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The Galvin property under the ultrapower axiom

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Abstract. We continue the study of the Galvin property from Benhamou, Garti, and Shelah (2023, *Proceedings of the American Mathematical Society* 151, 1301–1309) and Benhamou (2023, *Saturation properties in canonical inner models*, submitted). In particular, we deepen the connection between certain diamond-like principles and non-Galvin ultrafilters. We also show that any Dodd sound non p-point ultrafilter is non-Galvin. We use these ideas to formulate what appears to be the optimal large cardinal hypothesis implying the existence of a non-Galvin ultrafilter, improving on a result from Benhamou and Dobrinen (2023, *Journal of Symbolic Logic*, 1–34). Finally, we use a strengthening of the Ultrapower Axiom to prove that in all the known canonical inner models, a κ-complete ultrafilter has the Galvin property if and only if it is an iterated sum of p-points.

1 Introduction

In this paper, we study certain aspects of the *Galvin property* of ultrafilters.

Definition 1.1. Let U be a uniform ultrafilter over κ. We say that U has the Galvin property if for any sequence $\langle A_i \rangle_{i<2^\kappa}$, there is $I \in [2^\kappa]^\kappa$ such that $\bigcap_{i\in I} A_i \in U$.

More generally, if $\lambda \leq \kappa$ and U is a uniform ultrafilter over κ, we denote by $\mathrm{Gal}(U, \lambda, 2^\kappa)$ the statement that for any $\langle A_i \rangle_{i<2^\kappa}$, there is $I \in [2^\kappa]^\lambda$ such that $\bigcap_{i\in I} A_i \in U$. Galvin proved in 1973 every normal ultrafilter has the Galvin property. Gitik and Benhamou [7] recently improved this result to show that any product of κ-complete p-points over κ has the Galvin property.[1] Benhamou [1] then proved what appears to be a slight improvement of this result.

Theorem 1.2 *Suppose that U is Rudin–Keisler equivalent to an n-fold sum of κ-complete p-points (see Definition 2.4). Then U has the Galvin property.*

The main theorem of this paper shows that under natural combinatorial hypotheses which hold in all known canonical inner models, the converse of the above theorem is true.

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[1] A κ-complete ultrafilter U over κ is called p-point if every sequence $\langle A_i \mid i < \kappa\rangle \subseteq U$ has a measure-one pseudo-intersection; that is, there is $A \in U$ such that for every $i < \kappa$, $|A \setminus A_i| < \kappa$.

Main Theorem 1.1 *Assume the Ultrapower Axiom and that every irreducible ultrafilter is Dodd sound. If U is a κ-complete ultrafilter on κ with the Galvin property, then U is Rudin–Keisler equivalent to an iterated sum of κ-complete p-points on κ.*

The hypotheses of this theorem will be discussed and explained further later in the Introduction.

The study of the Galvin property is motivated by its presence in various areas of set theory and infinite combinatorics [2–8, 16]. One particularly noteworthy incarnation of the Galvin property is the maximal class in the Tukey order, which we shall now explain in more detail.

Definition 1.3. For two posets $(P, \leq_P), (Q, \leq_Q)$[2], we say that $P \leq_T Q$ if there is a cofinal map $f : Q \to P$.[3] We say that P, Q are Tukey equivalent and denote $P \equiv_T Q$, if $P \leq_T Q$ and $Q \leq_T P$.

The Tukey order finds its origins in the Moore–Smith convergence notions of nets and is of particular interest when considering the poset $(U, \supseteq)$, where U is an ultrafilter. The Tukey order restricted to ultrafilters over ω has been extensively studied by Isbell [20], Milovich [27, 28], Dobrinen and Todorcevic [12, 13, 15], Raghavan, Dobrinen, and Blass [10, 32], and many others. Lately, this investigation has been stretched to ultrafilters over uncountable cardinals and in particular to measurable cardinals by Benhamou and Dobrinen [2]. It turns out that the Tukey order on σ-complete ultrafilters over measurable cardinal behaves differently from the one on ω and requires a new theory to be developed. One of these differences revolves around the maximal class. For a given λ, a uniform ultrafilter U on κ is called *Tukey-top with respect to* λ if its Tukey class is above every λ-directed poset of size 2^κ. It turns out that an ultrafilter U is Tukey-top with respect to λ if and only if $\neg\mathrm{Gal}(U, \lambda, 2^\kappa)$. In particular, a uniform ultrafilter over κ is Tukey-top with respect to κ if and only if it is non-Galvin.

Working in ZFC (with no additional set theoretic hypotheses), Isbell [20] constructed ultrafilters on ω which are non-Galvin, this construction was accomplished independently by Juhász [23]. The first construction of non-Galvin ultrafilters over measurable cardinals is due to Garti, Shelah, and Benhamou [6], using the existence of Kurepa trees to prevent a certain ultrafilter from having the Galvin property. This connection between Kurepa trees and the Galvin property is further explored in this paper, where we define (Definition 3.3) a diamond-like principle $\diamondsuit^*_{\text{thin}}(W)$, and a slight weakening (Definition 3.12) of it that ensures that an ultrafilter is non-Galvin (Lemma 3.5).

In [2], Isbell's construction together with other features from [1] enabled the construction of a non-Galvin ultrafilter over a κ-compact cardinal. Here, we improve this initial large cardinal, isolate the notion of a *non-Galvin cardinal* (Definition 5.1), and prove the following.

[2] We shall abuse notation by suppressing the order in a poset.
[3] A map $f : Q \to P$ is called cofinal if for every cofinal set $B \subseteq Q$, $f''B$ is cofinal in P.

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Main Theorem 1.2 *Suppose that κ is a non-Galvin cardinal, then κ carries a κ-complete ultrafilter U such that $\neg\mathrm{Gal}(U,\kappa,\kappa^+)$. In particular, if in addition $2^\kappa = \kappa^+$, then U is a non-Galvin ultrafilter.*

We also prove that κ-compactness implies non-Galvinness (Theorem 5.7), that some degree of Dodd soundness implies it (Corollary 3.11), and that in the known canonical inner models, a κ-compact cardinal is a limit of non-Galvin cardinals (Proposition 6.9).

In [9], Gitik and Benhamou noted that although the existence of a non-Galvin ultrafilter is equiconsistent with a measurable cardinal, the latter assumption (measurability) does not outright imply that there is a non-Galvin ultrafilter. More precisely, in Kunen's model $L[U]$, since every σ-complete ultrafilter is Rudin–Keisler equivalent to a power of the normal ultrafilter U, Theorem 1.2 can be invoked to deduce the Galvin property for every σ-complete ultrafilter in $L[U]$. Being the simplest example of a canonical inner model which can accommodate a measurable cardinal, the result in $L[U]$ suggests that the Galvin property, like many other combinatorial properties of ultrafilters, has a rigid form in the canonical inner models. Indeed, the result from $L[U]$ was later generalized [1] to the Mitchell–Steel models $L[E]$ up to a measurable limit of superstrong cardinal[4] (see Theorem 1.2). These results in the inner models suggest the following question [1, Question 5.1].

Question 1.4. *Is there an inner model with a non-Galvin ultrafilter?*

In this paper, we take a more ambitious approach and work under the *Ultrapower Axiom* (UA)[5] which is a combinatorial principle discovered by Goldberg [17]. The advantage of UA is that with one simple axiom, which holds in all known canonical inner models, many of the usual principles are captured; for example, the linearity of the Mitchell order and instances of GCH. More relevant for our purposes, the presence of UA imposes rigidity on the structure of ultrafilters.

Theorem 1.5 (UA) *Let W be a σ-complete ultrafilter. Then W can be written as the n-fold sum of irreducible ultrafilters.*[6]

In [1], this kind of characterization, together with further fine structural properties of the Mitchell–Steel extender models $L[E]$ was already used to prove the following.

Theorem 1.6 *If $L[E]$ is an iterable Mitchell–Steel model containing no superstrong cardinals, then every κ-complete ultrafilter in $L[E]$ has the Galvin property.*

The point here is that in $L[E]$ every κ-complete ultrafilter takes the form of Theorem 1.2 and therefore satisfies the Galvin property.

The existence of canonical inner models with superstrong cardinals is open, though provable from widely believed conjectures: the fine structure for inner models

[4] A cardinal κ is *superstrong* if there is an elementary embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

[5] In this paper, we will use the structural consequences of UA rather than UA itself, so we choose not to provide the precise statement of the axiom, which can be found in [17].

[6] Recall the irreducible ultrafilters are those ultrafilters which are minimal in the Rudin–Frolík order. Equivalently, W is irreducible if there is no ultrapower embedding $j\colon V \to M$ and an ultrafilter $U \in M$ such that $j_W = (j_U)^M \circ j$.

with superstrong cardinals has been developed assuming iterability hypotheses [35]. Therefore, the current knowledge about canonical inner models does not quite reach the level where a κ-complete non-Galvin ultrafilter exists, although our results below show that the conditional canonical inner models built based on iterability hypotheses can contain non-Galvin ultrafilters.

Here, we shall prove the following stronger (in several senses) result.

Main Theorem 1.3 (UA) *Assume that every irreducible ultrafilter is Dodd sound (see Definition 2.1(6)). Then a uniform σ-complete ultrafilter over a regular cardinal has the Galvin property if and only if it is a D-limit of n-fold sums of κ-complete p-points over κ.*

We note that in the above theorem, the ultrafilter D might be just a σ-complete ultrafilter over a cardinals $\lambda < \kappa$ (see Theorem 2.10). By results of Schlutzenberg [34], in the Mitchell–Steel extender models $L[E]$, every irreducible ultrafilter is Dodd sound, so the assumption in the theorem holds in $L[E]$. Hence, Theorem 1.3 implies that in the canonical inner models of the form of $L[E]$, even above a superstrong cardinal, the n-fold sum of p-points, in fact, *characterizes* the ultrafilters with the Galvin property. This characterization implies, for example, that σ-complete ultrafilters over successor cardinals always possess the Galvin property (Corollary 6.2).

As a corollary, we obtain the characterization of the Tukey-top ultrafilters.

Corollary 1.7 (UA) *Assume that every irreducible ultrafilter is Dodd sound, then a σ-complete ultrafilter over a regular cardinal is Tukey-top if and only if it is not a D-sum of n-fold sums of κ-complete p-points over κ.*

This corollary may come as a bit of a surprise if one is familiar with the Tukey order on ω: Dobrinen and Raghavan proved independently that it is consistent that there are non-Tukey-top ultrafilters on ω that are not n-fold sums of p-points [10], more specifically, a generic ultrafilter for $P(\omega \times \omega)/\text{fin} \cdot \text{fin}$ is such an ultrafilter; this result was stretched by Dobrinen in [13, 14].

One might suspect that under these very restrictive assumptions, we again run into the situation where every κ-complete ultrafilter has the Galvin property, but by Theorem 1.2, a non-Galvin cardinal suffices to guarantee the existence of a non-Galvin ultrafilter. Our next result suggests that in the canonical inner models, non-Galvin cardinals are exactly the large cardinal assumption needed to ensure the existence of non-Galvin ultrafilters.

Main Theorem 1.4 (UA) *Assume that every irreducible ultrafilter is Dodd sound. If there is a κ-complete non-Galvin ultrafilter on an uncountable cardinal κ, then there is a non-Galvin cardinal.*

One feature which seems to require more effort is to obtain a non-Galvin ultrafilter which extends the club filter (i.e., q-point). The ultrafilters that were constructed in [2] from a κ-compact cardinal extended the club filter and it is not clear at this point whether a non-Galvin cardinal implies the existence of such ultrafilters. Nonetheless, in the canonical inner models, the implication holds. In fact, the existence of a non-Galvin ultrafilter is equivalent to the existence of a non-Galvin q-point.

Main Theorem 1.5 (UA) *Assume every irreducible ultrafilter is Dodd sound. Suppose κ is an uncountable cardinal that carries a κ-complete non-Galvin ultrafilter. Then the Ketonen least non-Galvin κ-complete ultrafilter on κ extends the closed unbounded filter.*

The organization of this paper is as follows:

- In Section 2, we collect some basic definitions and facts from the theory of ultrafilters.
- In Section 3, we establish the connection between non-Galvin ultrafilters and various diamond-like principles.
- In Section 4, we use partial soundness to conclude that some ultrafilter is non-Galvin and define the corresponding diamond $\diamond^{-}_{\text{thin}}$.
- In Section 5, we introduce the non-Galvin cardinals and prove Main Theorem 1.2.
- In Section 6, we work in the canonical inner models and prove Main Theorems 1.1, 1.3, 1.4, 1.5.
- In Section 7, we state some open questions and suggest further directions.

