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On the topological complexity and zero-divisor cup-length of real Grassmannians

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Topological complexity naturally appears in the motion planning in robotics. In this paper we consider the problem of finding topological complexity of real Grassmann manifolds $G_k(\mathbb{R}^n)$. We use cohomology methods to give estimates on the zero-divisor cup-length of $G_k(\mathbb{R}^n)$ for various $2 \leqslant k < n$, which in turn give us lower bounds on topological complexity. Our results correct and improve several results from Pavešić (*Proc. Roy. Soc. Edinb. A* **151** (2021), 2013–2029).

Keywords: Topological complexity; Grassmann manifold; zero-divisor cup-length

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1. Introduction

For a path-connected space X we denote its topological complexity by $\mathrm{TC}(X)$. In [**9**] the author considered the problem of finding $\mathrm{TC}(G_k(\mathbb{R}^n))$ for various $2 \leqslant k < n$ (in this paper, $G_k(\mathbb{R}^n)$ denotes the real Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$). Unfortunately, there is a problem with the proof of the main lemma of that paper (lemma 4.4) and the consequential results on the topological complexity (theorems 4.5, 4.8 and 4.12); see [**10**]. In this paper we reconsider this problem, and as an outcome correct and improve several results from [**9**]. As in [**9**], we use the *cohomology method* to obtain our results.

This paper closely follows and builds on the ideas presented in [**9**] (so, for background, motivation and all undefined notions, the reader is advised to consult [**9**]). Throughout the paper we will use, as much as possible, the notation from [**9**]. In particular, we will be working with the *unreduced* topological complexity, as defined by Farber in [**5**] (e.g. by this definition the topological complexity of a contractible space is equal to 1).

The paper is organized as follows. In § 2 we describe the cohomology method mentioned above and give an overview of the cohomology of real Grassmannians. In § 3 we consider the case $k = 2$. We obtain the exact value of the zero-divisor cup-length of $G_2(\mathbb{R}^{2^s+1})$ (denoted by $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+1}))$, and defined in § 2) for $s \geqslant 2$; additionally, for $s \geqslant 3$, $2^s + 4 \leqslant n \leqslant 2^{s+1}$ we prove a lower bound for $\mathrm{zcl}(G_2(\mathbb{R}^n))$. These results show that the value of the zero-divisor cup-length given in [**9**, theorem 4.5] is not correct; what is more interesting, our results improve lower bounds for topological complexity stated in the same theorem. Section 4 is devoted to the case $k = 3$. Separately, we prove lower bounds for $\mathrm{zcl}(G_3(\mathbb{R}^n))$ in the cases $n = 2^s + 1$ (for $s \geqslant 3$), and $2^s + 3 \leqslant n \leqslant 2^{s+1}$ (for $s \geqslant 2$). The first result shows that the corresponding result from [**9**, theorem 4.8] is not correct, and improves the stated lower bound for topological complexity of $G_3(\mathbb{R}^{2^s+1})$ (for $s \geqslant 5$). In § 5 we give a general lower bound for $\mathrm{zcl}(G_k(\mathbb{R}^n))$ (for $k \geqslant 4$). For $k \geqslant 9$ this result improves the bounds stated in [**9**, theorem 4.10].

2. Background and notation

As mentioned in the Introduction, to obtain our results we use the so-called *cohomology method*, which we now (briefly) explain.

Let $\Delta : X \to X \times X$ denote the diagonal map. Then the elements of

$$\mathrm{Ker}(\Delta^* : H^*(X \times X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2))$$

are called *zero-divisors*. Furthermore, the *zero-divisor cup-length* of X, denoted by $\mathrm{zcl}(X)$, is defined to be the maximum number of elements from $\mathrm{Ker}\,\Delta^*$ whose product is non-zero. In [**5**], Farber proved that $\mathrm{zcl}(X)$ gives a lower bound for $\mathrm{TC}(X)$, that is $\mathrm{TC}(X) \geqslant \mathrm{zcl}(X) + 1$. Hence, a lower bound for $\mathrm{zcl}(X)$ immediately gives a lower bound for $\mathrm{TC}(X)$. Note that for every $w \in H^*(X; \mathbb{Z}_2)$ the element

$$z(w) = w \otimes 1 + 1 \otimes w \in H^*(X \times X; \mathbb{Z}_2)$$

is in $\mathrm{Ker}\,\Delta^*$ (since $\Delta^*(z(w)) = w \cdot 1 + 1 \cdot w = 0$). Then, by [**2**, lemma 5.2], $\mathrm{Ker}\,\Delta^*$ is generated as an ideal by these elements, that is by the set $\{z(w) : w \in H^*(X; \mathbb{Z}_2)\}$. So, if $\mathrm{zcl}(X) = t$, then there are classes $x_1, x_2, \ldots, x_t \in H^*(X; \mathbb{Z}_2)$ such that $z(x_1)z(x_2) \cdots z(x_t) \neq 0$.

To get the best possible results on $\mathrm{TC}(G_k(\mathbb{R}^n))$ using the cohomology method, one requires fine understanding of the cohomology algebra $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. There are several ways to describe this algebra; in this paper we will use the one due to Borel (see [**1**]):

$$H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{k,n},$$

where $w_1, w_2, \ldots, w_k$ are the Stiefel–Whitney classes of the canonical k-dimensional vector bundle over $G_k(\mathbb{R}^n)$, and $I_{k,n} = (\overline{w}_{n-k+1}, \overline{w}_{n-k+2}, \ldots, \overline{w}_n)$ is the ideal generated by dual classes.

Although Borel's description of $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ appears simple enough, it turns out that performing concrete calculations in this algebra can be rather difficult. Hence, one usually needs to apply some additional methods and properties of $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. The following result gives an additive basis for this algebra (see, e.g. [**7**, **11**]).

Proposition 2.1. *The set* $B_{k,n-k} = \{w_1^{a_1} \cdots w_k^{a_k} : 0 \leqslant a_1 + \cdots + a_k \leqslant n - k\}$ *is an additive basis for* $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$.

The *height* of a class $c \in \widetilde{H}^*(X; \mathbb{Z}_2)$, denoted by $\mathrm{ht}(c)$, is the largest $m \in \mathbb{N}$ such that $c^m \neq 0$. For $k \geqslant 2$, the height of $w_1 \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ is obtained by Stong in

[12]: if $2 \leqslant k \leqslant n-k$ and s is the unique positive integer such that $2^s < n \leqslant 2^{s+1}$, then

$$\mathrm{ht}(w_1) = \begin{cases} 2^{s+1}-2, & \text{if } k=2 \text{ or } (k,n)=(3,2^s+1), \\ 2^{s+1}-1, & \text{otherwise.} \end{cases} \tag{2.1}$$

In this paper we will often use Stong's method from [12] for calculations in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ (later this method was generalized by Korbaš and Lörinc to all flag manifolds, see [8]). In what follows we briefly explain this method.

Let $\mathrm{Flag}(\mathbb{R}^n)$ denote the *(real) complete flag manifold* ($n \geqslant 2$). Denote by $e_i := w_1(\gamma_i)$ the first Stiefel–Whitney class of the canonical line bundle γ_i over $\mathrm{Flag}(\mathbb{R}^n)$, for $1 \leqslant i \leqslant n$. Then we have the map $\pi : \mathrm{Flag}(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$, given by

$$\pi(S_1,\ldots,S_k,S_{k+1},\ldots,S_n) = (S_1 \oplus \cdots \oplus S_k, S_{k+1} \oplus \cdots \oplus S_n).$$

The following result will be very useful for our calculations in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ (and $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$).

Proposition 2.2.

(1) *The set* $B_n = \{e_1^{a_1}e_2^{a_2}\cdots e_{n-1}^{a_{n-1}} : 0 \leqslant a_i \leqslant n-i\}$ *is an additive basis for* $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$.

(2) $\mathrm{ht}(e_i) = n-1$ *for* $1 \leqslant i \leqslant n$. *In particular* $e_i^n = 0$ *for* $1 \leqslant i \leqslant n$.

