text{for } j \neq k \\ b_i + b_k & \text{for } j = k \end{cases}$$

Then $B = \bigoplus_{j \in I}(c_j)$. By the above argument $c_k^2 = 0$, and so $b_i^2 + b_k^2 + b_i b_k + b_k b_i = 0$. The first 3 summands in the left hand side of the last inequality are zero by equality $(*)$, and so $b_k b_i = 0$. Therefore $b_j b_i = 0$ for all $i, j \in I$, and so $B^2 = 0$.

Conversely, let G be a bounded p-group. Then $G = \bigoplus_{i \in I}(a_i)$. Choose $k \in I$ such that $|a_k|$ is maximal. Let R be the ring with $R^+ = G$, and multiplication induced by the following products:

$$a_i a_j = \begin{cases} a_i, & j = k \\ 0, & j \neq k. \end{cases}$$

It is readily seen that R is E-associative.

Observe that the E-associative ring just constructed in the proof of Theorem 14 is not commutative as opposed to E-rings, which are all commutative, [6, Lemma 6]. The element a_k is a right unity in R, so an E-associative ring with right unity need not be an E-ring.

COROLLARY 15. *Let R be an E-associative ring. Then R_p is E-associative for every prime p. If $R_p^2 \neq 0$, then R_p is a direct summand of R^+.*

PROOF. Let B be a basic subgroup of R_p. Then $B = \bigoplus_{i<\omega} A_i$, with $A_i = \bigoplus Z(p^i)$. For every positive integer n, let $B_n = \bigoplus_{i=1}^n A_i$. Since B_n is a direct summand of R^+, Lemma 1 yields that B_n is an E-associative ring for every positive integer n, so B is E-associative by Lemma 7. If $B^2 = 0$, then $R_p^2 = 0$, [1, 1.4.6], and so R_p is E-associative. If $B^2 \neq 0$, then B is bounded by Theorem 14, and $R_p = B \oplus D$ with D a divisible group. Since B is a pure bounded subgroup of R^+, and D is divisible, it follows that R_p is a direct summand of R^+, [3, Theorem 27.5 and Theorem 21.2]. Therefore R_p is E-associative by Lemma 1. Actually, $D = 0$, by Lemma 13.

Corollary 12 and Theorem 14 yield a complete description of the additive groups of torsion E-associative rings, which by Corollary 15 is a description of the torsion part of an arbitrary E-associative ring. To determine the multiplicative structure of the torsion part of an E-associative ring R, it suffices to consider R a bounded p-ring. The bounded p-case is settled as follows:

THEOREM 16. *Let $G = \bigoplus_{i \in I}(a_i)$ be a bounded p-group, with $|a_i| = p^{k_i}$ for each $i \in I$, and let n be the greatest positive integer such that there exists $i \in I$ with $k_i = n$. For each $i \in I$ let m_i be an arbitrary integer satisfying $0 \leq m_i < p^{k_i}$ if $k_i > \frac{n}{2}$, and let*

$m_i = 0$ if $k_i \leq \frac{n}{2}$. A ring R with $R^+ = G$ is E-associative if and only if multiplication in R is determined by the following products:

$$a_i a_j = \begin{cases} m_j a_i & \text{if } k_i \leq k_j \\ n_i a_i & \text{if } k_i > k_j, \text{ with } n_i \text{ any integer satisfying } n_i \equiv m_j \pmod{p^{k_j}} \text{ and} \\ & n_i \equiv n_{i'} \pmod{p^{k'_i}} \text{ for all } i' \in I \text{ satisfying } k_i \geq k_{i'} > k_j \end{cases}$$

for all $i, j \in I$.

PROOF. Let R be an E-associative ring with $R^+ = G$, and let $i, j \in I$. There exists an integer m_j satisfying $0 \leq m_j < p^{k_j}$ such that $a_i a_j = m_j a_i$ if $k_i \leq k_j$ by Lemma 6. Suppose that $k_i > k_j$; Lemma 1 yields that $a_i a_j = n_i a_i$ for some integer n_i. Since $|p^{k_i - k_j} a_i| = |a_j|$, it follows from Lemma 6 that $p^{k_i-k_j} a_i \cdot a_j = m_j p^{k_i - k_j} a_i$. However $p^{k_i - k_j} a_i a_j = n_i p^{k_i - k_j} a_i$, and so $(n_i - m_j) p^{k_i - k_j} a_i = 0$, which implies that $n_i \equiv m_j \pmod{p^{k_j}}$. Let $i' \in I$ such that $k_i \geq k_{i'} > k_j$. As above $a_{i'} a_j = n_{i'} a_{i'}$. There exists $\varphi \in E(G)$ such that $\varphi(a_i) = a_{i'}$. Therefore $n_{i'} a_{i'} = a_{i'} a_j = \varphi(a_i) a_j = \varphi(a_i a_j) = \varphi(n_i a_i) = n_i \varphi(a_i) = n_i a'_i$. Hence $(n_i - n_{i'}) a_{i'} = 0$, and so $n_i \equiv n_{i'} \pmod{p^{k'_i}}$.