1.1 Notation

Our notation is mostly standard. Let κ be a cardinal, and let X be any set. Then $[X]^{\kappa} = \{Y \in P(X) \mid |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \in P(X) \mid |Y| < \kappa\}$. When X is a set of ordinals, we identify elements of $[X]^{<\kappa}$ with their increasing enumerations. We write ${}^{<\kappa}X$ for the set of all functions $f: \gamma \to X$, where $\gamma < \kappa$ and ${}^{\alpha}X$ for the set of all functions $f: \alpha \to X$. Let κ be regular. For two subsets of κ, we write $X \subseteq^* Y$ to denote that $X \setminus Y$ is bounded in κ. Similarly, for $f, g: \kappa \to \kappa$, we denote $f \leq^* g$ if there is $\alpha < \kappa$ such that for every $\alpha \leq \beta < \kappa$, $f(\beta) \leq g(\beta)$. We say that $C \subseteq \kappa$ is a *closed unbounded* (or *club*) subset of κ if it is a closed subset with respect to the order topology on κ and unbounded in the ordinals below κ. The *club filter* over κ is the filter:

$$\mathrm{Club}_{\kappa} := \{X \subseteq \kappa \mid X \text{ includes a closed unbounded subset of } \kappa\}.$$

If $f: A \to B$ is a function, then $f``(X) = \{f(x) \mid x \in X\}$ and $f^{-1}[Y] = \{a \in A \mid f(a) \in Y\}$.

2 Preliminaries

We only consider σ-complete ultrafilters over regular cardinals in this paper. We will, however, consider ultrafilters on κ that fail to be uniform or κ-complete. For a σ-complete ultrafilter U, we denote by M_U the transitive collapse of the ultrapower of the universe of sets by U and by $j_U: V \to M_U$ the usual ultrapower embedding. Given an elementary embedding $j: V \to M$ and an object $A \in M$, we let $\rho = \min\{\alpha \mid A \in V_{j(\alpha)}\}$ and define $D(j, A) := \{X \subseteq V_\rho \mid A \in j(X)\}$. When A is an ordinal, we will always replace V_ρ in the above definition by ρ. If M is any model of ZFC and f is a function or relation defined in the language of set theory, the relativization of f to this model is denoted by $(f)^M$; for example, if $\kappa \in M$, we might consider $(\kappa^+)^M$, V^M_κ, etc.

The primary large cardinals we will be interested in are measurable cardinals. We say that a cardinal κ is *measurable* if it carries a non-principal κ-complete ultrafilter. In the Introduction, we also mentioned the *compact cardinals,* which can be characterized using the filter extension property: we say κ has the λ*-filter extension* *property* if every κ-complete filter on λ can be extended to a κ-complete ultrafilter. A *κ-compact cardinal* is a cardinal κ which has that κ-filter extension property. For more background on large cardinals, we refer the reader to [25].

Definition 2.1 (Special properties of ultrafilters) Let U be an ultrafilter over a regular cardinal κ. We say that:

(1) A function f on κ is said to be *constant (mod U)* if there is a set $A \in U$ such that $f \restriction A$ is constant. A function f is *unbounded (mod U)* if $\forall \alpha < \kappa$, $f^{-1}[\alpha] \notin U$. A function f is *almost one-to-one (mod U)* if there is a set $A \in U$ such that $f \restriction A$ is almost one-to-one in the sense that for any x, $\{\alpha \in A : f(\alpha) = x\}$ is bounded below κ.
(2) U is a *p-point* if every function $f{:}\,\kappa \to \kappa$ which is unbounded $(\text{mod } U)$ is almost one-to-one $(\text{mod } U)$.[7]
(3) U is *μ-indecomposable* if for any function $f{:}\,\kappa \to \mu$, there is $\mu' < \mu$ such that $f^{-1}[\mu'] \in U$.
(4) U is *weakly normal* if whenever $f{:}\,A \to \kappa$ is such that $A \in U$ and f is regressive, there is $A' \subseteq A$, $A' \in U$ such that $f''[A']$ is bounded.[8]
(5) U is *α-sound* if the function $j^\alpha{:}\,P(\kappa) \to M_U$ defined by $j^\alpha(X) = j_U(X) \cap \alpha$ belongs to M_U.
(6) U is *Dodd sound* if it is $[\text{id}]_U$-sound.
(7) U is *κ-irreducible* if every ultrafilter W on an ordinal $\lambda < \kappa$ that is Rudin–Frolík below U is principal (see Definition 2.15).

Remark 2.2.

(1) The concept of Dodd soundness arose in inner model theory, where it serves as a strong form of the initial segment condition [33]. Though on first glance, it may appear quite different, the Dodd soundness of a mouse is essentially equivalent to the Dodd soundness of its last extender as defined above. The formulation of Dodd soundness given here is due to Goldberg [17].
(2) Note that if U is α-sound, then $\{j_U(A) \cap \alpha \mid A \subseteq \kappa\} \in M_U$. This is in fact equivalent. Indeed, if $\{j_U(A) \cap \alpha \mid A \subseteq \kappa\} \in M_U$, then it is the inverse of the transitive collapse of $\{j(S) \cap [\text{id}]_U \mid S \in P(\kappa)\}$.
(3) Note that if U is an ultrafilter over a regular cardinal κ, and $\lambda < \kappa$ is such that $\lambda \in U$, then automatically, U is a p-point as for any function $f : \kappa \to \kappa$, $f \restriction \lambda$ is bounded and hence there are no unbounded functions mod U.
(4) If U is irreducible and uniform on λ, then U is λ-irreducible.

Proposition 2.3 *Let $f{:}\,\kappa \to \kappa$ be any function, and let U be an ultrafilter over κ.*

(1) *f is unbounded mod U if and only if* $\sup_{\alpha<\kappa} j_U(\alpha) \leq [f]_U$.
(2) *f is almost one-to-one mod U if and only if there is a (monotone) function $g{:}\,\kappa \to \kappa$ such that* $j_U(g)([f]_U) = [g \circ f]_U \geq [id]_U$.

[7] Note that for κ-complete ultrafilters over κ this is equivalent to the definition of p-points using the existence of pseudo-intersections [24]. In general, without assuming κ-completeness, these definitions are not equivalent.

[8] The notion of decomposability and weak normality makes sense also for filters when requiring the sets to be positive instead of measure 1.

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Proof (1) is trivial. For (2), suppose that f is almost one-to-one on $A \in U$, and let for each $\alpha < \kappa$ $g(\alpha) = \sup f^{-1}[\alpha + 1] \cap A$. Then for each $\xi \in A$ $g(f(\xi)) = \sup f^{-1}[f(\xi) + 1] \cap A \geq \xi$, hence $[g \circ f]_U \geq [\mathrm{id}]_U$. For the other direction, let g be a monotone function such that $[g \circ f]_U \geq [\mathrm{id}]_U$. Then there is a set $A \in U$ such that for each $\alpha \in A$, $g \circ f(\alpha) \geq \alpha$. Hence, if $\beta \in f^{-1}[\alpha]$, then $g(\alpha) \geq g(f(\beta)) \geq \beta$, hence $f^{-1}[\alpha] \subseteq g(\alpha) + 1$. ∎

Definition 2.4. Suppose U is an ultrafilter over X and for each $\alpha \in X$, U_α is an ultrafilter over X_α. Define the limit

$$U\text{-}\lim \langle U_\alpha \rangle_{\alpha \in X} = \left\{ Y \subseteq X \mid \{\alpha \in X \mid Y \cap X_\alpha \in U_\alpha\} \in U \right\}$$

and the sum

$$\sum_U \langle U_\alpha \rangle_{\alpha \in X} = \left\{ Y \subseteq \cup_{\alpha \in X} \{\alpha\} \times X_\alpha \mid \{\alpha \in X \mid (Y)_\alpha \in U_\alpha\} \in U \right\},$$

where $(Y)_\alpha = \{\beta \in X_\alpha \mid (\alpha, \beta) \in Y\}$ is the αth fiber of Y.

The key property of sums is that they yield ultrafilters that represent iterated ultrapowers.

Lemma 2.5 [17, Corollary 5.2.7] *If U is an ultrafilter on X and $\langle W_\alpha \rangle_{\alpha \in X}$ is a sequence of ultrafilters, then letting $W^* = [\alpha \mapsto W_\alpha]_U$, $M_{\sum_U \langle W_\alpha \rangle_{\alpha \in X}} = (M_{W^*})^{M_U}$ and $j_{\sum_U \langle W_\alpha \rangle_{\alpha \in x}} = (j_{W^*})^{M_U} \circ j_U$. Moreover, $U\text{-}\lim \langle W_\alpha \rangle_{\alpha \in X} = j_U^{-1}[W^*]$.*

The sum construction is often used to obtain an ultrafilter representing an iterated ultrapower in this way, and in this context, the choice of the sequence $\langle W_\alpha \rangle_{\alpha \in X}$ representing W^* is usually irrelevant and distracting. For this reason, we introduce a notation that allows us to remain agnostic about this choice.

Definition 2.6. If U is an ultrafilter over X and M_U satisfies that W^* is an ultrafilter, then $U^\frown W^*$ denotes $\sum_U \langle W_\alpha \rangle_{\alpha \in X}$, where W_α is a sequence of ultrafilters such that $W^* = [\alpha \mapsto W_\alpha]_U$.

Technically, the definition of $U^\frown W^*$ depends on the choice of the underlying sets of W_α. This ambiguity causes no issues, however, since if W'_α is another sequence such that $W^* = [\alpha \mapsto W'_\alpha]_U$, then letting $Z = \sum_U \langle W_\alpha \rangle_{\alpha \in X}$ and $Z' = \sum_U \langle W'_\alpha \rangle_{\alpha \in X}$, there is a set $S \in Z \cap Z'$ such that $Z \cap P(S) = Z' \cap P(S)$.

Definition 2.7. We define recursively when U is an *n-fold sum of p-points*. W is a one-fold sum of p-points if W is a p-point. We say that W is an $(n+1)$-fold sum of p-points if there are n-fold sums of p-points U_α and a p-point ultrafilter U such that U is Rudin–Keisler equivalent to $\sum_U \langle U_\alpha \rangle_{\alpha < \kappa}$.

We shall now prove a slight improvement of the form of ultrafilters which have the Galvin property in Theorem 1.2, this will be turn out to be an exact characterization of the ultrafilters with the Galvin property under UA plus every irreducible is Dodd sound in Main Theorem 1.3. We need the definition of the modified diagonal intersection.

Definition 2.8. Suppose that W is a κ-complete ultrafilter over κ, and let $\pi_W : \kappa \to \kappa$ be the function which represents κ mod W. For a sequence $\langle A_i \rangle_{i<\kappa}$ of subsets of κ,

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we define the *modified diagonal intersection* by

$$\Delta^W_{i<\kappa} A_i = \{\alpha < \kappa \mid \forall i < \pi_W(\alpha),\ \alpha \in A_i\}.$$

Fact 2.9. *If W is a κ-complete ultrafilter over κ and $\langle A_i \rangle_{i<\kappa} \subseteq W$, then:*
(1) $\Delta^W_{i<\kappa} A_i \in W$.
(2) *For every $i_0 < \kappa$, $(\Delta^W_{i<\kappa} A_i)\setminus(\pi^{-1}[i_0+1]) \subseteq A_{i_0}$.*

Theorem 2.10 *Suppose that $\lambda < \kappa$, let D be any ultrafilter over λ, and let $\langle W_\xi \rangle_{\xi<\lambda}$ be a sequence of n-fold sums of κ-complete p-point ultrafilters over κ. Then $\sum_D \langle W_\xi \rangle_{\xi<\lambda}$ has the Galvin property.*

Proof Denote by $Z := \sum_D \langle W_\xi \rangle_{\xi<\lambda}$, and let us assume for simplicity of notation that $n = 2$. Hence, $Z = \sum_D \langle \sum_{U_\xi} \langle U_{\xi,\eta} \rangle_{\eta<\kappa} \rangle_{\xi<\lambda}$, where each U_ξ and $U_{\xi,\eta}$ is a κ-complete p-point over κ. For $A \in Z$, define

$$A^{(2)}_{i,j} = \{k < \kappa \mid \langle i, j, k \rangle \in A\},$$
$$A^{(1)}_i = \{j < \kappa \mid A^{(2)}_{i,j} \in U_{i,j}\},$$
$$A^{(0)} = \{i < \lambda \mid A^{(1)}_i \in U_i\}.$$

Note that

$$A \in \sum_D \langle \sum_{U_i} \langle U_{i,j} \rangle_{j<\kappa} \rangle_{i<\lambda} \Leftrightarrow \{i < \lambda \mid (A)_i \in \sum_{U_i} \langle U_{i,j} \rangle_{j<\kappa}\} \in D$$
$$\Leftrightarrow \{i < \lambda \mid \{j < \kappa \mid A^{(2)}_{i,j} \in U_{i,j}\} \in U_i\} \in D \Leftrightarrow A^{(0)} \in D.$$

For any $W \in \{U_i \mid i < \lambda\} \cup \{U_{i,j} \mid i < \lambda,\ \ j < \kappa\}$, choose $\pi_W : \kappa \to \kappa$ such that $[\pi_W]_W = \kappa$ and π_W is almost one-to-one. Such a function exists since W is a κ-complete p-point. Define $\rho_W : \kappa \to \kappa$ by

$$\rho_W(\alpha) = \sup \pi^{-1}_W[\alpha + 1] + 1.$$

Next, we define

$$\rho^{(1)}(\alpha) = \sup_{i<\alpha} \rho_{U_i}(\alpha), \text{ and } \rho^{(2)}(\alpha) = \sup_{i,j<\alpha} \rho_{U_{i,j}}(\alpha).$$

Note that $\rho^{(1)}, \rho^{(2)} : \kappa \to \kappa$ since κ is regular. Now we are ready to prove the theorem. Let $\langle A_i \rangle_{i<2^\kappa}$ be a sequence of sets in Z. Since $\lambda < \kappa$, we can assume without loss of generality that there is a set $A^{(0)}_* \in D$ such that for every $i < 2^\kappa$, $A^{(0)}_* = (A_i)^{(0)}$. Let $\mathcal{N}$ be an elementary substructure of $H(\theta)$ for some large enough θ such that:
(1) $|\mathcal{N}| = \kappa$.
(2) ${}^{<\kappa}\mathcal{N} \subseteq \mathcal{N}$.
(3) $\kappa \subseteq \mathcal{N}$ and $\kappa^+ \cap \mathcal{N} \in \kappa^+$.
(4) $\langle A_i \rangle_{i<2^\kappa} \in \mathcal{N}$.
Let $\alpha^* = \kappa^+ \cap \mathcal{N}$.