(3) *A monomial* $e_1^{a_1}e_2^{a_2}\cdots e_n^{a_n} \in H^{\binom{n}{2}}(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *is non-zero if and only if* $(a_1, a_2, \ldots, a_n)$ *is a permutation of the n-tuple* $(n-1, n-2, \ldots, 1, 0)$.

(4) *If* $u \in H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ *and*

$$v = e_1^{k-1}e_2^{k-2}\cdots e_{k-1}\cdot e_{k+1}^{n-k-1}e_{k+2}^{n-k-2}\cdots e_{n-1} \in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2),$$

then $\pi^*(u)\cdot v \in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, *and* $u \neq 0$ *if and only if* $\pi^*(u)\cdot v \neq 0$.

(5) *For* $1 \leqslant i \leqslant k$, $\pi^*(w_i)$ *is the i-th elementary symmetric polynomial in the variables* $e_1, e_2, \ldots, e_k$.

Heights of the classes $z(w_1)$ and $z(w_k)$ will be very useful in our calculations. In what follows we determine these values.

It turns out that if $\mathrm{ht}(w)$ is known, then $\mathrm{ht}(z(w))$ can easily be calculated. This is proven in lemma 4.3 from [9]. Namely, one has: if $w \in H^*(X;\mathbb{Z}_2)$ and t is the unique non-negative integer such that $2^t \leqslant \mathrm{ht}(w) < 2^{t+1}$, then

$$\mathrm{ht}(z(w)) = 2^{t+1}-1. \tag{2.2}$$

We will apply this identity for $X = G_k(\mathbb{R}^n)$, when $2 \leqslant k \leqslant n-k$. If $2^s < n \leqslant 2^{s+1}$, then (2.1) implies

$$\mathrm{ht}(z(w_1)) = 2^{s+1}-1. \tag{2.3}$$

On the other hand, proposition 2.1 implies $w_k^{n-k} \neq 0$, so $\mathrm{ht}(w_k) = n-k$ (by observing dimension we conclude that $w_k^{n-k+1} = 0$). Hence, if t is the unique non-negative

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integer such that $2^t \leqslant n-k < 2^{t+1}$, then (2.2) implies

$$\mathrm{ht}(z(w_k)) = 2^{t+1} - 1. \tag{2.4}$$

The following lemma will be particularly useful in § 3.

LEMMA 2.3. *Let m, k, $n \in \mathbb{N}$, $k < n$, and $d_1, \ldots, d_m \in \mathbb{N}$ be such that $d_1 + \cdots + d_m \geqslant 2k(n-k)$. If $x_i \in H^{d_i}(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ for $1 \leqslant i \leqslant m$, then*

$$z(x_1) \cdots z(x_m) = 0.$$

Proof. Note that the product $p = z(x_1) \cdots z(x_m)$ is the sum of certain classes of the form $x \otimes y + y \otimes x$, for some x, $y \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. Since p is in dimension at least $2k(n-k) = 2 \dim G_k(\mathbb{R}^n)$, so is $x \otimes y$, and hence x, $y \in H^{k(n-k)}(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ or $x \otimes y = y \otimes x = 0$. There is only one non-zero class in $H^{k(n-k)}(G_k(\mathbb{R}^n); \mathbb{Z}_2)$, namely w_k^{n-k} (by proposition 2.1), and hence $x \otimes y = y \otimes x = 0$ or $x \otimes y = w_k^{n-k} \otimes w_k^{n-k} = y \otimes x$. In both cases $x \otimes y + y \otimes x = 0$, which implies $p = 0$. □

Also, we recall some results from [9] that will be used in our calculations.

LEMMA 2.4.

(a) *If $2^s < n \leqslant 2^{s+1}$, then $w_1^{2^s} w_2^{n-2^s-1} \neq 0$ and $w_1^{2^s} w_2^{n-2^s} = 0$ in $H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$.*

(b) *If $2^s + 3 \leqslant n \leqslant 2^{s+1}$ and $t = n - 2^s$, then $w_1^{2^s} w_2^{2^s} w_3^{t-3} \neq 0$ in $H^*(G_3(\mathbb{R}^n); \mathbb{Z}_2)$.*

Throughout the paper we use the same notation as in this section.

Finally, let us say a few words on lemma 4.4 from [9] and our strategy that bypasses the application of this lemma. In lemma 4.4 from [9] the author assumes that $u_1, \ldots, u_n \in H^*(X; \mathbb{Z}_2)$ and $k_1, \ldots, k_n \in \mathbb{N}$ are such that $u_1^{k_1} \cdots u_n^{k_n} \neq 0$, and wants to prove that $A = z(u_1)^{2^{r_1}-1} \cdots z(u_n)^{2^{r_n}-1} \neq 0$, where r_i is the unique integer such that $2^{r_i-1} \leqslant k_i < 2^{r_i}$ for $1 \leqslant i \leqslant n$. For this he notices that after expanding A one summand is $u_1^{k_1} \cdots u_n^{k_n} \otimes u_1^{2^{r_1}-k_1-1} \cdots u_n^{2^{r_n}-k_n-1}$, which is nonzero, and from this immediately concludes that $A \neq 0$. As we will see in the proofs of our results, the problem is that the set

$$S = \{(l_1, \ldots, l_n) \,:\, 0 \leqslant l_i \leqslant 2^{r_i} - 1,\, u_1^{l_1} \cdots u_n^{l_n} = u_1^{k_1} \cdots u_n^{k_n}\}$$

can contain more than one element, and hence that the corresponding summands of A with the first coordinate equal to $u_1^{k_1} \cdots u_n^{k_n}$ may cancel out. So, in our proofs we choose the n-tuple $(k_1, \ldots, k_n)$ a bit more carefully to ensure that

$$\sum_{(l_1,\ldots,l_n)\in S} u_1^{2^{r_1}-l_1-1} \cdots u_n^{2^{r_n}-l_n-1} \neq 0$$

and that this further leads to $A \neq 0$ (note: in our applications the degree of $z(u_i)$ in A will not always be $2^{r_i} - 1$, so we will have slightly different formulas than the one given above).

3. Zero-divisor cup-length of $G_2(\mathbb{R}^n)$

Let s be the unique integer such that $2^s < n \leqslant 2^{s+1}$. In this section we consider $\operatorname{zcl}(G_2(\mathbb{R}^n))$. We note that propositions 3.7 and 3.10, that we prove in this section, show that the corresponding results of [9, theorem 4.5] are not correct (see also remark 3.9). Fortunately, correct versions give better lower bounds for the topological complexity of $G_2(\mathbb{R}^n)$.

We will compare our results with the following upper bound from [9] (this result is a consequence of a general result from [3, theorem 1]).

PROPOSITION 3.1. *If $1 \leqslant k < n$, then $\operatorname{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k)$. In fact, if $k \neq 1$ and $(k, n) \neq (2, 2^d+1)$ for all $d \in \mathbb{N}$, then $\operatorname{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k)-1$.*

3.1. Preliminary lemmas

Let n be a positive integer and $n = \sum_{i=0}^{t} \alpha_i \cdot 2^i$, where $\alpha_i \in \{0, 1\}$ for $0 \leqslant i \leqslant t$ and $\alpha_t = 1$, its representation in base 2. Then we write $n := (\alpha_t, \ldots, \alpha_1, \alpha_0)_2$.

As we use $\mathbb{Z}_2$ coefficient the following special case of Lucas' theorem will be particularly useful to us: if $n := (\alpha_t, \ldots, \alpha_1, \alpha_0)_2$ and $m := (\beta_r, \ldots, \beta_1, \beta_0)_2$, then

$$\binom{n}{m} \equiv 1 \pmod 2 \qquad \text{if and only if} \qquad t \geqslant r \quad \text{and} \quad \alpha_i \geqslant \beta_i \text{ for } 0 \leqslant i \leqslant r.$$

We will use the following two consequences of Lucas' theorem throughout the paper. Let $w \in H^*(X; \mathbb{Z}_2)$. By Lucas' theorem, $\binom{2^m}{i}$ is even for $1 \leqslant i \leqslant 2^m - 1$, and so

$$z(w)^{2^m} = (w \otimes 1 + 1 \otimes w)^{2^m} = w^{2^m} \otimes 1 + 1 \otimes w^{2^m}.$$

On the other hand, by Lucas' theorem $\binom{2^m-1}{i}$ is odd for all $0 \leqslant i \leqslant 2^m - 1$, and hence

$$z(w)^{2^m-1} = (w \otimes 1 + 1 \otimes w)^{2^m-1} = \sum_{i=0}^{2^m-1} w^i \otimes w^{2^m-1-i}.$$

We will also need the following result.