Conversely, let R be a ring with $R^+ = G$, and multiplication induced by the above products. Let $\varphi \in E(G)$, and let $i, j \in I$. If $k_i \leq k_j$ then $\varphi(a_i a_j) = \varphi(m_j a_i) = m_j \varphi(a_i)$. Since $|\varphi(a_i)| \leq |a_i| \leq p^{k_j}$, it follows from Lemma 7 that $\varphi(a_i) a_j = m_j \varphi(a_i)$, and so $\varphi(a_i a_j) = \varphi(a_i) a_j$. If $k_i > k_j$ then $\varphi(a_i a_j) = n_i \varphi(a_i)$. If $|\varphi(a_i)| \leq |a_j|$, then $\varphi(a_i) a_j = m_j \varphi(a_i)$ by Lemma 7. Since $n_i \equiv m_j \pmod{p^{k_j}}$, it follows that $\varphi(a_i) a_j = n_i \varphi(a_i) = \varphi(a_i a_j)$. It remains to consider $p^{k_i} \geq |\varphi(a_i)| > p^{k_j}$. In this case $\varphi(a_i) = \sum_{t=1}^m s_t a_t$, with $|a_t| \leq p^{k_i}$, and s_t an integer for all $1 \leq t \leq m$. Since $\varphi(a_i) a_j = \sum_{t=1}^m s_t (a_t a_j) = \sum_{t=1}^m s_t n_t a_t$, and $\varphi(a_i a_j) = n_i (\sum_{t=1}^m s_t a_t)$, it suffices to show that $n_i a_t = n_t a_t$ for all $1 \leq t \leq m$. This follows from the fact that $n_i \equiv n_t \pmod{p^{k_t}}$.

The torsion free case. The E-associative rings with completely decomposable torsion free additive groups will now be determined. First some notation will be introduced. Let $G = \bigoplus_{i \in I} (e_i)_*$ be a completely decomposable torsion free group. The elements e_i will be chosen so that $h(e_i) = t(e_i)$ for all $i \in I$. Let

$$J = \big\{ j \in I \mid t(e_j) \text{ is minimal in the type-set of } G,\, t(e_j) \text{ is idempotent, and for } i \in I \text{ such that } t(e_i) \text{ is incomparable with } t(e_j), \text{ there does not exist } k \in I \text{ such that } t(e_k) \geq t(e_i), \text{ and } t(e_k) \geq t(e_j) \big\}.$$

LEMMA 17. *Let G be as above, let R be an E-associative ring with $R^+ = G$, and let $j \in I$. There exists a rational number r_j such that $r_j e_j \in (e_j)_*$, and $e_i e_j = r_j e_i$ for all $i \in I$ for which $t(e_i) \geq t(e_j)$. If $t(e_j)$ is not idempotent, then $r_j = 0$.*

PROOF. $e_j^2 \in (e_j)_*$ by Lemma 1, so $e_j^2 = r_j e_j$, with r_j a rational number satisfying $r_j e_j \in (e_j)_*$. Let $i \in I$ such that $t(e_i) \geq t(e_j)$. There exists $\varphi \in E(G)$ such that $\varphi(e_j) = e_i$. Therefore $e_i e_j = \varphi(e_j) e_j = \varphi(e_j^2) = \varphi(r_j e_j) = r_j \varphi(e_j) = r_j e_i$. If $t(e_j)$ is not idempotent, then $t(r_j e_j) = t(e_j^2) > t(e_j)$ which implies that $r_j = 0$.

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THEOREM 18. *Let G and J be as above, and let R be a ring with $R^+ = G$. For every $j \in J$ let r_j be a rational number such that $r_je_j \in (e_j)_*$. Then R is E-associative if and only if multiplication in R is determined by the following products:*

$$e_ie_j = \begin{cases} r_je_i & \text{if } j \in J,\ t(e_i) \geq t(e_j) \\ 0 & \text{otherwise} \end{cases}.$$

PROOF. Let R be an E-associative ring with $R^+ = G$, and let $i,j \in I$. By Lemma 17, there exists a rational number r_j such that $r_je_j \in (e_j)_*$ and $e_ie_j = r_je_i$ if $t(e_i) \geq t(e_j)$. Since $e_ie_j \in (e_i)_*$ by Lemma 1, and $t(e_ie_j) \geq t(e_j)$, it follows that $e_ie_j = 0$ if $t(e_i) \not\geq t(e_j)$. It remains to show that $r_j = 0$ for $j \notin J$. If $t(e_j)$ is not minimal in the type-set of G, then there exists $i \in I$ such that $t(e_i) < t(e_j)$, and so $e_ie_j = 0$. There exists $\varphi \in E(G)$ such that $\varphi(e_i) = e_j$. Therefore $r_je_j = e_j^2 = \varphi(e_i)e_j = \varphi(e_ie_j) = 0$, and so $r_j = 0$. If $t(e_j)$ is not idempotent, then $r_j = 0$ by Lemma 17. Suppose there exist $i,k \in I$ such that $t(e_i)$ and $t(e_j)$ are incomparable, but $t(e_k) \geq t(e_i)$, and $t(e_k) \geq t(e_j)$. Then $e_ke_j = r_je_k$. Since $t(e_i) \not\geq t(e_j)$ it follows that $e_ie_j = 0$. There exists $\varphi \in E(G)$ such that $\varphi(e_i) = e_k$. Therefore $r_je_k = e_ke_j = \varphi(e_i)e_j = \varphi(e_ie_j) = 0$, and so $r_j = 0$.