Claim 2.11 *For every $\langle \alpha_1, \alpha_2 \rangle \in [\kappa]^2$ and $\delta < \alpha^*$, there is $\delta < \beta < \alpha^*$ such that:*
(1) $\forall i \in (A_*)^{(0)}, (A_\beta)^{(1)}_i \cap \alpha_1 = (A_{\alpha^*})^{(1)}_i \cap \alpha_1$.
(2) $\forall i \in (A_*)^{(0)} \forall j < \alpha_1,\ (A_\beta)^{(2)}_{i,j} \cap \alpha_2 = (A_{\alpha^*})^{(2)}_{i,j} \cap \alpha_2$.

Proof Consider the statement

$$\phi(\alpha_1,\alpha_2,\delta) \equiv \exists\beta > \delta\ (1) \wedge (2)$$

$H(\theta) \models \phi(\alpha_1,\alpha_2,\delta)$ as witnessed by α^* and since $\alpha_1,\alpha_2,\delta \in \mathcal{N}$, the elementarity of $\mathcal{N}$ implies that there is such $\beta \in \mathcal{N}$ and, in particular, $\beta < \alpha^*$. ∎

Define a sequence $\langle \mu_i \mid i < \kappa\rangle$ inductively, suppose that $\langle \mu_j \mid j < i\rangle$ was defined. Let $\delta = \sup_{j<i}\mu_j + 1 \in \mathcal{N}$ and apply the claim to δ and

$$\alpha_1 = \rho^{(1)}(i), \text{ and } \alpha_2 = \rho^{(2)}(i)$$

to produce $\mu_i > \delta$ (and thus $\mu_i \neq \mu_j$ for all $j < i$). We claim that

$$\bigcap_{i<\kappa} A_{\mu_i} \in \sum_D(\langle \sum_{U_i}\langle U_{i,j}\rangle_{j<\kappa}\rangle_{i<\lambda}.$$

To see this, we define for every $\xi \in (A_*)^{(0)}$,

$$(A_*)^{(1)}_\xi = (A_{\alpha^*})^{(1)}_\xi \cap \Delta^{U_\xi}_{i<\kappa}(A_{\mu_i})^{(1)}_\xi \setminus \rho_{U_\xi}(\xi)$$

and for every $\xi \in (A_*)^{(0)}$, $\eta \in (A_*)^{(1)}_\xi$, define

$$(A_*)^{(2)}_{\xi,\eta} = (A_{\alpha^*})^{(2)}_{\xi,\eta} \cap \Delta^{U_{\xi,\eta}}_{i<\kappa}(A_{\mu_i})^{(2)}_{\xi,\eta} \setminus \rho_{U_{\xi,\eta}}(\eta).$$

Let

$$A_* = \bigcup_{\xi\in A^{(0)}_*}\ \bigcup_{\eta\in(A_*)^{(1)}_\xi} \{\xi\}\times\{\eta\}\times(A_*)^{(2)}_{\xi,\eta}.$$

Claim 2.12 *For every $\langle\alpha,\beta,\gamma\rangle \in A_*$, and for every $i < \kappa$, $\alpha \in (A_{\mu_i})^{(0)}$, $\beta \in (A_{\mu_i})^{(1)}_\alpha$ and $\gamma \in (A_{\mu_i})^{(2)}_{\alpha,\beta}$.*

Proof of claim. Let $\langle\alpha,\beta,\gamma\rangle \in A_*$. By definition of A_*, $\alpha \in (A_*)^{(0)}$, $\beta \in (A_*)^{(1)}_\alpha$ and $\gamma \in (A_*)^{(2)}_{\alpha,\beta}$. In particular,

$$(*) \quad \alpha < \pi_{U_\alpha}(\beta) \text{ and } \beta < \pi_{U_{\alpha,\beta}}(\gamma).$$

For $i < \kappa$, we note first that $\alpha \in (A_{\mu_i})^{(0)}$ since we assume $(A_{\mu_i})^{(0)} = (A_*)^{(0)}$. Now to see that $\beta \in (A_{\mu_i})^{(1)}_\alpha$, split into cases. If $i < \pi_{U_\alpha}(\beta)$, then $\beta \in (A_{\mu_i})^{(1)}_\alpha$ by the definition of the modified diagonal intersection. If $i \geq \pi_{U_\alpha}(\beta)$, then $\beta < \rho_{U_\alpha}(i)$. Also, by $(*)$, $\alpha < \pi_{U_\alpha}(\beta) \leq i$ and therefore $\rho_{U_\alpha}(i) \leq \sup_{\alpha<i}\rho_{U_\alpha}(i) = \rho^{(1)}(i)$. By the choice of μ_i, (1) of Claim 2.11

$$\beta \in (A_{\alpha^*})^{(1)}_\alpha \cap \rho^{(1)}(i) = (A_{\mu_i})^{(1)}_\alpha \cap \rho^{(1)}(i).$$

Finally for γ, if $i < \pi_{U_{\alpha,\beta}}(\gamma)$, then $\gamma \in (A_{\mu_i})^{(2)}_{\alpha,\beta}$. If $i \geq \pi_{U_{\alpha,\beta}}(\gamma)$, then as in the previous paragraph, $\beta < \pi_{U_{\alpha,\beta}}(\gamma) \leq i$ and thus

$$\gamma < \rho_{U_{\alpha,\beta}}(i) \leq \rho^{(2)}(i).$$

We conclude that $\gamma \in (A_{\alpha^*})^{(2)}_{\alpha,\beta} \cap \rho^{(2)}(i)$. By the choice of μ_i and (2) of Claim 2.11, $\gamma \in (A_{\mu_i})^{(2)}_{\alpha,\beta} \cap \rho^{(2)}(i)$. ∎

By the claim, that for every $\langle\alpha,\beta,\gamma\rangle \in A_*$ and every $i < \kappa$, $\langle\alpha,\beta,\gamma\rangle \in A_{\mu_i}$, namely $A_* \subseteq \bigcap_{i<\kappa} A_{\mu_i}$. Finally, we note that $A_* \in Z$. Indeed, $(A_*)^{(0)} \in D$ by the choice of $(A_*)^{(0)}$. Also, for every $i < \kappa$, and $\alpha \in (A_*)^{(0)}$, $\alpha \in (A_{\mu_i})^{(0)}$ and so $(A_{\mu_i})^{(1)}_\alpha \in U_\alpha$. We conclude $(A_*)^{(1)}_\alpha \in U_\alpha$. Also, for $\beta \in (A_*)^{(1)}_\alpha$, $\beta \in (A_{\mu_i})^{(1)}_\alpha$ and therefore $(A_{\mu_i})^{(2)}_{\alpha,\beta} \in U_{\alpha,\beta}$. It follows that $(A_*)^{(2)}_{\alpha,\beta} \in U_{\alpha,\beta}$. Hence, $A_* \in Z$, and in particular, $\bigcap_{i<\kappa} A_{\mu_i} \in Z$.

Recall that the sequence of $\langle U_\alpha\rangle_{\alpha\in X}$ is called *discrete* if there is a sequence of pairwise disjoint sets $\langle A_\alpha\rangle_{\alpha\in X}$ such that $A_\alpha \in U_\alpha$. We say that $\langle U_\alpha\rangle_{\alpha\in X}$ is discrete mod U, if there is $Y \in U$, $Y \subseteq X$ such $\langle U_\alpha\rangle_{\alpha\in Y}$ is discrete.

Fact 2.13. *$\sum_U \langle U_\alpha\rangle_{\alpha<\kappa} \equiv_{RK} U\text{-}\lim\langle U_\alpha\rangle_{\alpha<\kappa}$ iff $\langle U_\alpha\rangle_{\alpha<\kappa}$ is discrete mod U.*

Proposition 2.14 *If U is a p-point ultrafilter, then any sequence $\langle U_\alpha\rangle_{\alpha<\kappa}$ of distinct κ-complete ultrafilters is discrete mod U.*

Proof See [24, Corollary 5.15]. ∎

Definition 2.15. (Orderings of ultrafilters) Let U, W be ultrafilters over ordinals κ, λ (resp.) define:

(1) The *Rudin–Keisler order* by $U \leq_{RK} W$ if there is a function $\pi : \lambda \to \kappa$ such that $U = \{B \subseteq \kappa \mid \pi^{-1}[B] \in W\}$.
(2) The *Rudin–Frolík* order by $U \leq_{RF} W$ if there is a set $I \in U$ and a discrete sequence $\langle W_i\rangle_{i\in I}$ of ultrafilters over κ such that $W = U\text{-}\lim\langle W_i\rangle_{i\in I}$.
(3) The *Ketonen order* by $U <_{\Bbbk} W$ if $j''_W U$ is contained in a countably complete ultrafilter U^* of M_W such that $[id]_W \in U^*$.

For more background on ultrafilters, their orderings, and the Ultrapower Axiom, we refer the reader to [24] and [17].

We also record here the definition and basic properties of the *canonical functions*.

Definition 2.16. For every $\eta < \kappa^+$, we fix a cofinal sequence $\langle\eta_i\rangle_{i<\mathrm{cf}(\eta)}$. Define recursively the canonical functions $f_\alpha : \kappa \to \kappa$ for $\alpha < \kappa^+$ as follows: $f_0 = 0$ is the constant function with value 0. Given f_α, define $f_{\alpha+1}(x) = f_\alpha(x) + 1$. For limit $\eta < \kappa^+$, we split into cases:

(1) If $\mathrm{cf}(\eta) < \kappa$, define $f_\eta(x) = \sup_{i<\mathrm{cf}(\eta)} f_{\eta_i}(x)$.
(2) If $\mathrm{cf}(\eta) = \kappa$, define $f_\eta(x) = \sup_{i<x} f_{\eta_i}(x)$.

It is not hard to see that the canonical functions are increasing modulo the bounded ideal, but the main reason we are interested in those functions is the following.

Proposition 2.17 *Let $k : N \to M$ be an elementary embedding (not necessarily definable in N) with critical point κ. Then for every $\alpha < (\kappa^+)^N$, $k(f_\alpha)(\kappa) = \alpha$.*

Proof By induction on α. Clearly, for $\alpha = 0$, $k(f_0)(\kappa) = 0$ and if $k(f_\alpha)(\kappa) = \alpha$, then by elementarity $k(f_{\alpha+1})(\kappa) = \alpha + 1$. For limit η, if $\mathrm{cf}(\eta) < \kappa$, then the functions used in the definition of f_η are $\langle f_{\eta_i}\rangle_{i<\mathrm{cf}(\eta)}$ are pointwise mapped by k; that is, $k(\langle f_{\eta_i} \mid i <$

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$\mathrm{cf}(\eta)\rangle) = \langle k(f_{\eta_i}) \mid i < \mathrm{cf}(\eta)\rangle$. It follows by elementarity and the definition of f_η that $k(f_\eta)(\kappa) = \sup_{i<\mathrm{cf}(\eta)} k(f_{\eta_i})(\kappa)$. Hence, by the induction hypothesis, $k(f_\eta)(\kappa) = \sup_{i<\mathrm{cf}(\eta)} \eta_i = \eta$. If $\mathrm{cf}(\eta) = \kappa$, then the sequence $\langle f_{\eta_i} \mid i < \kappa\rangle$ is stretched by k to $k(\langle f_{\eta_i} \mid i < \kappa\rangle) = \langle f'_{\eta_i} \mid i < k(\kappa)\rangle$ but for every $i < \kappa$, as $k(i) = i$, we have $f'_{\eta_i} = k(f_{\eta_i})$. Again by the definition of f_η, elementarity, and the induction hypothesis, we conclude that

$$k(f_\eta)(\kappa) = \sup_{i<\kappa} f'_{\eta_i}(\kappa) = \sup_{i<\kappa} k(f_{\eta_i})(\kappa) = \sup_{i<\kappa} \eta_i = \eta. \qquad \blacksquare$$

3 Diamond-like principle and the Galvin property

In [6], a relation between Kurepa trees and the Galvin property has been established to construct a κ-complete non-Galvin ultrafilter. In this section, we exploit the deep connection between Kurepa trees and diamond principles which was first observed by Jensen [22], to find new combinatorial properties of ultrafilters which ensures the Galvin property.