LEMMA 3.2. *Let n be a non-negative integer. Then:*

(a) *$\binom{2n}{n}$ is odd if and only if $n = 0$;*

(b) *$\binom{2n}{n+1}$ is odd if and only if $n = 2^{t+1} - 1$ for some $t \in \mathbb{N}_0$.*

Proof. Part (a) immediately follows from Lucas' theorem.

For part (b) we note that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$ is the n-th Catalan number. Then the result follows from part (a) and the fact that C_n (for $n \geqslant 1$) is odd if and only if $n = 2^{t+1} - 1$ for some $t \in \mathbb{N}_0$ (see [4]). □

LEMMA 3.3. *Let $0 \leqslant m \leqslant n-2$ and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1-m} \in \mathbb{Z}_2$. Then:*

(a) *$\sum_{i=0}^{n-1-m} \alpha_i e_1^{m+i} e_2^{n-1-i} = 0$ in $H^*(\operatorname{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ iff $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1-m}$;*

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(b) *for a polynomial* $p \in H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ *in classes* e_1 *and* e_2 *one has*

$$p \cdot e_3^{n-3} e_4^{n-4} \cdots e_{n-1} = 0 \quad \textit{in} \quad H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$$

if and only if $p = 0$ *in* $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$.

Proof. (a) By proposition 2.1 from [6] we have $e_2^{n-1} = e_1^{n-1} + e_1^{n-2} e_2 + \cdots + e_1 e_2^{n-2}$ (we use this proposition for $m = 1$, $k = n-1$ and $i = n-2$). Since $e_1^n = 0$ (by proposition 2.2(2)), we have

$$\sum_{i=0}^{n-1-m} \alpha_i e_1^{m+i} e_2^{n-1-i} = \sum_{i=1}^{n-1-m} (\alpha_i + \alpha_0) e_1^{m+i} e_2^{n-1-i}.$$

Since $e_1^{m+1} e_2^{n-2}$, $e_1^{m+2} e_2^{n-3}$, ..., $e_1^{n-1} e_2^m$ are in the additive basis B_n (from proposition 2.2(1)), the last sum is zero if and only if $\alpha_1 + \alpha_0 = \alpha_2 + \alpha_0 = \cdots = \alpha_{n-1-m} + \alpha_0 = 0$, i.e. if and only if $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1-m}$.

(b) As in part (a) we use the identities $e_2^{n-1} = e_1^{n-1} + e_1^{n-2} e_2 + \cdots + e_1 e_2^{n-2}$ and $e_1^n = e_2^n = 0$ to express p in the form $\sum \alpha_{i,j} e_1^i e_2^j$, where $\alpha_{i,j} \in \{0, 1\}$, $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-2$. Then $\sum \alpha_{i,j} e_1^i e_2^j e_3^{n-3} e_4^{n-4} \cdots e_{n-1}$ ($= p e_3^{n-3} e_4^{n-4} \cdots e_{n-1}$) is a sum of the elements from the basis B_n from proposition 2.2(1); so this sum is zero if and only if $\alpha_{ij} = 0$ for all i, j, i.e. if and only if $p = 0$ (since p is also represented in the basis B_n).

□

REMARK 3.4. We will use the following consequence of part a) of this lemma. Let $p = \sum_{i=0}^{b-a} \alpha_i e_1^{a+i} e_2^{b-i} \in H^{a+b}(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ for some $0 \leqslant a \leqslant n-2$, $a \leqslant b \leqslant n-1$. If there exist $0 \leqslant i' \neq i'' \leqslant b-a$ such that $\alpha_{i'} = 0$ and $\alpha_{i''} = 1$, then $p \neq 0$.

Furthermore, if $q \in H^c(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$, where $c \leqslant 2n-3$, is written as a sum of some monomials of the form $e_1^i e_2^j$, then after removing all summands with $i \geqslant n$ or $j \geqslant n$ (since they are 0 by proposition 2.2(2)), we get that q is written in the same way as p above.

LEMMA 3.5. *If* $2^s < n \leqslant 2^{s+1}$ *and* $a, b \in \mathbb{N}_0$ *are such that* $a + 2b = 2(n-2)$, *then* $w_1^a w_2^b \neq 0$ *in* $H^{2n-4}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ *if and only if*

$$(a, b) = (2^{l+1} - 2, n - 2^l - 1) \quad \textit{for some } 0 \leqslant l \leqslant s.$$

Proof. By proposition 2.2(4), $w_1^a w_2^b \neq 0$ in $H^{2n-4}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ if and only if

$$\pi^*(w_1^a w_2^b) e_1 e_3^{n-3} \cdots e_{n-1} = (e_1 + e_2)^a (e_1 e_2)^b e_1 e_3^{n-3} \cdots e_{n-1} \neq 0$$

in $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$. After expanding we have

$$(e_1 + e_2)^a (e_1 e_2)^b e_1 e_3^{n-3} \cdots e_{n-1} = e_3^{n-3} \cdots e_{n-1} \sum_{i=0}^{a} \binom{a}{i} e_1^{i+1+b} e_2^{a-i+b}.$$

Note that by proposition 2.2(3) the only non-zero monomials in this sum are the ones for i that satisfies $(i+1+b,\ a-i+b) \in \{(n-1,\ n-2),\ (n-2,\ n-1)\}$ and $\binom{a}{i}$ is odd, i.e. $i \in \{n-2-b,\ n-3-b\}$ and $\binom{a}{i}$ is odd.

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If $i = n-2-b$, then $\binom{a}{i} = \binom{2(n-2-b)}{n-2-b} = \binom{2m}{m}$ (here $2m = 2(n-2-b) = a$). By lemma 3.2 this number is odd only if $m=0$, i.e. $(a, b) = (0, n-2)$.

Let us now consider the case $i = n-3-b$. Then $\binom{a}{i} = \binom{2(n-2-b)}{n-3-b} = \binom{2m}{m-1} = \binom{2m}{m+1}$ (again $2m = 2(n-2-b) = a$). By lemma 3.2 this number is odd if and only if $m = 2^l - 1$ for some $l \geqslant 1$. Then $a = 2^{l+1} - 2$ and $b = n - 2^l - 1 \geqslant 0$, which completes our proof. □

REMARK 3.6. If $w_1^a w_2^b \neq 0$ and $a + 2b = 2(n-2)$, then, by proposition 2.1, $w_1^a w_2^b = w_2^{n-2}$ (since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$).

3.2. Some exact values

In this section we calculate $\mathrm{zcl}(G_2(\mathbb{R}^n))$ for $n = 2^s + 1$.

In the proof of the main result we will use the following observation. Let $n \geqslant 4$. Then, by proposition 2.1, every class in $H^1(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ is of the form αw_1, $\alpha \in \mathbb{Z}_2$, while every class in $H^2(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ is of the form $\beta w_1^2 + \gamma w_2$, $\beta, \gamma \in \mathbb{Z}_2$. Since $z(w_1^2) = z(w_1)^2$, we conclude: if $\mathrm{zcl}(G_2(\mathbb{R}^n)) = t$, then there are $a, b, c \in \mathbb{N}_0$ such that $z(w_1)^a z(w_2)^b z(x_1) \cdots z(x_c) \neq 0$, where $a + b + c = t$ and $x_1, \ldots, x_c$ are some classes of $H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ each in dimension at least 3.

PROPOSITION 3.7. *For $s \geqslant 2$ and $n = 2^s + 1$ one has*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) = 2^{s+1} + 2^s - 4 \quad \textit{and} \quad \mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s - 3.$$

Proof. First, we prove that $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-3} \neq 0$. After expanding, we consider all summands of the form $w_2^{n-2} \otimes x$, for some $x \in H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$. By lemma 3.5 each such summand is of the form $w_1^{2^{l+1}-2} w_2^{2^s-2^l} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3}$ (for $l \geqslant 2$) with coefficient $\binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s-3}{2^s-2^l}$. By Lucas' theorem each of these binomial coefficients is odd, so $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-3}$ contains $w_2^{n-2} \otimes \sum_{l=2}^{s} w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3}$. Since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ (by proposition 2.1), it is enough to prove $\sum_{l=2}^{s} w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3} \neq 0$ (in $H^*(G_2(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$).