Conversely, suppose that R is a ring with $R^+ = G$, and multiplication in R is determined by the above products. Let $\varphi \in E(G)$, and let $i,j \in I$. If $j \in J$, and $t(e_i) \geq t(e_j)$, then $\varphi(e_ie_j) = r_j\varphi(e_i)$. Since $t[\varphi(e_i)] \geq t(e_i)$, it follows that $\varphi(e_i) = \sum_{k=1}^m s_ke_k$ with s_k a rational number, and $t(e_k) \geq t(e_i)$ for all $1 \leq k \leq m$. Hence $\varphi(e_i)e_j = \sum_{k=1}^m s_k(e_ke_j) = r_j(\sum_{k=1}^m s_ke_k) = r_j\varphi(e_i) = \varphi(e_ie_j)$. If $j \in J$, and $t(e_i) \not\geq t(e_j)$ then $\varphi(e_ie_j) = \varphi(0) = 0$. Since $t(e_i) \not< t(e_j)$ by the minimality of $t(e_j)$ it follows that $t(e_i)$ and $t(e_j)$ are incomparable. Since $t[\varphi(e_i)] \geq t(e_i)$, either $\varphi(e_i) = 0$, or $t[\varphi(e_i)] \not\geq t(e_j)$. In either case, $\varphi(e_i)e_j = 0 = \varphi(e_ie_j)$.

If $j \notin J$, then $\varphi(e_ie_j) = \varphi(e_i)e_j = 0$.

An argument similar to that used in the proof of Lemma 7 yields:

LEMMA 19. *Let R be a ring with R^+ a separable torsion free group. Then R is E-associative if and only if every (finite rank) completely decomposable direct summand of R^+ is E-associative.*

A generalization. E-associativity is just one of a set of ring properties which will now be defined.

DEFINITION. Let $n \geq 2$ be a positive integer, let i be a fixed integer, $1 \leq i \leq n$, and let $\sigma \in S_n$. A ring R is a (σ, i, n)-ring if $\varphi(x_1 \cdots x_n) = x_{\sigma(1)} \cdots x_{\sigma(i-1)}\varphi(x_{\sigma(i)})x_{\sigma(i+1)} \cdots x_{\sigma(n)}$ for all $\varphi \in E(R^+)$, and all $x_1, \ldots, x_n \in R$.

The E-associative rings are precisely the $(1,1,2)$-rings, where 1 is the identity in S_2.

THEOREM 20. *Let R be a ring with unity. The following are equivalent:*
(1) R is a (σ, i, n)-ring for all $n \geq 2$, all $1 \leq i \leq n$, and all $\sigma \in S_n$.
(2) There exists a positive integer $n \geq 2$, an integer $1 \leq i \leq n$, and $\sigma \in S_n$ such that R is a (σ, i, n)-ring.

(3) R is an E-ring.

PROOF. Clearly 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3): Let R be a (σ, i, n)-ring, and let $\varphi \in E(R^+)$. It suffices to show that either $\varphi(x) = \varphi(1)x$ for all $x \in R$, or that $\varphi(x) = x \cdot \varphi(1)$ for all $x \in R$, [6, p. 65, Lemma 6 and Definition]. If $i = n$, then choose $x_{\sigma(1)} = x$, and $x_j = 1$ for all $1 \leq j \leq n$ with $j \neq \sigma(1)$. Then $\varphi(x) = \varphi(x_1 \cdots x_n) = x_{\sigma(1)} \cdots x_{\sigma(n-1)} \varphi(x_{\sigma(n)}) = x\big(\varphi(1)\big)$. If $i \neq n$, then choose $x_{\sigma(n)} = x$, and $x_j = 1$ for all $1 \leq j \leq n$ with $j \neq \sigma(n)$. Then $\varphi(x) = \varphi(x_1 \cdots x_n) = x_{\sigma(1)} \cdots \varphi(x_{\sigma(n)}) = \varphi(1)x$.

3) $\Rightarrow$ 1): Let R be an E-ring. Then R is commutative, and $\varphi(x) = \varphi(1)x$ for all $\varphi \in E(R^+)$, and all $x \in R$, [6, Lemma 6]. This clearly implies that $\varphi(x_1 \cdots x_n) = x_{\sigma(1)} \cdots \varphi(x_{\sigma(i)}) \cdots x_{\sigma(n)} = \varphi(1)x_1 \cdots x_n$ for every positive integer $n \geq 2$, all $1 \leq i \leq n$, all $\sigma \in S_n$, and all $x_1, \ldots, x_n \in R$.

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