Definition 3.1. Let S be a stationary set. $\diamond^*(S)$ is the assertion that there is a sequence $\langle \mathcal{A}_\alpha\rangle_{\alpha\in S}$ such that $\mathcal{A}_\alpha \subseteq P(\alpha)$ and:

(1) $|\mathcal{A}_\alpha| \leq \alpha$.
(2) For every $X \subseteq \kappa$, there is a club C such that for each $\alpha \in C \cap S$, $C \cap \alpha, X \cap \alpha \in \mathcal{A}_\alpha$.

Proposition 3.2 *If $\diamond^*(S)$ holds, then any ultrafilter U over a regular cardinal κ satisfying $Club_\kappa \cup \{S\} \subseteq U$ and $\mathrm{cf}^{M_U}([id]_U) \leq \mathrm{crit}(j_U)$ must be non-Galvin.*

Proof Suppose otherwise, and let C_X for every $X \subseteq \kappa$ be the club guaranteed by item (2) of $\diamond^*(S)$. Then $C_X \in U$. Also, for each $\alpha \in S$, let $\langle I^\alpha_i\rangle_{i<\mathrm{cf}(\alpha)}$ be a partition of $\mathcal{A}_\alpha$ such that $|I^\alpha_i| < \alpha$. Now for each $X \subseteq \kappa$, consider the function $f_X : C_X \cap S \to \kappa$ defined by $f_X(\alpha) = i < \mathrm{cf}(\alpha)$ for the unique i such that $X \cap \alpha \in I^\alpha_i$. Since $cf^{M_U}([id]_U) \leq \mathrm{crit}(j_U)$, there is a function $\pi : \kappa \to On$ such that $i < \mathrm{cf}(\alpha) \leq \pi(\alpha)$ and $[\pi]_U = \mathrm{crit}(j_U)$. It follows that there is $A_X \subseteq C_X \cap S$, $A_X \in U$ and $\gamma_X < \kappa$ such that for every $\alpha \in A_X$, $f_X(\alpha) = \gamma_X$. There are 2^κ-many subsets with the same $\gamma_X = \gamma^*$. Now apply Galvin's property to those 2^κ-many sets in order find κ-many distinct subsets of κ, $\langle X_\xi\rangle_{\xi<\kappa}$ for which $A^* := \bigcap_{\xi<\kappa} A_{X_\xi} \in U$. Now for each $\alpha \in A^* \cap S$, $|I^\alpha_{\gamma^*}| < \alpha$. Since κ is regular, we may apply Födor's lemma to find a stationary set $S' \subseteq A^* \cap S$ and $\theta < \kappa$ such that $|I^\alpha_{\gamma^*}| = \theta$ for each $\alpha \in S'$. Consider $\langle X_i\rangle_{i<\theta^+}$ and for each $i \neq j < \theta^+$, let $\beta_{i,j} < \kappa$ be high enough so that $X_i \cap \beta_{i,j} \neq X_j \cap \beta_{i,j}$. Take any $\alpha \in S' \setminus \sup_{i\neq j<\theta^+} \beta_{i,j}$. To reach a contradiction, note that on one hand, since $\alpha \in S'$, $|I^\alpha_{\gamma^*}| = \theta$. On the other hand, for every $i \neq j < \theta^+$, $X_i \cap \alpha \in I^\alpha_{\gamma^*}$ and the sets $X_i \cap \alpha$ are all distinct. $\blacksquare$

Let us introduce a similar guessing principle $\diamond^*_{\mathrm{thin}}(U)$ to the one above, which can be formulated in terms of the ultrapower and does not involve the club filter. Then we will prove that $\diamond^*_{\mathrm{thin}}(U)$ implies that U is non-Galvin.

Definition 3.3. An ultrafilter W on a regular cardinal κ satisfies $\diamond^*_{\mathrm{thin}}(W)$ if there is a sequence of sets $\langle \mathcal{A}_\alpha\rangle_{\alpha<\kappa}$ such that:

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(1) For all $A \subseteq \kappa$, for W-almost all α, $A \cap \alpha \in \mathcal{A}_\alpha$.
(2) $\alpha \mapsto |\mathcal{A}_\alpha|$ is not almost one-to-one mod W.

The sequence $\langle \mathcal{A}_\alpha \rangle_{\alpha<\kappa}$ is called a $\diamond^*_{\text{thin}}(U)$*-sequence.*

In the ultrapower, this is expressed as follows.

Lemma 3.4 *$\diamond^*_{\text{thin}}(U)$ is equivalent to the existence of a set $A \in M_U$ such that:*

(1) $\{j_U(S) \cap [id]_U \mid S \subseteq \kappa\} \subseteq A$.
(2) There is no function $f: \kappa \to \kappa$ such that $j_U(f)(|A|^M) \geq [id]_U$.

Proof The witnessing $\diamond^*_{\text{thin}}(U)$-sequence is just the sequence $\langle \mathcal{A}_\alpha \rangle_{\alpha<\kappa}$ representing A in M_U. Clearly, condition (1) is equivalent to the fact that for every $S \subseteq \kappa$, $\{\alpha < \kappa \mid S \cap \alpha \in \mathcal{A}_\alpha\} \in U$. By Proposition 2.3, condition (2) is equivalent to the function $\alpha \mapsto |\mathcal{A}_\alpha|$ not being almost one-to-one mod U. ∎

Lemma 3.5 *If $\diamond^*_{\text{thin}}(W)$, then W is non-Galvin.*

Proof Assume toward contradiction that W has the Galvin property. Enumerate $\mathcal{A}_\alpha = \{A_{\alpha,i} \mid i < |\mathcal{A}_\alpha|\}$. For every set X, there is $B_X \in W$ such that for every for every $\alpha \in B_X$, $X \cap \alpha \in \mathcal{A}_\alpha$. By our assumption, there are κ-many distinct sets $\{X_i \mid i < \kappa\}$ such that $B := \bigcap_{i<\kappa} B_{X_i} \in W$. Note that the key property of B is that for every $i < \kappa$ and for all $\alpha \in B$, $X_i \cap \alpha \in \mathcal{A}_\alpha$. Since the function $\alpha \mapsto |\mathcal{A}_\alpha|$ is not almost one-to-one mod W, there is $\theta < \kappa$ and an unbounded subset $B' \subseteq B$ such that for every $\alpha \in B'$, $|\mathcal{A}_\alpha| = \theta$. Consider $\{X_i \mid i < \theta^+\}$. For every $i \neq j < \theta^+$, find $\alpha_{i,j} < \kappa$ such that $X_i \cap \alpha_{i,j} \neq X_j \cap \alpha_{i,j}$ and take $\alpha^* = \sup_{i,j<\theta^+} \alpha_{i,j}$. By regularity of κ, $\alpha^* < \kappa$. Since B' is unbounded there exists some $\beta^* \in B'$ with $\beta^* > \alpha^*$. It follows that for every $i < \theta^+$, $X_i \cap \beta^* \in \mathcal{A}_{\beta^*}$, and also for every $i \neq j$, since $\alpha_{i,j} < \beta^*$, $X_i \cap \beta^* \neq X_j \cap \beta^*$. It follows that $i \mapsto X_i \cap \beta^*$ is a one-to-one function from θ^+ into $\mathcal{A}_{\beta^*}$. This contradicts the fact that $\beta^* \in B'$ and thus $|\mathcal{A}_{\beta^*}| = \theta$. ∎

Corollary 3.6 *Suppose that κ is regular and U is an ultrafilter extending the club filter on κ. Assume that there is a sequence of sets $\langle \mathcal{A}_\alpha \rangle_{\alpha<\kappa}$ such that:*

(1) For every $\alpha < \kappa$, $|\mathcal{A}_\alpha| < \alpha$.
(2) For every $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha \in \mathcal{A}_\alpha\} \in U$.

Then $\diamond^*_{\text{thin}}(U)$ holds and, in particular, U is non-Galvin.

Proof It remains to show that $\alpha \mapsto |\mathcal{A}_\alpha|$ is not one-to-one on a set in U. If $A \in U$, then A is stationary since $\text{Club}_\kappa \subseteq U$. By Födor applied to the function $\alpha \mapsto |\mathcal{A}_\alpha|$ restricted to A, there is an unbounded subset $S' \subseteq A$ and $\theta < \kappa$ such that for every $\alpha \in S'$, $|\mathcal{A}_\alpha| = \theta$. In particular, $\alpha \mapsto |\mathcal{A}_\alpha|$ is not almost one-to-one on A. ∎

The most important class of ultrafilters which satisfy $\diamond^*_{\text{thin}}$ are the non p-point Dodd sound ultrafilters as will be proven in Lemma 3.8. To prove that lemma, we will need the following characterization due to Goldberg of Dodd sound ultrafilters [17, Theorem 4.3.26]:

Theorem 3.7 *A uniform ultrafilter U on an ordinal δ is Dodd sound if and only if there is a sequence $\langle \mathcal{A}_\alpha \rangle_{\alpha<\delta}$ such that for any sequence $\langle S_\alpha \subseteq \alpha \rangle_{\alpha<\delta}$, the following are equivalent:*

(a) *There is a set $S \subseteq \kappa$ such that for U-almost every α, $S \cap \alpha = S_\alpha$.*
(b) *For U-almost every α, $S_\alpha \in \mathcal{A}_\alpha$.*

Lemma 3.8 *Let κ be regular, and let U a non p-point Dodd sound ultrafilter, then $\diamondsuit^*_{\text{thin}}(U)$.*

Proof Assume that U is a non p-point Dodd sound ultrafilter. Let $\langle \mathcal{A}_\alpha \rangle_{\alpha<\kappa}$ be the sequence obtained by Theorem 3.7. Note that for every $S \subseteq \kappa$, the sequence $\langle S \cap \alpha \rangle_{\alpha<\kappa}$ satisfies condition (a) of Theorem 3.7. By the theorem, we conclude that for U-almost every α, $S \cap \alpha \in \mathcal{A}_\alpha$. It follows that $j_U(S) \cap [id]_U \in [\alpha \mapsto \mathcal{A}_\alpha]$ and $\{j_U(S) \cap [id]_U \mid S \subseteq \kappa\} \subseteq [\alpha \mapsto \mathcal{A}_\alpha]_U$. Similarly, from the implication (b) to (a) we deduce that $[\alpha \mapsto \mathcal{A}_\alpha]_U \subseteq \{j_U(S) \cap [\text{id}]_U \mid S \subseteq \kappa\}$. By Dodd soundness, the function $j^{[\text{id}]_U}: P(\kappa) \to \{j_U(S) \cap [\text{id}]_U \mid S \subseteq \kappa\}$ defined by $j^{[\text{id}]_U}(S) = j(S) \cap [\text{id}]_U$ belongs to M_U. Thus $M_U \models |[\alpha \mapsto \mathcal{A}_\alpha]_U| = 2^\kappa$. Finally, $\alpha \mapsto |\mathcal{A}_\alpha|$ cannot be an almost one-to-one function mod U: otherwise, the class of any unbounded function $\kappa \leq [\pi]_U$ would also be an almost one-to-one mod U. To see this, suppose that $[\tau]_U = \kappa$, then $[\alpha \mapsto 2^{\tau(\alpha)}]_U = 2^\kappa$ and by our assumption, this is represented by an almost one-to-one function mod U[9]. Let $X \in U$ be the set witnessing that $\alpha \mapsto 2^{\tau(\alpha)}$ is almost one-to-one mod U. Also we let $Y \in U$ be such that for every $\alpha \in Y$, $\tau(\alpha) \leq \pi(\alpha)$. We claim that $\pi \restriction X \cap Y$ is almost one-to-one as for any $\gamma < \kappa$,

$$\{\alpha < \kappa \mid \pi(\alpha) < \gamma\} \cap X \cap Y \subseteq \{\alpha < \kappa \mid \tau(\alpha) < \gamma\} \cap X \cap Y \subseteq$$
$$\subseteq \{\alpha < \kappa \mid 2^{\tau(\alpha)} \leq 2^\gamma\} \cap X \cap Y.$$

The right most set is bounded by the choice of X. We conclude that U is a p-point contradiction. ∎

Note that an ultrafilter U satisfying $\diamondsuit^*_{\text{thin}}(U)$ need not be Dodd sound since by Lemma 3.4, we only cover the set $\{j_U(S) \cap [id]_U \mid S \subseteq \kappa\}$. However, at least for κ-complete Dodd sound ultrafilters, the second requirement of $\diamondsuit^*_{\text{thin}}(U)$ regarding the function $\alpha \mapsto |\mathcal{A}_\alpha|$ is equivalent to U not being a p-point.