Note that by lemma 2.4, $w_1^{2^s} w_2 = 0$, and so $w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3} = 0$ for $2 \leqslant l \leqslant s-1$. Hence, it is enough to prove that $w_1 w_2^{2^s-3} = w_1 w_2^{n-4} \neq 0$, which follows from the fact that $w_1 w_2^{n-4}$ is in the additive basis $B_{2,n-2}$ (proposition 2.1). So, $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+1})) \geqslant 2^{s+1} + 2^s - 4$.

Let us now prove that $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+1})) \leqslant 2^{s+1} + 2^s - 4$. Suppose that this is not the case and let $a, b, c \in \mathbb{N}_0$ and $x_1, \ldots, x_c \in H^*(G_2(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$ be some classes each in dimension at least 3, such that $a + b + c \geqslant 2^{s+1} + 2^s - 3$ and $z(w_1)^a z(w_2)^b z(x_1) \cdots z(x_c) \neq 0$. By lemma 2.3, we have $a + 2b + 3c \leqslant 4(2^s - 1) - 1 = 2^{s+2} - 5$, and hence $b + 2c \leqslant 2^s - 2$. Furthermore, since $z(w_1)^{2^{s+1}} = 0$ (by (2.3)), we have $a \leqslant 2^{s+1} - 1$ and hence $b + c = (a + b + c) - a \geqslant 2^s - 2$. This implies $b = 2^s - 2$ and $c = 0$. Finally, $a + b + c \geqslant 2^{s+1} + 2^s - 3$ and $a \leqslant 2^{s+1} - 1$ imply $a = 2^{s+1} - 1$.

So, it is enough to prove $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-2} = 0$. Suppose that this is not the case. Note that the dimension of A is $2^{s+1} - 1 + 2(2^s - 2) = 4(n-2) - 1$, so every summand of A is of the form $x' \otimes x''$ where one of the classes x' and x''

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has dimension $2(n-2)$ and the other $2(n-2)-1$. Note that, by proposition 2.1, the only class in $H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ of dimension $2(n-2)$ (resp. $2(n-2)-1$) is w_2^{n-2} (resp. $w_1w_2^{n-3}$). By symmetry, this and $A \neq 0$ imply $A = w_2^{n-2} \otimes w_1w_2^{n-3} + w_1w_2^{n-3} \otimes w_2^{n-2}$. Now, we proceed as in the first part of the proof to prove that the coefficient of $w_2^{n-2} \otimes w_1w_2^{n-3}$ in A is zero. By lemma 3.5 each such summand in $A = z(w_1)^{2^{s+1}-1}z(w_2)^{2^s-2}$ is of the form $w_1^{2^{l+1}-2}w_2^{2^s-2^l} \otimes w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-2}$ (for some $1 \leqslant l \leqslant s$) with coefficient $\binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s-2}{2^s-2^l}$. By Lucas' theorem this coefficient is 1, so it is enough to prove $\sum_{l=1}^{s} w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-2} = 0$.

Again, by lemma 2.4, $w_1^{2^s}w_2 = 0$, so $w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-2} = 0$ for $2 \leqslant l \leqslant s-1$. Hence, the previous sum is equal to $w_1^{2^{s+1}-3} + w_1w_2^{2^s-2}$. By (2.1), $w_1^{2^{s+1}-3} \neq 0$, so $w_1^{2^{s+1}-3} = w_1w_2^{n-3} = w_1w_2^{2^s-2}$, and hence $A = 0$. □

Remark 3.8. By proposition 3.1, $\mathrm{TC}(G_2(\mathbb{R}^{2^s+1})) \leqslant 2^{s+2}-4$, so there is a gap of 2^s-1 between our lower bound and this bound. For example, $9 \leqslant \mathrm{TC}(G_2(\mathbb{R}^5)) \leqslant 12$.

Remark 3.9. Ideas from this paper can be used to prove the following:

(1) If $s \geqslant 1$, then $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+2})) = 3\cdot 2^s - 2$ (one has $z(w_1)^{2^{s+1}-2}z(w_2)^{2^s} \neq 0$). So, by proposition 3.1, $3\cdot 2^s - 1 \leqslant \mathrm{TC}(G_2(\mathbb{R}^{2^s+2})) \leqslant 2^{s+2}-1$.

(2) If $s \geqslant 2$, then $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+3})) = 3\cdot 2^s$ (one has $z(w_1)^{2^{s+1}-1}z(w_2)^{2^s+1} \neq 0$). So, by proposition 3.1, $3\cdot 2^s + 1 \leqslant \mathrm{TC}(G_2(\mathbb{R}^{2^s+3})) \leqslant 2^{s+2}+3$.

Complete proofs of these results can be found in the extended version of this paper which is available on the author's website.

3.3. General bounds for zcl($G_2(\mathbb{R}^n)$)

Let $2^s+4 \leqslant n \leqslant 2^{s+1}$ and $t = n-2^s$. Also, we assume $s \geqslant 3$ (i.e. $n \neq 8$). Furthermore, let r be the unique integer such that $2^{r-1} < t \leqslant 2^r$. Since $t \geqslant 4$, we have $r \geqslant 2$. Let j be the smallest positive integer such that the digit on position j in the binary representation of $t-2$ is equal to 1 (j is well-defined since $t-2 \geqslant 2$); in other words, $t-2$ has the binary representation of the following form

$$t-2 = 2^m + \alpha_{m-1}2^{m-1} + \cdots + \alpha_{j+1}2^{j+1} + 2^j + \alpha_0,$$

for some $\alpha_0, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{m-1} \in \{0,1\}$ and $1 \leqslant j \leqslant m$. Since $2^m \leqslant t-2 \leqslant 2^r-2 \leqslant 2^s-2$, we additionally have $1 \leqslant j \leqslant m < r \leqslant s$.

Proposition 3.10. *If n, s, t, r and j are as above, then*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^r - \varepsilon - 2$$

and $\mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^r - \varepsilon - 1$, *where* $\varepsilon = \begin{cases} 2^j, & \text{if } t \text{ is even} \\ 2^j+1, & \text{otherwise.} \end{cases}$

Proof. It is enough to prove that $z(w_1)^{2^{s+1}-1}z(w_2)^{2^s+2^r-\varepsilon-1} \neq 0$. After expanding, we consider all summands of the form $w_2^{n-2} \otimes x$, for some $x \in H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$.

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By lemma 3.5 each such summand is of the form $w_1^{2^{l+1}-2}w_2^{2^s+t-2^l-1} \otimes w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^r+2^l-\varepsilon-t}$, $0 \leqslant l \leqslant s$, with coefficient $\alpha_l = \binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s+2^r-\varepsilon-1}{2^s+t-2^l-1} = \binom{2^s+2^r-\varepsilon-1}{2^s+t-2^l-1}$. (Note: if $2^r+2^l-\varepsilon-t<0$, then $2^s+2^r-\varepsilon-1<2^s+t-2^l-1$ and hence $\alpha_l=0$, so there is no need to discard summands $\alpha_l w_1^{2^{l+1}-2}w_2^{2^s+t-2^l-1} \otimes w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^r+2^l-\varepsilon-t}$ when $2^r+2^l-\varepsilon-t<0$.) Since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ (by proposition 2.1), it is enough to prove

$$A=\sum_{l=0}^{s}\alpha_l w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^r+2^l-\varepsilon-t} \neq 0 \quad \text{in } H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2).$$

Let us first consider the case when t is even. Then $\varepsilon=2^j$. Note that $2^s+2^r-2^j-1=2^s+2^{r-1}+2^{r-2}+\cdots+2^{j+1}+2^{j-1}+2^{j-2}+\cdots+1$ $(j<r)$. So, by Lucas' theorem, α_0 and α_s are even (since both 2^s+t-2 and $t-1$ have digit 1 on the j-th position in the binary representation), while α_j is odd (since $2^s+t-1-2^j$ has digit 0 on the j-th position in the binary representation).