Proposition 3.9 *Let κ be measurable and U be a κ-complete Dodd sound ultrafilter over κ, and let $[\alpha \mapsto \mathcal{A}_\alpha]_U = \{j_U(S) \cap [id]_U \mid S \subseteq \kappa\}$. Then U is a non p-point ultrafilter if and only if the function $\alpha \mapsto |\mathcal{A}_\alpha|$ is not almost one-to-one mod U.*

Proof One direction follows from the previous lemma. Let us prove the other, note that $\alpha \mapsto |\mathcal{A}_\alpha|$ cannot be bounded on a set in U, just otherwise, suppose that $\theta < \kappa$ is such that $B^* := \{\alpha < \kappa \mid |\mathcal{A}_\alpha| \leq \theta\} \in U$. Take any θ^+-many sets $\{X_i \mid i < \theta^+\}$ such that there is $\gamma < \kappa$ such that for all $i \neq j < \theta^+$, $X_i \cap \gamma \neq X_j \cap \gamma$. For each $i < \theta^+$, Denote by $B_i := \{\alpha < \kappa \mid X_i \cap \alpha \in \mathcal{A}_\alpha\} \in U$. By κ-completeness and fineness, there is $\gamma^* \in B^* \cap (\bigcap_{i<\theta^+} B_i) \setminus \gamma$. It follows that $|\mathcal{A}_{\gamma^*}| = \theta$ but also for each $i < \theta^+$, $X_i \cap \gamma^* \in \mathcal{A}_{\gamma^*}$ are all distinct sets. Contradiction. We conclude that $\alpha \mapsto |\mathcal{A}_\alpha|$ is an unbounded function mod U which is also not almost one-to-one according to (1). Hence, U is not a p-point. ∎

We cannot drop the κ-completeness assumption here.

[9] Being an almost one-to-one function mod U is clearly a property of an equivalence class mod U.

Example 3.10. Suppose that W is a fine normal ultrafilter over $P_\kappa(\lambda)$ for $\kappa < \lambda$, where λ is a regular cardinal. By [17, Theorems 4.4.37 and 4.4.25], there is a Dodd sound non uniform ultrafilter U on λ (and therefore p-point) which is Rudin–Keisler equivalent to W. Note that there is no function which is unbounded (and therefore no function which is almost one-to-one) mod U. In particular, $\alpha \mapsto |\mathcal{A}_\alpha|$ is not almost one-to-one mod U. Also, note that U satisfies $\diamondsuit^*_{\text{thin}}(U)$ and therefore is an example of a non-Galvin ultrafilter over λ which is uniform and not λ-complete.

Corollary 3.11 *If U is a non p-point, Dodd sound ultrafilter over a regular cardinal κ, then U is non-Galvin.*

In attempt to pinpoint the exact guessing principle that catches non-Galvinness, we note that the usage of $\diamondsuit^*_{\text{thin}}(W)$ in the argument of Lemma 3.5 can be replaced with the following weakening.

Definition 3.12. Let $\kappa \leq \lambda \leq 2^\kappa$. An ultrafilter W on a regular cardinal κ satisfies $\diamondsuit^*_{\text{par}}(W,\lambda)$ if there is a sequence of sets $\langle X_\alpha\rangle_{\alpha<\lambda}$, $A \in M_W$ such that:

(1) $\{j_U(X_\alpha) \cap [id]_U \mid \alpha < \lambda\} \subseteq A$.
(2) For any function $f : \kappa \to \kappa$, $j_U(f)(|A|^{M_W}) < [id]_W$.

Clearly, $\diamondsuit^*_{\text{thin}}(W)$ implies $\diamondsuit^*_{\text{par}}(W, 2^\kappa)$ which in turn imply $\diamondsuit^*_{\text{par}}(W,\lambda)$ for any $\lambda \in [\kappa, 2^\kappa]$.

Proposition 3.13 *$\diamondsuit^*_{\text{par}}(W,\lambda)$ implies that $\neg\mathrm{Gal}(W,\kappa,\lambda)$.*

Proof The argument of Lemma 3.5 gives this stronger result. ■

The principle $\diamondsuit^*_{\text{par}}(W,\lambda)$ is equivalent to the existence of a set $K \subseteq P(\kappa)$ of size λ and a sequence $\langle A_\alpha\rangle_{\alpha<\kappa}$ such that:

(1) For every $X \in K$, $\{\alpha < \kappa \mid X \cap \alpha \in A_\alpha\} \in W$.
(2) The function $\alpha \mapsto |A_\alpha|$ is not almost one-to-one mod W.

The referee pointed out to us the strong similarity of $\diamondsuit^*_{\text{par}}$ to the notion of pseudo-Kurepa families due to Todorcevic [36]. Indeed, many of the initial segments of the sets in K must be equal in order for the sets A_α of asymptotically bounded cardinality to exist.

Next, we would like to provide two closure properties of the class of ultrafilters satisfying $\diamondsuit^*_{\text{thin}}$.

Lemma 3.14 *Suppose U is an ultrafilter on κ and Z is the U-limit of a discrete sequence of ultrafilters W_ξ on κ such that $\diamondsuit^*_{\text{thin}}(W_\xi)$. Then $\diamondsuit^*_{\text{thin}}(Z)$.*

Proof Fix a partition of κ into sets $S_\xi \in W_\xi$. For each $\xi < \kappa$, let $\langle\mathcal{A}^\xi_\alpha\rangle_{\alpha<\kappa}$ witness that $\diamondsuit^*_{\text{thin}}(W_\xi)$. Then let $\mathcal{A}_\alpha = \mathcal{A}^\xi_\alpha$, where $\xi < \kappa$ is unique such that $\alpha \in S_\xi$. Fixing $A \subseteq \kappa$, we would like to show that $B := \{\alpha < \kappa \mid A \cap \alpha \in \mathcal{A}_\alpha\} \in U\text{-}\lim\langle W_\xi\rangle_{\xi<\kappa}$. For any $\xi < \kappa$, then $B_\xi := \{\alpha \in S_\xi \mid A \cap \alpha \in \mathcal{A}^\xi_\alpha\} \in W_\xi$. Since for each $\alpha \in S_\xi$, $\mathcal{A}_\alpha = \mathcal{A}^\xi_\alpha$, we conclude that $B_\xi \subseteq B$ and therefore $B \in W_\xi$. It follows that $B \in U\text{-}\lim\langle W_\xi\rangle_{\xi<\kappa}$. It remains to show that $c(\alpha) = |\mathcal{A}_\alpha|$ is not almost one-to-one on any set $B \in W$. Suppose otherwise, and let $B \in W$ witness that c is almost one-to-one. Pick any $\xi < \kappa$ such that $B \in W_\xi$ to reach a contradiction note that $B \cap S_\xi \in W_\xi$, and the function c is almost one-to-one

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on this set. However, for every $\alpha \in B \cap S_\xi$, $\mathcal{A}^\xi_\alpha = \mathcal{A}_\alpha = c(\alpha)$ and so $\alpha \mapsto |\mathcal{A}^\xi_\alpha|$ is almost one-to-one on $B \cap S_\xi$, contradicting $\diamondsuit^*_{\text{thin}}(W_\xi)$. ∎

Lemma 3.15 *Suppose U is an n-fold sum of p-points on κ and $\langle W_\xi \rangle_{\xi<\kappa}$ is a sequence of (not necessarily discrete) κ-complete ultrafilters on κ such that $\diamondsuit^*_{\text{thin}}(W_\xi)$. Then letting $Z = U\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa}$, we have $\diamondsuit^*_{\text{thin}}(Z)$.*

Proof We first consider the case that U is a p-point. Then replace U with $U_W = D(j_U, W)$, where W is the point in M_U represented by $\xi \mapsto W_\xi$. Note that U_W is Rudin–Keisler below an ultrafilter on κ which implies that U_W concentrates on a set of (κ-complete) ultrafilters of size κ. By enumerating those ultrafilters W'_ξ for $\xi < \kappa$, we can shift U_W to an ultrafilter U' on κ such that $[\text{id}]_{U_W}$ is identified with $[\xi \mapsto W'_\xi]_{U'}$. Also, note that $U' - \lim W'_\xi = U - \lim W_\xi$ since the factor map $k\colon M_{U'} \to M_U$ sends $k([\xi \mapsto W'_\xi]_{U'}) = W$ and thus

$$X \in U' - \lim \langle W'_\xi \rangle_{\xi<\kappa} \Leftrightarrow j_{U'}(X) \in [\xi \mapsto W'_\xi]_{U'} \Leftrightarrow$$

$$\Leftrightarrow j_U(X) = k(j_{U'}(X)) \in W \Leftrightarrow X \in U - \lim \langle W_\xi \rangle_{\xi<\kappa}.$$

Since $U' \leq_{RK} U$, and U is a p-point, U' is also a p-point (see [24, Corollary 2.8]). The sequence $\langle W'_\xi \rangle_{\xi<\kappa}$ represents the identity in U', it is one-to-one mod U', since all the W'_ξ's are κ-complete, by Proposition 2.14 the sequence is discrete on a set in U'.[10] This allows us to apply the previous lemma, obtaining thin diamond for $U'\text{-}\lim \langle W'_\xi \rangle_{\xi<\kappa} = U\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa}$.

Now suppose the lemma is true for n-fold sums of p-points, and we will prove it when U is an $(n+1)$-fold sum. We can fix a p-point D such that U is the D-limit of a sequence of n-fold sum p-points U_ξ on κ. As in the previous paragraph, since D is a p-point, we may assume that the U_ξ's are discrete. Let $U^* = [\xi \mapsto U_\xi]_D$, then by elementarity, $M_D \vDash U^*$ is an n-fold sum of p-points. Applying the induction hypothesis in M_D to U^* and the ultrafilters $j_D(\langle W_\xi \rangle_{\xi<\kappa}) = \langle Z^*_\xi \rangle_{\xi<j_D(\kappa)}$, we conclude that $Z^* = U^*\text{-}\lim \langle Z^*_\xi \rangle_{\xi<j_D(\kappa)}$ satisfies $\diamondsuit^*_{\text{thin}}(Z^*)$. Let $[\alpha \mapsto Z_\alpha]_D = Z^*$ and assume without loss of generality that for every $\alpha < \kappa$, $\diamondsuit^*_{\text{thin}}(Z_\alpha)$ holds. We claim that

$$(*)\quad Z = D\text{-}\lim \langle Z_\alpha \rangle_{\alpha<\kappa} = U\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa}$$

from which it follows that $\diamondsuit^*_{\text{thin}}(Z)$, by the argument of the previous paragraph. To see $(*)$, since we assumed that the U_α's are discrete, by the theory of sums and limits of ultrapower

$$j_{\sum_D \langle U_\alpha \rangle_{\alpha<\kappa}} = j_{D\text{-}\lim \langle U_\alpha \rangle_{\alpha<\kappa}} = j_{U^*} \circ j_D \text{ and } [\text{id}]_{D\text{-}\lim \langle U_\alpha \rangle_{\alpha<\kappa}} = [\text{id}]_{U^*},$$

hence

$$X \in D\text{-}\lim \langle Z_\alpha \rangle_{\alpha<\kappa} \Leftrightarrow j_D(X) \in Z^* = U^*\text{-}\lim \langle Z^*_\xi \rangle_{\xi<j_D(\kappa)} \Leftrightarrow$$

[10] Note that even if the W_ξ's we started with were not distinct, the W'_ξ's will be distinct on a set in U'. For example, if $W_\xi = W_0$ for every ξ, then U_W is the principle ultrafilter concentrating on $\{W_0\}$ and thus U' is principle and $W_0 = W'_\xi$. It is still true that on a measure one set in U', i.e., $\{0\}$, the sequence $\langle W'_\xi \rangle_{\xi<\kappa}$ is distinct. In this case, the lemma is trivial as $Z = W_0$.

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$$\Leftrightarrow j_{U^*}(j_D(X)) \in j_{U^*}(j_D(\langle W_\xi \rangle_{\xi<\kappa}))([\mathrm{id}]_{U^*}) \Leftrightarrow$$

$$\Leftrightarrow j_{D\text{-}\lim \langle U_\alpha \rangle_{\alpha<\kappa}}(X) \in j_{D\text{-}\lim \langle U_\alpha \rangle_{\alpha<\kappa}}(\langle W_\xi \rangle_{\xi<\kappa})([\mathrm{id}]_{D\text{-}\lim \langle U_\alpha \rangle_{\alpha<\kappa}}) \Leftrightarrow$$

$$\Leftrightarrow X \in (D\text{-}\lim \langle U_\alpha \rangle_{\alpha<\kappa})\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa} \Leftrightarrow X \in U\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa}. \qquad \blacksquare$$

4 Partial Dodd soundness and skies

A finer analysis of the diamond-like principles of the previous section reveals that partial soundness suffices for an ultrafilter to be non-Galvin. To better understand this improvement, let us prove the following theorem in terms of general elementary embeddings.

Theorem 4.1 *Suppose that $j\colon V \to M$ is an elementary embedding with* $\mathrm{crit}(j) = \kappa$ *such that $\lambda = \sup\{j(f)(\kappa) \mid f\colon \kappa \to \kappa\}$ and $\{j(A) \cap \lambda \mid A \subseteq \kappa\} \in M$. Then there is ξ such that $D := D(j, \xi)$ and $\neg\mathrm{Gal}(D, \kappa, 2^\kappa)$.*

Remark 4.2. Note that from the assumptions of the theorem it follows that $\lambda < j(\kappa)$, indeed, if $\lambda = j(\kappa)$, then since we are assuming $\{j(A) \cap \lambda \mid A \subseteq \kappa\} \in M$, we have $j''P(\kappa) \in M$ and therefore $\{j(f)(\kappa) \mid f : \kappa \to \kappa\} \in M$. It follows that $M \vDash cf(j(\kappa)) = 2^\kappa$. But by elementarity, $M \vDash j(\kappa)$ is regular. Contradiction.

Proof Denote $\mathcal{A} = \{j(A) \cap \lambda \mid A \subseteq \kappa\} \in M$. Enumerate V_κ in V, $f\colon \kappa \to V_\kappa$ such that for every $x \in V_\kappa$, $f^{-1}[x]$ is unbounded in κ. Since $\mathcal{A} \in (V_{j(\kappa)})^M$, there is $j(\kappa) > \xi \geq \lambda$ such that $j(f)(\xi) = \mathcal{A}$. By similar arguments, we can ensure that there are some functions $g, h : \kappa \to \kappa$ such that for the same ξ we will also have $\kappa = j(g)(\xi)$ and $\lambda = j(h)(\xi)$. Let $D = D(j, \xi)$, $j_D\colon V \to M_D$ be the ultrapower, and let $k_D\colon M_D \to M$ be the factor map $k_D([\phi]_D) = j(\phi)(\xi)$. Note that

$$\lambda = k_D([h]_D),\ \kappa = k_D([g]_D),\ \mathcal{A} = k_D([f]_D)$$

and therefore $\kappa, \lambda, \mathcal{A} \in Im(k_D)$. It follows that $\mathrm{crit}(k_D) > \kappa$ and $[g]_D = \kappa$. Since

$$k_D([h]_D) = \lambda \leq \xi = k_D([id]_D),$$

the elementarity of k_D implies that $[h]_D \leq [id]_D$. Recall that for any function $\phi\colon \kappa \to \kappa$, $j(\phi)(\kappa) < \lambda$ thus by elementarity of k_D,

$$(*) \text{ for any function } \phi\colon \kappa \to \kappa,\ j_D(\phi)(\kappa) < [h]_D.$$

By our initial assumption, $\lambda > j(\alpha \mapsto 2^\alpha)(\kappa) = 2^\kappa$ and since $M \vDash |\mathcal{A}| = 2^\kappa$,

$$M_D \vDash |[f]_D| = 2^{[g]_D} < [h]_D.$$

Denote by $B_\alpha = f(\alpha)$, note that and fix a set $X^* \in D$ such that if $\alpha \in X^*$ then $|B_\alpha| = 2^{g(\alpha)} < h(\alpha)$. Pick any 2^κ distinct subsets of κ, $\langle A_\alpha \rangle_{\alpha<2^\kappa}$, then $j(A_\alpha) \cap \lambda \in \mathcal{A}$ and by elementarity $j_D(A_\alpha) \cap \lambda' \in B$. It follows that

$$X_\alpha := \{\xi < \kappa \mid A_\alpha \cap h(\xi) \in B_\xi\} \in D.$$

We claim that $\langle X_\alpha \rangle_{\alpha<2^\kappa}$ witness that $\neg\mathrm{Gal}(U, \kappa, 2^\kappa)$. Otherwise, there is $I \in [2^\kappa]^\kappa$ such that $X_I := \cap_{i\in I} X_i \in D$. Let us argue that there must be $\theta < \kappa$ such that

$$\sup\{h(\xi) : \xi \in X_I \cap X^*,\ 2^{g(\xi)} < \theta\} = \kappa.$$

To see this, assume otherwise, then for each $\theta < \kappa$, we can define

$$\rho(\theta) = \sup\{h(\xi) \mid \xi \in X_I \cap X^*,\ 2^{g(\xi)} \leq 2^{\theta}\}$$

then $\rho{:}\,\kappa \to \kappa$ is well-defined. Since $2^{j_D(g)([id]_D)} = 2^{[g]_D} = 2^{\kappa}$, we conclude that $j_D(\rho)(\kappa) \geq j_D(h)([\mathrm{id}]_D) = [h]_D$, contradicting $(*)$. We proceed as before, find $\beta \in X_I \cap X^*$ such that the restriction of θ-many of the sets in I to $h(\beta)$ are distinct. This produces a contradiction. ■

Let us define the concept of a *sky of an elementary embedding at* δ, which was first considered in the case that $\delta = \omega$ by Puritz [30, 31] and generalized to measurable cardinals later by Kanamori [24]. This concept will enable us to simplify our future definitions.

Definition 4.3. Let $j : V \to M$ be an elementary embedding where M is transitive, and let κ be any cardinal. We define a transitive relation on $[\sup(j''\kappa), j(\kappa))$: $\alpha \preceq \beta$ if there is a function $f : \kappa \to \kappa$ such that $j(f)(\beta) \geq \alpha$. We derive the equivalence relation $\alpha \equiv \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. A *sky of* j *at* κ is a $\equiv$-equivalence class. We denote by $sky(\alpha)$ the sky of α at κ for the unique κ such that $\alpha \in [\sup(j''\kappa), j(\kappa))$.

Note that the only interesting situation is when κ is not a continuity point of j. Since M is transitive, $\prec$ is a well-defined well-ordering of the skies. Moreover, since $\alpha \leq \beta$ implies $\alpha \preceq \beta$, then each sky is a half-open interval.

Suppose now that U is a σ-complete ultrafilter over κ. It is clear that for any $\alpha < j_U(\kappa)$, $\alpha \preceq [id]_U$ as $\alpha = [f]_U$ for some $f : \kappa \to \kappa$ and therefore $\alpha = j_U(f)([id]_U)$. So $sky([id]_U)$ is the maximal sky. This simple observation, together with Proposition 2.3, leads to an elegant characterization of p-points in terms of skies.

Corollary 4.4 *Let U be a σ-complete ultrafilter over κ, then U is a p-point if and only if j_U has a unique sky at κ.*

We can now reformulate Theorem 4.1 in terms of skies.

Corollary 4.5 *Suppose that U is a κ-complete, λ-sound ultrafilter over κ such that λ is the least element of the second sky at κ. Then U is non-Galvin.*

Proof By the definition of ξ in the proof of Theorem 4.1, we can choose $\xi = [\mathrm{id}]_U$ and the theorem ensures that $U = D(j_U, [\mathrm{id}]_U)$ is non-Galvin. ■

Note that a embedding j with critical point κ has at least two skies at κ if and only if $\sup\{j(f)(\kappa) \mid f{:}\,\kappa \to \kappa\} < j(\kappa)$.

Corollary 4.6 *Suppose that there is a superstrong embedding $j{:}\,V \to M$ with* $\mathrm{crit}(j) =$ *κ and at least two skies. Then κ carries a non-Galvin ultrafilter.*

The reason that $\diamondsuit^*_{\mathrm{thin}}$ (Definition 3.1) is not equivalent to Dodd soundness is that we are only trying to cover $\{j_U(S) \cap [id]_U \mid S \subseteq \kappa\}$ with a set A in M_U, while in Dodd soundness we need the actual set $\{j_U(S) \cap [id]_U \mid S \subseteq \kappa\}$ to be in M_U. Let us call this property *covering soundness*. The innovation here is to work with covering λ-soundness which is just the ability to cover $\{j_U(S) \cap \lambda \mid S \subseteq \kappa\}$.

However, without any further assumptions, we can always take $P^{M_U}(\lambda)$ as our covering set, so covering λ-soundness is always true. What makes $\diamondsuit^*_{\mathrm{thin}}$ nontrivial is

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the second requirement that there is no function $f : \kappa \to \kappa$ such that $j_U(f)(|A|^{M_U}) \geq [id]_U$. This rules out our previous example of $P^{M_U}(\lambda)$ or any other trivial example. Equipped with our new terminology of skies, we note that (2) of Definition 3.1 is in fact equivalent to $|A|^M$ not laying the top sky (namely $sky(|A|^{M_U}) \prec sky([id]_U)$).

Assuming (full) λ-soundness, the results of this section ensure that the "covering" set could be chosen to be precisely $\{j_U(S) \cap \lambda \mid S \subseteq \kappa\}$. Moreover, with this choice, the M_U-cardinality of the covering set is 2^κ. Then, under the assumption on λ in Theorem 4.1, there is no function $f : \kappa \to \kappa$ such that $j_U(f)(2^\kappa) \geq \lambda$.

Bearing the idea of skies in mind, we see the following common theme: if A is the covering set and λ is the degree of covering soundness, then $sky(|A|^{M_U}) \prec sky(\lambda)$. Let us formulate a diamond-like principle which generalize both Theorem 4.1 and $\diamondsuit^*_{\text{thin}}(U)$. It corresponds to covering λ-soundness, allowing λ to lay in an arbitrary sky (except the least one). This diamond-like principle is essential to prove the characterization of σ-complete non-Galvin ultrafilters.

Definition 4.7. Let U be an ultrafilter over a regular cardinal κ. $\diamondsuit^-_{\text{thin}}(U)$ is the statement that there is $A \in M_U$ and $\lambda < j_U(\kappa)$ such that:

(1) $\{j_U(S) \cap \lambda \mid S \subseteq \kappa\} \subseteq A$.

(2) There is no function $f : \kappa \to \kappa$ such that $j_U(f)(|A|^M) \geq \lambda$[11].

Clearly, $\diamondsuit^*_{\text{thin}}(U)$ implies $\diamondsuit^-_{\text{thin}}(U)$ by taking $\lambda = [id]_U$.

Corollary 4.8 *If U is an ultrafilter over a regular cardinal κ which is λ-sound where λ is such that for every function $f : \kappa \to \kappa$, $j_U(f)(\kappa) < \lambda$, then $\diamondsuit^-_{\text{thin}}(U)$.*

Proof By λ-soundness of U, $A := \{j_U(S) \cap \lambda \mid S \subseteq \kappa\} \in M_U$ and $M_U \vDash |A| = 2^\kappa$. There cannot be a function $g : \kappa \to \kappa$ such that $j_U(g)(2^\kappa) \geq \lambda$, since otherwise, the function $g'(\alpha) = g(2^\alpha)$ would be a function from κ to κ such that $j_U(g')(\kappa) \geq \lambda$, contradicting the assumptions of the corollary. ■

Theorem 4.9 *$\diamondsuit^-_{\text{thin}}(U)$ implies that U is non-Galvin.*

Proof Fix any $\langle X_\alpha \rangle_{\alpha<2^\kappa}$ sequence of distinct subsets of κ. $[\alpha \mapsto A_\alpha]_U = A$ and $[f]_U = \lambda = j_U(f)([id]_U)$. By our assumption,

$$B_\alpha = \{\xi < \kappa \mid X_\alpha \cap f(\xi) \in A_\alpha\} \in U.$$

We claim that $\langle B_\alpha \rangle_{\alpha<2^\kappa}$ witness that $\neg\mathrm{Gal}(U, \kappa, 2^\kappa)$. Otherwise, there is $I \in [2^\kappa]^\kappa$ such that $B_I := \cap_{i \in I} B_i \in U$. Consider the map $\xi \mapsto |A_\xi|$, note that $|A_\xi| \leq \pi(\xi)$ where $j_U(\pi)([\mathrm{id}]_U) = |A|$, and therefore there must be $\theta < \kappa$ such that

$$\sup\{f(\xi) : \xi \in B_I,\ \pi(\xi) < \theta\} = \kappa.$$

Just assume otherwise, then for each $\theta < \kappa$, we can define

$$g(\theta) = \sup\{f(\xi) \mid \xi \in B_I,\ \pi(\xi) \leq \theta\}$$

then $g : \kappa \to \kappa$ is well-defined. Since $j_U(\pi)([\mathrm{id}]_D) = |A|$ we conclude that $j_U(g)(|A|) \geq j_U(f)([\mathrm{id}]_D) = \lambda$, contradicting condition (2). Now the continuation is as before, we find $\beta \in B_I$ such that $f(\beta)$ is high enough so that the restriction of θ^+-many of the sets in I to $f(\beta)$ are distinct. This produces a contradiction. ■

[11] That is, $sky(|A|^M) \prec sky(\lambda)$.

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The advantage of using the class of ultrafilters satisfying $\diamondsuit^-_{\text{thin}}(U)$ over the class satisfying $\diamondsuit^*_{\text{thin}}$, is that is it upward closed with respect to the Rudin–Keisler ordering.