Let us denote $\tau=2^r-2^j-t$. Note that $t-2+2^j \leqslant 2^r$, i.e. $\tau \geqslant -2$. By proposition 2.2.(4), $A \neq 0$ if and only if

$$\sum_{l=0}^{s}\alpha_l(e_1+e_2)^{2^{s+1}-2^{l+1}+1}(e_1e_2)^{2^l+\tau}\cdot e_1\cdot e_3^{n-3}e_4^{n-4}\cdots e_{n-1}\neq 0,$$

and, by part (b) of lemma 3.3, if and only if

$$p_1=\sum_{l=0}^{s}\alpha_l(e_1+e_2)^{2^{s+1}-2^{l+1}+1}(e_1e_2)^{2^l+\tau}\cdot e_1\neq 0.$$

To prove that $p_1\neq 0$ we will use remark 3.4, i.e. we write p_1 as in remark 3.4 and find suitable indices i' and i'' (as in that remark). We denote

$$\begin{aligned} q_1 &= \sum_{l=0}^{s}\alpha_l(e_1+e_2)^{2^{s+1}-2^{l+1}}(e_1e_2)^{2^l+\tau} = \sum_{l=0}^{s}\alpha_l(e_1^{2^{l+1}}+e_2^{2^{l+1}})^{2^{s-l}-1}(e_1e_2)^{2^l+\tau} \\ &= \sum_{l=0}^{s}\alpha_l\sum_{i=0}^{2^{s-l}-1}e_1^{i\cdot 2^{l+1}+2^l+\tau}e_2^{(2^{s-l}-1-i)\cdot 2^{l+1}+2^l+\tau}. \end{aligned}$$

Let us observe a monomial $e_1^ae_2^b$ that appears in the inner sum for l. Then $a+b=2^{s+1}+2\tau$ and $a-b=(2i+1-2^{s-l})2^{l+1}$, i.e. $2^{l+1}\parallel a-b$ for $s\neq l$ (i.e. $2^{l+1}\mid a-b$ and $2^{l+2}\nmid a-b$) and $a=b$ for $s=l$; so, $e_1^ae_2^b$ appears only once in q_1 and its coefficient is α_l. Now, since α_s is even this implies that the coefficient of $(e_1e_2)^{2^s+\tau}$ in q_1 is 0, and since α_0 is even that the coefficients of $e_1^{2^s+\tau-1}e_2^{2^s+\tau+1}$ and $e_1^{2^s+\tau+2^j-1}e_2^{2^s+\tau-2^j+1}$ in q_1 are 0. On the other hand, since α_j is odd the coefficient of $e_1^{2^s+\tau+2^j}e_2^{2^s+\tau-2^j}$ in q_1 is 1.

Now, we expand $p_1=(e_1^2+e_1e_2)q_1$. Note that the degree of each monomial in p_1 is $2^{s+1}+2\tau+2=2^{s+1}+2^{r+1}-2t-2^{j+1}+2\leqslant 2^{s+1}+4(t-1)-2t-2=2n-6$, and hence, after removing all monomials of the form $e_1^ae_2^b$ when $a\geqslant n$ or

$b \geqslant n$, we get p_1 written as in remark 3.4. Let us observe a monomial $e_1^a e_2^b$ in p_1. By the previous identity, its coefficient is the sum of coefficients of $e_1^{a-2} e_2^b$ and $e_1^{a-1} e_2^{b-1}$ in q_1. So, the coefficient of $(e_1 e_2)^{2^s+\tau+1}$ is 0, while the coefficient of $e_1^{2^s+\tau+2^j+1} e_2^{2^s+\tau-2^j+1}$ is 1. Since $2^s+\tau+2^j+1 = 2^s+2^r-t+1 \leqslant 2^s+t-1 = n-1$, the degrees of e_1 and e_2 in these monomials are less than n, so we can apply lemma 3.3 and remark 3.4 to conclude $p_1 \neq 0$.

Finally, we consider the case when t is odd. Then $\varepsilon = 2^j+1$. Note that $2^s + 2^r - 2^j - 2 = 2^s + 2^{r-1} + 2^{r-2} + \cdots + 2^{j+1} + 2^{j-1} + 2^{j-2} + \cdots + 2$, while $t-2 = 2^{j+1}t' + 2^j + 1 < 2^r \leqslant 2^s$ for some $t' \geqslant 0$. So, by Lucas' theorem, we have that $\alpha_0 = \binom{2^s+2^r-2^j-2}{2^s+2^{j+1}t'+2^j+1}$ and $\alpha_1 = \binom{2^s+2^r-2^j-2}{2^s+2^{j+1}t'+2^j}$ are even, while

$$\alpha_2 = \binom{2^s+2^r-2^j-2}{2^s+t-5} = \binom{2^s+2^{r-1}+\cdots+2^{j+1}+2^{j-1}+\cdots+2}{2^s+2^{j+1}t'+2^{j-1}+2^{j-2}+\cdots+2}$$

is odd.

Let us denote $\theta = 2^r - 2^j - t - 1$. Note that $2^j + t - 2 \leqslant 2^r + 1$, i.e. $\theta \geqslant -4$. By proposition 2.2.(4), $A \neq 0$ if and only if

$$\sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}+1} (e_1 e_2)^{2^l+\theta} \cdot e_1 \cdot e_3^{n-3} e_4^{n-4} \dots e_{n-1} \neq 0,$$

and, by lemma 3.3(b), if and only if $p_2 = \sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}+1} (e_1 e_2)^{2^l+\theta} e_1$ is non-zero. Let us denote

$$q_2 = \sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}} (e_1 e_2)^{2^l+\theta} = \sum_{l=0}^{s} \alpha_l (e_1^{2^{l+1}} + e_2^{2^{l+1}})^{2^{s-l}-1} (e_1 e_2)^{2^l+\theta}.$$

Now, as in the previous part of the proof we conclude: the coefficients of $e_1^{2^s+\theta-1} e_2^{2^s+\theta+1}$, $e_1^{2^s+\theta-2} e_2^{2^s+\theta+2}$ and $e_1^{2^s+\theta-3} e_2^{2^s+\theta+3}$ in q_2 are 0 (since α_0 and α_1 are even); the coefficient of $e_1^{2^s+\theta-4} e_2^{2^s+\theta+4}$ in q_2 is 1 (since α_2 is odd). So, in the polynomial $p_2 = (e_1^2 + e_1 e_2) q_2$ the coefficient of $e_1^{2^s+\theta} e_2^{2^s+\theta+2}$ is 0, while the coefficient of $e_1^{2^s+\theta-2} e_2^{2^s+\theta+4}$ is 1. Since the total degree of each monomial of p_2 is $2^{s+1} + 2\theta + 2 = 2^{s+1} + 2^{r+1} - 2^{j+1} - 2t \leqslant 2^{s+1} + 4t - 8 - 2t = 2n - 8$ and $2^s + \theta + 4 = 2^s + 2^r - 2^j - t + 3 \leqslant 2^s + 2^r - t + 1 \leqslant 2^s + t - 1 = n - 1$, we can apply lemma 3.3 and remark 3.4 to conclude $p_2 \neq 0$. □

4. Zero-divisor cup-length of $G_3(\mathbb{R}^n)$

Let s be the unique integer such that $2^s < n \leqslant 2^{s+1}$. In this section we give some bounds for $\mathrm{zcl}(G_3(\mathbb{R}^n))$.

In the following proposition we consider the case $n = 2^s + 1$. This result will show that the corresponding result of [**9**, theorem 4.8] is not correct (see also remark 4.2). Fortunately, this proposition gives a better lower bound for topological complexity.

PROPOSITION 4.1. *Let $n = 2^s + 1$, where $s \geqslant 3$. Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^{s-2} - 7 \textit{ and } \mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^{s-2} - 6.$$

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Proof. It is enough to show $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^{s-1}+2^{s-2}-2} z(w_3)^{2^{s-1}-4} \neq 0$.