Lemma 4.10 *Suppose that $\diamondsuit^-_{\text{thin}}(U)$ holds and $U \leq_{RK} W$, then $\diamondsuit^-_{\text{thin}}(W)$ holds.*

Proof Let $k : M_U \to M_W$ be an elementary embedding such that $j_W = k \circ j_U$ and A, λ witnessing $\diamondsuit^-_{\text{thin}}(U)$. For every $S \subseteq \kappa$, we have

$$j_W(S) \cap k(\lambda) = k(j_U(S) \cap \lambda) \in k(A).$$

Hence $\{j_W(S) \cap k(\lambda) \mid S \subseteq \kappa\} \subseteq k(A) \in M_W$. By elementarity, $|k(A)|^{M_W} = k(|A|^{M_U})$. Suppose toward contradiction that there is a function $g : \kappa \to \kappa$ such that $j_W(g)(k(|A|^{M_U})) \geq k(\lambda)$, then $k(j_U(g)(|A|)) \geq k(\lambda)$ and by elementarity if k, $j_U(g)(|A|) \geq \lambda$, contradiction. ■

Lemma 4.11 *Suppose that Z is an ultrafilter on κ which is the U-limit of a discrete sequence of ultrafilters W_ξ on κ and such that $\diamondsuit^-_{\text{thin}}(W_\xi)$. Then $\diamondsuit^-_{\text{thin}}(Z)$.*

Proof Fix a partition of κ into sets $S_\xi \in W_\xi$. For each $\xi < \kappa$, let $\langle \mathcal{A}^\xi_\alpha \rangle_{\alpha<\kappa}$ and f_ξ witness that $\diamondsuit^-_{\text{thin}}(W_\xi)$. Then let $\mathcal{A}_\alpha = \mathcal{A}^\xi_\alpha$, where $\xi < \kappa$ is unique such that $\alpha \in S_\xi$ and $f(\alpha) = f_\xi(\alpha)$. Let $A \subseteq \kappa$, we would like to show that $B := \{\alpha < \kappa \mid A \cap f(\alpha) \in \mathcal{A}_\alpha\} \in U\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa}$. Take any $\xi < \kappa$, then $B_\xi := \{\alpha \in S_\xi \mid A \cap f_\xi(\alpha) \in \mathcal{A}^\xi_\alpha\} \in W_\xi$. Since for each $\alpha \in S_\xi$ and $f(\alpha) = f_\xi(\alpha)$, $\mathcal{A}_\alpha = \mathcal{A}^\xi_\alpha$, we conclude that $B_\xi \subseteq B$ and therefore $B \in W_\xi$. It follows that $B \in U\text{-}\lim \langle W_\xi \rangle_{\xi<\kappa}$. It remains to show that $c(\alpha) = |\mathcal{A}_\alpha|$ is in a lower sky than f. Suppose otherwise and let $g : \kappa \to \kappa$ such that for some $B \in W$, $\alpha \in B \to g(c(\alpha)) \geq f(\alpha)$. Pick any $\xi < \kappa$ such that $B \in W_\xi$ to reach a contradiction note that $B \cap S_\xi \in W_\xi$, and for every $\alpha \in B \cap S_\xi$, $g(|A^\xi|_\alpha) = g(c(\alpha)) \geq f(\alpha) = f_\xi(\alpha)$. However, the sky $\alpha \mapsto |\mathcal{A}^\xi_\alpha|$ is below the sky of f_ξ, contradicting the choice of f_ξ. ■

For a non-discrete sequence, we have the following.

Lemma 4.12 *Suppose that Z is an ultrafilter over κ which is Rudin–Keisler equivalent to $\sum_U \langle W_\xi \rangle_{\xi<\lambda}$, where U is any ultrafilter over $\lambda \leq \kappa$ and W_ξ's are ultrafilters over κ such that $\diamondsuit^-_{\text{thin}}(W_\xi)$ holds. Then $\diamondsuit^-_{\text{thin}}(Z)$ holds.*

Proof Let $W^* = [\xi \mapsto W_\xi]_U$. By our assumption,

$$M_U \models W^* \text{ is an ultrafilter over } j_U(\kappa) \text{ and } \diamondsuit^-_{\text{thin}}(W^*).$$

Let $j_{W^*} : M_U \to M_{W^*}$ be the ultrapower of M_U by W^*. It follows that there is $A \in M_{W^*}$ and $\lambda < j_{W^*}(j_U(\kappa))$ such that $\{j_{W^*}(S) \cap \lambda \mid S \in P(j_U(\kappa))^{M_U}\} \subseteq A$ and there is no function $f : j_U(\kappa) \to j_U(\kappa) \in M_U$ such that $j_{W^*}(f)(|A|^{M_{W^*}}) \geq \lambda$. Note that $M_{W^*} = M_{\sum_U \langle W_\xi \rangle_{\xi<\lambda}}$ and $j_{\sum_U \langle W_\xi \rangle_{\xi<\lambda}} = j_{W^*} \circ j_U$. We claim that A and λ witness that $\diamondsuit^-_{\text{thin}}(\sum_U \langle W_\xi \rangle_{\xi<\lambda})$. Indeed, for any $X \subseteq \kappa$, $j_U(X) \in P(j_U(\kappa))^{M_U}$ and therefore $j_{W^*}(j_U(X)) \cap \lambda \in A$. Similarly, for any function $f : \kappa \to \kappa$, $j_U(f) : j_U(\kappa) \to j_U(\kappa) \in M_U$ and therefore $j_{W^*}(j_U(f))(|A|^{M_{W^*}}) < \lambda$. ■

5 Non-Galvin cardinals

As pointed out in the Introduction, a measurable cardinal does not imply the existence of a non-Galvin ultrafilter [9]. In [1], the question regarding which large cardinal

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properties imply the existence of non-Galvin ultrafilters was raised and in [2] a κ-compact cardinal was proven to carry such an ultrafilter. We open this section with a new large cardinal property.

Definition 5.1. κ is called *non-Galvin cardinal* if there are elementary embeddings $j\colon V \to M$, $i\colon V \to N$, $k\colon N \to M$ such that:

(1) $k \circ i = j$.
(2) $\mathrm{crit}(j) = \kappa$, $\mathrm{crit}(k) = i(\kappa)$.
(3) ${}^{\kappa}N \subseteq N$ and ${}^{\kappa}M \subseteq M$.
(4) There is $A \in M$ such that $i''\kappa^+ \subseteq A$ and $M \vDash |A| < i(\kappa)$.

Note that by condition (4), $\kappa \subseteq A$ and that A can be chosen so that $\min(A \setminus \kappa) = i(\kappa)$.

The next proposition implies that we may assume that the embedding j in the definition of non-Galvin cardinals is an ultrapower embedding and the embedding i is an extender ultrapower derived from it.

Proposition 5.2 *Suppose that $j\colon V \to M$, $i\colon V \to N$, $k\colon N \to M$ and $A \in M$ are as in Definition 5.1. Then there is a κ-complete ultrafilter U over V_κ and $\rho < j_U(\kappa)$ which, together with the ultrapower by the (κ, ρ)-extender E derived from j_U and $[id]_U$, witnesses that κ is non-Galvin. Namely, the following hold:*

(1) $k_E \circ j_E = j_U$.
(2) $\mathrm{crit}(j_U) = \kappa$, $\mathrm{crit}(k_E) = \rho = j_E(\kappa)$.
(3) ${}^{\kappa}M_E \subseteq M_E$ *and* ${}^{\kappa}M_U \subseteq M_U$.
(4) $j_E''\kappa^+ \subseteq [id]_U$ *and* $M_U \vDash |[id]_U| < j_E(\kappa)$.

Proof We may assume $\sup(A) = \sup i''\kappa^+$ and $A \cap i(\kappa) = \kappa$. Let U be the ultrafilter derived from j using A. Let $\bar{A} = [\mathrm{id}]_U$, and let $k_U : M_U \to M$ be the unique elementary embedding with $k_U \circ j_U = j$ and $k_U(\bar{A}) = A$. Note that κ and $i(\kappa)$ are in the range of k_U since these ordinals are definable in M using A as a parameter: κ is the least ordinal not in A, and $i(\kappa) = |\sup(A)|^M$. Therefore $k_U(\kappa) = \kappa$. Let ρ be such that $k_U(\rho) = i(\kappa)$.

Let E be the extender of length ρ derived from j_U. Let $k_E : M_E \to M_U$ denote the unique factor embedding with $k_E \circ j_E = j_U$ and $k_E \restriction \rho = \mathrm{id}$.

$$\begin{tikzcd}
& M_E \arrow[r, "k_E"] & M_U \arrow[dd, "k_U"] \\
V \arrow[ur, "j_E"] \arrow[dr, "i"'] \arrow[urr, bend left, "j_U"] \arrow[drr, bend right, "j"'] & & \\
& N \arrow[r, "k"'] & M
\end{tikzcd}$$

We will verify (1), (2), (3), and (4). Of course, (1) is true essentially by the definition of k_E.

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For (2), note that $\mathrm{crit}(j_U) = \kappa$ since $k_U \circ j_U = j$ and $k_U(\kappa) = \kappa$. The fact that $\mathrm{crit}(k_E) \geq \rho$ follows from the definition of k_E. To see $\mathrm{crit}(k_E) = \rho$ and $k_E(\rho) = j_U(\kappa)$, we will show that[12]

$$\mathrm{Hull}^{M_U}(j_U'' V \cup \rho) \cap j_U(\kappa) = \rho.$$

This will establish that $\mathrm{crit}(k_E) = \rho$ and $k_E(\rho) = j_U(\kappa)$ since k_E is the inverse of the transitive collapse of $\mathrm{Hull}^{M_U}(j_U'' V \cup \rho)$. To prove this equality, it suffices to show the inclusion $\mathrm{Hull}^{M_U}(j_U'' V \cup \rho) \cap j_U(\kappa) \subseteq \rho$.

Since $k \circ i = j$ and since $\mathrm{crit}(k) = i(\kappa)$, we have $\mathrm{Hull}^M(j'' V \cup i(\kappa)) \subseteq k'' N$, and therefore $\mathrm{Hull}^M(j'' V \cup i(\kappa)) \cap j(\kappa) \subseteq k'' N \cap j(\kappa) = i(\kappa)$. Since

$$k_U''[\mathrm{Hull}^{M_U}(j_U'' V \cup \rho)] \subseteq \mathrm{Hull}^M(j'' V \cup i(\kappa)),$$

we have

$$\mathrm{Hull}^{M_U}(j_U'' V \cup \rho) \subseteq k_U^{-1}[\mathrm{Hull}^M(j'' V \cup i(\kappa))].$$

In particular,

$$\begin{aligned}\mathrm{Hull}^{M_U}(j_U'' V \cup \rho) \cap j_U(\kappa) &\subseteq k_U^{-1}[\mathrm{Hull}^M(j'' V \cup i(\kappa)) \cap j(\kappa)] \\ &= k_U^{-1}(i(\kappa)) = \rho.\end{aligned}$$

Since $k_E(\rho) = j_U(\kappa) > \rho$ and $k_E \circ j_E = j_U$, it follows $\rho = j_E(\kappa)$. Note also that $\rho < j_U(\kappa)$, and so the fact that $k_E(\rho) = j_U(\kappa)$ implies $k_E(\rho) \neq \rho$ and hence $\mathrm{crit}(k_E) = \rho$.

For (3), the inner model M_U is closed under κ-sequences since it is the ultrapower of V by a κ-complete ultrafilter. The inner model M_E is closed under κ-sequences by Lemma 5.3, since $\mathrm{cf}(\rho) > \kappa$ and ${}^\kappa\rho \subseteq M_E$. To see that $\mathrm{cf}(\rho) > \kappa$, note that $\mathrm{cf}(i(\kappa)) > \kappa$ since N satisfies that $i(\kappa)$ is regular and N is closed under κ-sequences. Therefore M satisfies that $\mathrm{cf}(i(\kappa)) > \kappa$. By the elementarity of k_U, and since $k_U(\rho) = i(\kappa)$, M_U satisfies $\mathrm{cf}(\rho) > \kappa$. Here, we use that $k_U(\kappa) = \kappa$.

Finally, we verify (4). By elementarity of k_U, since $|A| < i(\kappa)$, we have $|\bar{A}| < \rho$. So we just have to show that $j_E'' \kappa^+ \subseteq \bar{A}$. Suppose $\alpha \in j_E'' \kappa^+$. We claim that $k_U(\alpha) \in \mathrm{ran}(i)$. Let $\preceq$ be a well order of κ of order type $j_E^{-1}(\alpha)$. Then $j_E(\preceq)$ has order type α. Note that

$$k_U(j_E(\preceq)) = k_U(j_U(\preceq) \cap \rho) = j(\preceq) \cap i(\kappa) = i(\preceq).$$

Thus $k_U(\alpha)$, which is the order type of $k_U(j_E(\preceq))$ is equal to the order type of $i(\preceq)$, which is in the range of i. It follows that $(k_U \circ j_E)'' \kappa^+ \subseteq i'' \kappa^+ \subseteq A$. Since $k_U(\bar{A}) = A$, we conclude that $j_E'' \kappa^+ \subseteq k_U^{-1}[A] \subseteq \bar{A}$. ∎

The proof of the following lemma, which was cited in the previous proposition, appears in [18, Lemma 2.9].