First, we prove that $w_1^{2^s} w_3 = 0$. By proposition 2.2, this follows from

$$\begin{aligned} p_3 &= \pi^*(w_1^{2^s} w_3) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1 + e_2 + e_3)^{2^s} (e_1 e_2 e_3) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s+3} e_2^2 e_3 + e_1^3 e_2^{2^s+2} e_3 + e_1^3 e_2^2 e_3^{2^s+1}) e_4^{n-4} \cdots e_{n-1} = 0. \end{aligned}$$

Since $w_1^{2^s} w_3 = 0$, we have

$$\begin{aligned} A &= z(w_1)^{2^s-1} z(w_2)^{2^{s-1}+2^{s-2}-2} z(w_1^{2^s}) z(w_3)^{2^{s-1}-4} \\ &= z(w_1)^{2^s-1} z(w_2)^{2^{s-1}+2^{s-2}-2} (w_1^{2^s} \otimes w_3^{2^{s-1}-4} + w_3^{2^{s-1}-4} \otimes w_1^{2^s}). \end{aligned}$$

Let us observe all classes of the form $w_3^{n-3} \otimes x$ for some $x \in H^*(G_3(\mathbb{R}^n); \mathbb{Z}_2)$ after expanding the expression for A; since w_3^{n-3} is the only non-zero class in $H^{3(n-3)}(G_3(\mathbb{R}^n); \mathbb{Z}_2)$ (by proposition 2.1), to prove that A is non-zero it is enough to show that the sum of all such x is non-zero. To do so, we determine all monomials x' and x'' in classes w_1 and w_2, such that $w_1^{2^s} x' = w_3^{n-3} = w_3^{2^s-2}$ and $w_3^{2^{s-1}-4} x'' = w_3^{2^s-2}$.

Let $x' = w_1^a w_2^b$ be such that $w_1^{2^s+a} w_2^b = w_3^{2^s-2}$. Then $a + 2b = 2(2^s - 3)$. We use proposition 2.2:

$$\begin{aligned} p_1 &= \pi^*(w_1^{2^s+a} w_2^b) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s} + e_2^{2^s} + e_3^{2^s})(e_1 + e_2 + e_3)^a (e_1 e_2 + e_2 e_3 + e_3 e_1)^b e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= e_3^{2^s} (e_1 + e_2)^a (e_1 e_2)^{b+1} e_1 e_4^{n-4} \cdots e_{n-1} \\ &= e_3^{2^s} \sum_{i=0}^{a} \binom{a}{i} e_1^{i+b+2} e_2^{a-i+b+1} \cdot e_4^{n-4} \cdots e_{n-1}. \end{aligned}$$

Note that by proposition 2.2(3) the only non-zero monomials in this sum are the ones for i that satisfies $(i + b + 2, a - i + b + 1) \in \{(2^s - 1, 2^s - 2), (2^s - 2, 2^s - 1)\}$ and $\binom{a}{i}$ is odd, i.e. $i \in \{2^s - 3 - b, 2^s - 4 - b\}$ and $\binom{a}{i}$ is odd.

If $i = 2^s - 3 - b$, then $\binom{a}{i} = \binom{2(2^s-3-b)}{2^s-3-b} = \binom{2\delta}{\delta}$ (here $2\delta = 2(2^s - 3 - b) = a$). By lemma 3.2, this number is odd only if $\delta = 0$, i.e. $(a, b) = (0, 2^s - 3)$. Let us now consider the case $i = 2^s - 4 - b$. Then $\binom{a}{i} = \binom{2(2^s-3-b)}{2^s-4-b} = \binom{2\delta}{\delta-1} = \binom{2\delta}{\delta+1}$. Again, by lemma 3.2, this number is odd only if $\delta = 2^l - 1$, and hence $a = 2^{l+1} - 2$ and $b = 2^s - 2^l - 2$ for some $1 \leqslant l \leqslant s - 1$.

Let us now go back to our expression for A. Here we only consider pairs (a, b) that satisfy $a \leqslant 2^s - 1$ and $b \leqslant 2^{s-1} + 2^{s-2} - 2$; hence $b = 2^s - 2^l - 2$ only if $l \in \{s - 2, s - 1\}$, so we have two pairs to consider: $(a, b) \in \{(2^{s-1} - 2, 2^{s-1} + 2^{s-2} - 2), (2^s - 2, 2^{s-1} - 2)\} = P$.

Next, let $x'' = w_1^{a'} w_2^{b'}$ be such that $w_1^{a'} w_2^{b'} w_3^{2^{s-1}-4} = w_3^{2^s-2}$. We denote the set of all such pairs (a', b') with P'. Clearly, if $(a', b') \in P'$, then $a' + 2b' = 3(2^{s-1} + 2)$, and hence $a' + b' \geqslant 3(2^{s-2} + 1)$; also, by observing A, it is clear that $a' \leqslant 2^s - 1$.

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Now, to prove that A is non-zero, it is enough to prove that B is non-zero, where B is equal to

$$\sum_{(a,b)\in P} w_1^{2^s-1-a} w_2^{2^{s-1}+2^{s-2}-2-b} w_3^{2^{s-1}-4} + \sum_{(a',b')\in P'} w_1^{2^s+2^s-1-a'} w_2^{2^{s-1}+2^{s-2}-2-b'}.$$

By proposition 2.2.(4), this is equivalent to $p = \pi^*(B)e_1^2 e_2 e_4^{n-4}\cdots e_{n-1} \neq 0$. In what follows we will be working with the additive basis

$$\widetilde{B}_{2^s+1} = \{e_1^{a_1} e_2^{a_2}\cdots e_{2^s}^{a_{2^s}} \mid a_1 \leqslant 2^s-1,\ a_2 \leqslant 2^s-2,\ a_3 \leqslant 2^s,\ a_i \leqslant 2^s+1-i,\ i \geqslant 4\}$$

for $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, given by proposition 2.2(1) and the canonical homeomorphism $\sigma : \mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ defined by

$$\sigma(L_1, L_2, L_3, L_4, L_5, \ldots, L_n) = (L_3, L_1, L_2, L_4, L_5, \ldots, L_n).$$

Let $d_{3,n-3} = e_1^2 e_2 e_4^{n-4}\cdots e_{n-1}$. Then

$$\begin{aligned} p_2 &= \pi^*\left(\sum_{(a,b)\in P} w_1^{2^s-1-a} w_2^{2^{s-1}+2^{s-2}-2-b} w_3^{2^{s-1}-4}\right) d_{3,n-3} \\ &= \pi^*(w_1^{2^{s-1}+1} w_3^{2^{s-1}-4} + w_1 w_2^{2^{s-2}} w_3^{2^{s-1}-4}) d_{3,n-3} \\ &= ((e_1+e_2+e_3)^{2^{s-1}} + (e_1e_2+e_2e_3+e_3e_1)^{2^{s-2}}) \\ &\quad \cdot (e_1+e_2+e_3)(e_1e_2e_3)^{2^{s-1}-4} d_{3,n-3}. \end{aligned}$$

Note that the monomials of p_2 belong to $\widetilde{B}_{2^s+1}$; indeed, the degree of e_1 in each monomial is at most $2^{s-1}+1+2^{s-1}-4+2 = 2^s-1$, the degree of e_2 is at most $2^{s-1}+1+2^{s-1}-4+1 = 2^s-2$, and the degree of e_3 is at most $2^{s-1}+1+2^{s-1}-4 = 2^s-3$. In particular, each monomial of p_2 is not divisible by $e_3^{2^s}$. Finally, $p_2 \neq 0$ since $e_1^{2^s-1} e_2^{2^{s-1}-3} e_3^{2^{s-1}-4} e_4^{n-4}\cdots e_{n-1}$ has coefficient 1 in p_2.

On the other hand,

$$\begin{aligned} p_3 &= \pi^*\left(\sum_{(a',b')\in P'} w_1^{2^s+2^s-1-a'} w_2^{2^{s-1}+2^{s-2}-2-b'}\right) d_{3,n-3} \\ &= \sum_{(a',b')\in P'} (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1+e_2+e_3)^{2^s-1-a'} \\ &\quad \cdot (e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}+2^{s-2}-2-b'} d_{3,n-3} \\ &= \sum_{(a',b')\in P'} e_3^{2^s}(e_1+e_2)^{2^s-1-a'}(e_1e_2)^{2^{s-1}+2^{s-2}-2-b'} d_{3,n-3}. \end{aligned}$$

Since $a'+b' \geqslant 3(2^{s-2}+1)$, the degree of e_1 (resp. e_2) in each monomial of this sum is at most $2^s+2^{s-1}+2^{s-2}-1-a'-b' \leqslant 2^s-4$ (resp. $2^s+2^{s-1}+2^{s-2}-2-a'-b' \leqslant 2^s-5$), and hence, after expansion, each monomial (if any) of p_3 is in $\widetilde{B}_{2^s+1}$ and divisible by $e_3^{2^s}$ (note: it is possible that $p_3 = 0$).

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Hence, p_2 and p_3 do not have any common monomials from $\widetilde{B}_{2^s+1}$, and so there are no cancellations between monomials of p_2 and p_3. Now, $p_2 \neq 0$ implies $p = p_2 + p_3 \neq 0$. □

Remark 4.2. Ideas from this paper can be used to prove the following: if $s \geqslant 4$, then $\mathrm{zcl}(G_3(\mathbb{R}^{2^s+2})) \geqslant 7 \cdot 2^{s-1}$ (one has $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s+2^{s-1}} z(w_3) \neq 0$). Hence, $\mathrm{TC}(G_3(\mathbb{R}^{2^s+2})) \geqslant 7 \cdot 2^{s-1} + 1$. Complete proof of this result can be found in the extended version of this paper which is available on the author's website.

Proposition 4.3. *Let $s \geqslant 2$, $n = 2^s + t \leqslant 2^{s+1}$, $t \geqslant 3$ and $2^{r-1} < t \leqslant 2^r$. Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+2} - 2^r - 1 \quad \textit{and} \quad \mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 2^{s+2} - 2^r.$$

Also, if $t - 3 \geqslant 2^{s-1}$, then $\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 7 \cdot 2^{s-1} - 1$ and $\mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 7 \cdot 2^{s-1}$.

Proof. For the first inequality it is enough to show

$$A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^{s+1}-2^{r+1}} z(w_3)^{2^r} \neq 0.$$

Note that $w_1^{2^s} w_3^{2^r} = 0$. Indeed, this follows from proposition 2.2(4), $e_i^{2^s+2^r} = 0$ for $i \in \{1, 2, 3\}$ and the following calculations:

$$\begin{aligned} p_1 &= \pi^*(w_1^{2^s} w_3^{2^r}) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s} + e_2^{2^s} + e_3^{2^s})(e_1 e_2 e_3)^{2^r} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s+2^r} e_2^{2^r} e_3^{2^r} + e_1^{2^r} e_2^{2^s+2^r} e_3^{2^r} + e_1^{2^r} e_2^{2^r} e_3^{2^s+2^r}) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} = 0. \end{aligned}$$

Similarly, one proves that $w_2^{2^s} w_3^{2^r} = 0$, $w_1^{2^s} w_2^{2^s+2^r} = 0$ and $w_1^{2^s+2^r} w_2^{2^s} = 0$.

Note that $2^r \geqslant t \geqslant 3$ implies $r \geqslant 2$. Now, we consider the cases $2 \leqslant r \leqslant s-1$ and $r = s$ separately.

Case 1: $2 \leqslant r \leqslant s-1$. We have

$$\begin{aligned} A &= z(w_1)^{2^s-1} z(w_1)^{2^s} z(w_2)^{2^s-2^{r+1}} z(w_2)^{2^s} z(w_3)^{2^r} \\ &= z(w_1)^{2^s-1} z(w_2)^{2^s-2^{r+1}} (w_1^{2^s} w_2^{2^s} \otimes w_3^{2^r} + w_3^{2^r} \otimes w_1^{2^s} w_2^{2^s}). \end{aligned}$$

Since $2^s - 1 = 2^{s-1} + \cdots + 2^{r+1} + 2^r + 2^r - 1$ and $2^s - 2^{r+1} = 2^{s-1} + \cdots + 2^{r+1}$, in a similar way we get

$$A = z(w_1)^{2^r-1} (w_1^{2^s} w_2^{2^s} \otimes w_1^{2^s-2^r} w_2^{2^s-2^{r+1}} w_3^{2^r} + w_1^{2^s-2^r} w_2^{2^s-2^{r+1}} w_3^{2^r} \otimes w_1^{2^s} w_2^{2^s}).$$

Since the dimension of $w_1^{2^s} w_2^{2^s}$ is greater than the dimension of the class $w_1^{2^s-2^r} w_2^{2^s-2^{r+1}} w_3^{2^r}$, after expanding the expression for A, there is only one summand with the first coordinate in dimension $3 \cdot 2^s + 2^r - 1$, and this summand is $w_1^{2^s+2^r-1} w_2^{2^s} \otimes w_1^{2^s-2^r} w_2^{2^s-2^{r+1}} w_3^{2^r}$. Hence, it is enough to prove that $w_1^{2^s+2^r-1} w_2^{2^s} \neq 0$ and $w_1^{2^s-2^r} w_2^{2^s-2^{r+1}} w_3^{2^r} \neq 0$.

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First, we prove that $w_1^{2^s+2^r-1}w_2^{2^s} \neq 0$. Since $e_i^{2^{s+1}} = 0$ for $i \in \{1, 2, 3\}$ (by proposition 2.2(2)), by proposition 2.2(4) it is enough to prove that

$$\begin{aligned} p_2 &= \pi^*(w_1^{2^s+2^r-1}w_2^{2^s})e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1+e_2+e_3)^{2^r-1}(e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1e_2+e_2e_3+e_3e_1)^{2^s}e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1+e_2+e_3)^{2^r-1}(e_1e_2e_3)^{2^s}e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= \pi^*(w_1^{2^r-1}w_3^{2^s})e_1^2e_2e_4^{n-4}\cdots e_{n-1} \end{aligned}$$

is non-zero in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, i.e. that $w_1^{2^r-1}w_3^{2^s}$ is non-zero in $H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$. Observe the inclusion $i : G_3(\mathbb{R}^{n-2^s}) \subset G_3(\mathbb{R}^n)$. Note that the height of $i^*(w_1)$ in $H^*(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ is 2^r-1 (by (2.1)). So, let x be a class in $H^*(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ such that $i^*(w_1)^{2^r-1}x \in H^{3(n-2^s-3)}(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ is non-zero (this class exists by Poincare's duality); further, let $\widetilde{x} \in H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ be such that $i^*(\widetilde{x}) = x$. Then, by [**12**, lemma 1], the value of $w_1^{2^r-1}\widetilde{x}\cdot w_3^{2^s}$ is the same as the value of $i^*(w_1^{2^r-1}\widetilde{x}) = i^*(w_1)^{2^r-1}x$, which is non-zero. Hence, $w_1^{2^r-1}w_3^{2^s} \neq 0$.

Finally, we prove that $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r} \neq 0$. This will immediately follow from the identity $w_1^{2^s-2^r}w_2^{2^s-2^r}w_3^{2^r} = w_1^{2^s}w_2^{2^s} = w_3^{2^s} \neq 0$, which we now prove. Since $e_i^{2^s+2^r} = 0$ for $i \in \{1, 2, 3\}$, by proposition 2.2(4) this follows from (here $d_{3,n-3} = e_1^2e_2e_4^{n-4}\cdots e_{n-1}$)

$$\begin{aligned} p_3 &= \pi^*(w_1^{2^s-2^r}w_2^{2^s-2^r}w_3^{2^r})d_{3,n-3} \\ &= (e_1+e_2+e_3)^{2^s-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^s-2^r}(e_1e_2e_3)^{2^r}d_{3,n-3} \\ &= (e_1+e_2+e_3)^{2^{s-1}}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}} \\ &\quad \cdot (e_1+e_2+e_3)^{2^{s-1}-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^r}(e_1e_2e_3)^{2^r}d_{3,n-3} \\ &= (e_1+e_2+e_3)^{2^{s-1}-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^r}(e_1e_2e_3)^{2^{s-1}+2^r}d_{3,n-3} \\ &= \ \ldots \\ &= (e_1e_2e_3)^{2^{s-1}+2^{s-2}+\cdots+2^r+2^r}d_{3,n-3} \\ &= (e_1e_2e_3)^{2^s}d_{3,n-3} \\ &= (e_1+e_2+e_3)^{2^s}(e_1e_2+e_2e_3+e_3e_1)^{2^s}d_{3,n-3} \\ &= \pi^*(w_1^{2^s}w_2^{2^s})d_{3,n-3}. \end{aligned}$$

Since $w_3^{2^s} \in B_{3,n-3}$, we have $w_3^{2^s} \neq 0$, which completes our proof.

Case 2: $r = s$. Then $A = z(w_1)^{2^s-1}(w_1^{2^s} \otimes w_3^{2^s} + w_3^{2^s} \otimes w_1^{2^s})$. Since after expanding A there is only one summand with the first coordinate in dimension $2^{s+2}-1$, and this summand is $w_1^{2^s-1}w_3^{2^s} \otimes w_1^{2^s}$, it is enough to prove $w_1^{2^s-1}w_3^{2^s} \neq 0$ and $w_1^{2^s} \neq 0$. The second follows from $w_1^{2^s} \in B_{3,n-3}$, and the first one is proven after the calculations for p_2.

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Suppose now that $t-3 \geqslant 2^{s-1}$. We will prove that

$$B = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s} z(w_3)^{2^{s-1}} \neq 0,$$

which implies $\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^{s-1} - 1$.

Let us observe all summands of B with the first coordinate in dimension $9 \cdot 2^{s-1}$. Note that

$$B = z(w_1)^{2^s-1} z(w_1^{2^s}) z(w_2^{2^s}) z(w_3^{2^{s-1}}),$$

so the only monomial of this form is $w_1^{2^s} w_2^{2^s} w_3^{2^{s-1}} \otimes w_1^{2^s-1}$, and hence it is enough to prove that $w_1^{2^s} w_2^{2^s} w_3^{2^{s-1}} \neq 0$ and $w_1^{2^s-1} \neq 0$. This follows from lemma 2.4 (indeed, since $t-3 \geqslant 2^{s-1}$, both monomials divide $w_1^{2^s} w_2^{2^s} w_3^{t-3} \neq 0$). □

5. Zero-divisor cup-length of $G_k(\mathbb{R}^n)$

In this section we give a lower bound for $G_k(\mathbb{R}^n)$ for $k \geqslant 4$.

PROPOSITION 5.1. *Let $4 \leqslant k < n$ and $2^s + k \leqslant n \leqslant 2^{s+1}$. Then*

$$\mathrm{zcl}(G_k(\mathbb{R}^n)) \geqslant (\lceil \log_2 k \rceil + 1) \cdot 2^s - 1 \quad \textit{and} \quad \mathrm{TC}(G_k(\mathbb{R}^n)) \geqslant (\lceil \log_2 k \rceil + 1) \cdot 2^s.$$

Proof. Let $2^{r-1} < k \leqslant 2^r$. Then $\lceil \log_2 k \rceil = r$, so it is enough to prove

$$A = z(w_1)^{2^{s+1}-1} \prod_{i=1}^{r-1} z(w_{2^i})^{2^s} = z(w_1)^{2^s-1} \prod_{i=0}^{r-1} z(w_{2^i}^{2^s}) \neq 0.$$

First, let us prove that $p = \prod_{i=0}^{r-2} w_{2^i}^{2^s}$ is non-zero in $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. Let $d_{k,n-k} = e_1^{k-1} \cdots e_{k-1} e_{k+1}^{n-k-1} \cdots e_{n-1}$. Since $e_i^{2^{s+1}} = 0$ for $1 \leqslant i \leqslant k$ (by proposition 2.2(2)) and $k' := \sum_{i=0}^{r-2} 2^i = 2^{r-1} - 1 < k$ we have

$$\begin{aligned} p_1 &= \pi^* \left(\prod_{i=0}^{r-2} w_{2^i}^{2^s} \right) d_{k,n-k} \\ &= \prod_{i=0}^{r-2} \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{2^i} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{2^i}}^{2^s} \right) d_{k,n-k} \\ &= [2^0, 2^1, \ldots, 2^{r-2}] \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{k'} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{k'}}^{2^s} \right) d_{k,n-k}, \end{aligned}$$

where $[2^0, 2^1, \ldots, 2^{r-2}] = \binom{2^0+2^1+\cdots+2^{r-2}}{2^0}\binom{2^1+\cdots+2^{r-2}}{2^1} \cdots \binom{2^{r-2}}{2^{r-2}}$ denotes the multinomial coefficient. By Lucas' theorem, this coefficient is odd. Also, for $1 \leqslant i \leqslant k$ the degree of e_i in each monomial in the last expression for p_1 is at most $2^s + k - i \leqslant n - i$, so all monomials in this expression are distinct members of the basis B_n for $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$, and hence $p_1 \neq 0$. So, by proposition 2.2(4), $p \neq 0$.

Now, let us observe all summands after expanding A with first coordinate in dimension $(2^{r-1}-1) \cdot 2^s$. The dimension of p is $(2^{r-1}-1) \cdot 2^s$, and it is easy to see

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that the only term of this form is $p \otimes w_1^{2^s-1} w_{2^{r-1}}^{2^s}$. So, to finish the proof it is enough to prove $w_1^{2^s-1} w_{2^{r-1}}^{2^s} \neq 0$. In fact, we prove that $w_1^{2^s} w_{2^{r-1}}^{2^s} \neq 0$. Since $e_i^{2^{s+1}} = 0$ for $1 \leqslant i \leqslant k$, we have

$$\begin{aligned} p_2 &= \pi^* \left(w_1^{2^s} w_{2^{r-1}}^{2^s} \right) d_{k,n-k} \\ &= \left(e_1^{2^s} + e_2^{2^s} + \cdots + e_k^{2^s} \right) \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{2^{r-1}} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{2^{r-1}}}^{2^s} \right) d_{k,n-k} \\ &= \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{2^{r-1}+1} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{2^{r-1}+1}}^{2^s} \right) d_{k,n-k}. \end{aligned}$$

Now, as above, $2^s + k \leqslant n$ implies that all monomials in the last expression for p_2 are distinct members of the basis B_n for $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$, and hence $p_2 \neq 0$. By proposition 2.2(4), it follows that $w_1^{2^s} w_{2^{r-1}}^{2^s} \neq 0$. □

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References

1 A. Borel. La cohomologie mod 2 de certains espaces homogenes. *Commun. Math. Helv.* **27** (1953), 165–197.

2 D. Cohen and A. Suciu. Boundary manifolds of projective hypersurfaces. *Adv. Math.* **206** (2006), 538–566.

3 A. Costa and M. Farber. Motion planning in spaces with small fundamental groups. *Commun. Contemp. Math.* **12** (2010), 107–119.

4 Ö. Eğecioğlu. The parity of the Catalan numbers via lattice paths. *Fibonacci Quart.* **21** (1983), 65–66.

5 M. Farber. Topological complexity of motion planning. *Discrete Comput. Geom.* **29** (2003), 211–221.

6 J. González, B. Gutiérrez, D. Gutiérrez and A. Lara. Motion planning in real flag manifolds. *Homol. Homotopy Appl.* **82** (2016), 359–375.

7 J. Jaworowski, An additive basis for the cohomology of real Grassmannians, *Lecture Notes in Math.* Vol. 1474 (Springer-Verlag, Berlin, 1991), pp. 231–234.

8 J. Korbaš and J. Lörinc. The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds. *Fundam. Math.* **178** (2003), 143–158.

9 P. Pavešić. Topological complexity of real Grassmannians. *Proc. Roy. Soc. Edinb. A* **151** (2021), 2013–2029.

10 P. Pavešić, Erratum to 'Topological complexity of real Grassmannians', submitted.

11 Z. Z. Petrović, B. I. Prvulović and M. Radovanović. Multiplication in the cohomology of Grassmannians via Gröbner bases. *J. Algebra* **438** (2015), 60–84.

12 R. E. Stong. Cup products in Grassmannians. *Topol. Appl.* **13** (1982), 103–113.

https://doi.org/10.1017/prm.2022.15 Published online by Cambridge University Press