Lemma 5.3 *Suppose E is an extender of length ρ with $\mathrm{crit}(j_E) = \kappa$. If ${}^\kappa\rho \subseteq M_E$, then ${}^\kappa M_E \subseteq M_E$. In particular, if E is the extender of length ρ derived from an elementary embedding $j : V \to M$ where ${}^\kappa M \subseteq M$, $\mathrm{cf}(\rho) > \kappa$, and $M \vDash \rho^\kappa = \rho$, then ${}^\kappa M_E \subseteq M_E$.*

Let us turn to the proof of Main Theorem 1.2.

[12] For a model M, $\mathrm{Hull}^M(A)$ denotes the usual closure of the class $A \subseteq M$ under the Skolem functions of M, which in the case of an ultrapower simplifies to $\mathrm{Hull}^{M_U}(A) = \{j_U(f)(\xi) \mid f : \kappa \to V, \xi \in A\}$.

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Theorem 5.4 *Suppose that κ is a non-Galvin cardinal. Then there exists a κ-complete ultrafilter U over κ such that $\neg\mathrm{Gal}(U,\kappa,\kappa^+)$. In particular, if $2^\kappa = \kappa^+$ then U is non-Galvin.*

Proof We use the notation of 5.1. As before, we can fix an ordinal $\nu < j(\kappa)$ such that for some sequence $\vec{A} = \langle A_\alpha \rangle_{\alpha<\kappa}$ such that $A = j(\vec{A})_\nu$ and for some sequence $\vec{\kappa} = \langle \kappa_\alpha \rangle_{\alpha<\kappa}$, $i(\kappa) = j(\vec{\kappa})_\nu$. Let $U = D(j,\nu)$ be the ultrafilter on κ derived from j using ν. Since $\mathrm{crit}(j) = \kappa$, U is a κ-complete ultrafilter over κ. We will show $\neg\mathrm{Gal}(U,\kappa,\kappa^+)$.

Let $\langle f_\xi \rangle_{\xi<\kappa^+}$ denote the sequence of canonical functions on κ (see Definition 2.16). For $\xi < \kappa^+$, define

$$B_\xi = \{\alpha < \kappa : f_\xi(\kappa_\alpha) \in A_\alpha\}.$$

Note that $B_\xi \in U$ since

$$j(B_\xi) = \{\alpha < j(\kappa) : j(f_\xi)(j(\vec{\kappa})_\alpha) \in j(\vec{A})_\alpha\}$$

and

$$j(f_\xi)(j(\vec{\kappa})_\nu) = j(f_\xi)(i(\kappa)) = k(i(f_\xi))(i(\kappa)) = i(\xi) \in A = j(\vec{A})_\nu.$$

The point here is that in N, $\vec{g} = i(\vec{f})$ is the sequence of canonical functions on $i(\kappa)$, and since $\mathrm{crit}(k) = i(\kappa)$, by Proposition 2.17, for any $\eta < i(\kappa^+)$, $k(g_\eta)(i(\kappa)) = \eta$. The fact that $k(i(f_\xi))(i(\kappa)) = i(\xi)$ follows from this observation when $\eta = i(\xi)$ (and thus $i(f_\xi) = g_{i(\xi)}$).

Suppose $\sigma \subseteq \kappa^+$ and $\bigcap_{\xi\in\sigma} B_\xi \in U$. We must show that $|\sigma| < \kappa$. Since $|A|^M < i(\kappa)$, it suffices to show that $i(\sigma) \subseteq A$: then $\mathrm{ot}(i(\sigma)) < \mathrm{ot}(A) < i(\kappa)$, and hence $N \models \mathrm{ot}(i(\sigma)) < i(\kappa)$, which by elementarity implies $\mathrm{ot}(\sigma) < \kappa$.

The proof that $i(\sigma) \subseteq A$ is similar to the calculation in the previous paragraph: Since $\bigcap_{\xi\in\sigma} B_\xi \in U$, for all $\eta \in j(\sigma)$, $j(\vec{f})_\eta(i(\kappa)) \in A$. Fix $\xi \in i(\sigma)$, and we will prove that $\xi \in A$. We have $k(\xi) \in j(\sigma)$, so $j(\vec{f})_{k(\xi)}(i(\kappa)) \in A$. But $j(\vec{f})_{k(\xi)} = k(g_\xi)$, hence $k(g_\xi)(i(\kappa)) = \xi$. It follows that $\xi \in A$. ∎

Remark 5.5. Note that in condition (4) the Definition 5.1 of non-Galvin cardinal it is important to work with κ^+ instead of 2^κ for there are no canonical functions in general up to 2^κ.

Remark 5.6. As proven in [2], if κ is κ-compact then there are 2^{2^κ}-many κ-complete non-Galvin ultrafilters that extend the closed unbounded filter on κ. On the other hand, assuming the Ultrapower Axiom and that every irreducible ultrafilter is Dodd sound, the least non-Galvin cardinal carries a unique non-Galvin ultrafilter that extends the closed unbounded filter on κ. Under these assumptions, if κ carries distinct non-Galvin ultrafilters extending the closed unbounded filter, then the Ketonen least distinct such ultrafilters are precisely the least two extensions of the closed unbounded filter concentrating on singular cardinals (see the proof of Theorem 6.6). These ultrafilters are irreducible (and in fact are Mitchell points) by [17, Corollary 8.2.13 and Proposition 8.3.39]. Therefore $D_0 \lhd D_1$, so κ carries a non-Galvin ultrafilter in $\mathrm{Ult}(V,D_1)$, and so κ is not the least non-Galvin cardinal.

As a first upper bound for the non-Galvin cardinals we have the following.

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Theorem 5.7 *If κ is κ-compact, then κ is a non-Galvin cardinal.*

Proof Let U be a normal ultrafilter on κ. Since $|P^{M_U}(P_\kappa(\kappa^+))| = 2^\kappa$, there is a transitive model M with

$$P^{M_U}(P_\kappa(\kappa^+)) \subseteq M, \ |M| = 2^\kappa.$$

By Hayut's result [19, Corollary 6], there is a transitive model N, an elementary embedding $j_0 : M \to N$, with $\mathrm{crit}(j_0) = \kappa$ along with some $s \in N$, $s \subseteq j_0(\kappa)^+$ such that $j_0'' \kappa^+ \subseteq s$ with $|s|^N < j_0(\kappa)$. Define $\mathcal{W}$ the κ-complete ultrafilter on $P_\kappa(\kappa^+)$ derived from j_0 and s. Note that $\mathcal{W}$ is fine since $j_0''\kappa^+ \subseteq s$ and it measures all the subsets of $P_\kappa(\kappa^+)$ in M_U. Let $j_{\mathcal{W}} : M_U \to M_{\mathcal{W}}$ be the ultrapower of M_U by $\mathcal{W}$ defined in V, and $j : V \to M_{\mathcal{W}}$ be the embedding $j = j_{\mathcal{W}} \circ j_U$. Let $\lambda = j_{\mathcal{W}}(\kappa) < j(\kappa)$, and let E be the extender of length λ derived from j.

Claim 5.8 *E is also the extender of length λ derived from $j_{\mathcal{W}}$.* ∎

Proof For any $X \subseteq \kappa$, we have that

$$j(X) \cap \lambda = j_{\mathcal{W}}(j_U(X)) \cap j_{\mathcal{W}}(\kappa) = j_{\mathcal{W}}(j_U(X) \cap \kappa) = j_{\mathcal{W}}(X).$$

Thus for all $\alpha < \lambda$, $\alpha \in j(X)$ iff $\alpha \in j_{\mathcal{W}}(X)$. ∎

Finally, let $i : V \to N_E$ be the ultrapower of V by E and $A = [\mathrm{id}]_{\mathcal{W}} \in M_{\mathcal{W}}$. We claim that i, j, A witness that κ is a non-Galvin cardinal. Indeed, $i(\kappa) \geq \lambda$. To see that $i(\kappa) \leq \lambda$, we compute the ultrapower i' of M_U by E, and since M_U is closed under κ-sequences, it follows that $i(\kappa) = i'(\kappa)$. By the previous claim, $j_{\mathcal{W}}$ also factors through i' and thus $j_{\mathcal{W}}(\kappa) = k'(i'(\kappa)) \geq i'(\kappa) = i(\kappa)$, as wanted.

By the usual argument about the derived extender, the factor map $k : N_E \to M_{\mathcal{W}}$ has critical point $i(\kappa)$ (see, for example, [21, Lemma 20.29(ii)]). Also, $M_{\mathcal{W}} \models |A| < j_{\mathcal{W}}(\kappa) = i(\kappa)$ and since $\mathcal{W}$ is fine, $j_{\mathcal{W}}''\kappa^+ \subseteq A$.

Claim 5.9 *For every $\alpha < \kappa^+$, $i(\alpha) = j_{\mathcal{W}}(\alpha)$.*

Proof Note that $i(U) \in N_E$ is a normal measure on $i(\kappa)$, let $X \in i(U)$ be any set, $k(X) \in j(U) = j_{\mathcal{W}}(j_U(U))$. Note that $j_{\mathcal{W}}(j_U(U))$ is generated by $j_{\mathcal{W}}'' j_U(U)$ by Theorem 6 and Corollary 8 of [11, Section 3]. Therefore, there is a set $Y \in j_U(U)$ such that $j_{\mathcal{W}}(Y) \subseteq k(X)$. Since U is normal, there is a set $A \in U$ such that $j_U(A) \subseteq^* Y$ and $j(A) \subseteq^* k(X)$, which in turn implies that $i(A) \subseteq^* X$. Now we note that $i(A) \in R$, where R is the N_E-ultrafilter (external) derived from k and $j_{\mathcal{W}}(\kappa)$. We conclude that $i(U) \subseteq R$ and thus that $i(U) = R$ (as two N_E-ultrafilters). So k factors through $j_{i(U)}$ and $k' : M_{i(U)} \to M_{\mathcal{W}}$ has critical point $> j_{\mathcal{W}}(\kappa)$ (since $k'(j_{\mathcal{W}}(\kappa)) = k'([\mathrm{id}]_{i(U)}) = k(id)(j_{\mathcal{W}}(\kappa)) = j_{\mathcal{W}}(\kappa)$). To conclude the claim, let $\alpha < \kappa^+$ and $f : \kappa \to \kappa$ be the canonical function such that $j_U(f)(\kappa) = \alpha$, then

$$j_{\mathcal{W}}(\alpha) = j_{\mathcal{W}}(j_U(f)(\kappa)) = j(f)(j_{\mathcal{W}}(\kappa)) = k(i(f))(j_{\mathcal{W}}(\kappa)).$$

By elementarity, $i(f) : i(\kappa) \to i(\kappa)$ is the canonical function for $i(\alpha)$. Since $j_{i(U)}$ is the ultrapower by a normal ultrafilter over $j_{\mathcal{W}}(\kappa)$, we conclude that

$$k(i(f))(j_{\mathcal{W}}(f)) = k'(j_{i(U)}(i(f)))(j_{\mathcal{W}}(\kappa)) = k'(i(\alpha)) = i(\alpha)$$

as desired. ∎

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κ is superstrong with an inaccessible target (which simply means that there is an elementary $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $V_{j(\kappa)} \subseteq M$, and $j(\kappa)$ is inaccessible in V), then by the argument of 4.1, κ is a non-Galvin cardinal. Moreover, any subcompact cardinal is a limit of cardinals that are superstrong with an inaccessible target.

Hayut proved [19] that κ^+-Π^1_1-subcompactness implies κ-compactness and he conjectures that these notions are equiconsistent.[13] So morally speaking, κ-compact cardinals should be strictly greater than non-Galvin ultrafilters in the large cardinal hierarchy. In the next section, we will see that at least under UA this is the case. Finally, we establish the connection between Dodd soundness and non-Galvin cardinals.

Lemma 5.10 *Suppose that U is a κ-complete non p-point λ-sound ultrafilter, and let E be the (κ, λ)-extender derived from j_U and $\lambda = \sup\{j_U(f)(\kappa) \mid f : \kappa \to \kappa\}$. Then $j''_E 2^\kappa \in M_U$ and moreover j_U, j_E, k_E and $j''_E 2^\kappa$ witness that κ is a non-Galvin cardinal.*

Proof Derive the extender E from λ, i.e., $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$, where E_a is an ultrafilter over $[\kappa]^{|a|}$ defined by

$$E_a = \{X \subseteq [\kappa]^{|a|} \mid a \in j(X)\}.$$

By λ-soundness of U, $E \in M_U$ and we let $i = j_E : M \to M_E$. Note that $j''_E P(\kappa)$ can be calculated in M_U and therefore $j''_E P(\kappa) \in M_U$. Also, note that $j_E(\kappa) \geq \lambda$ and since $E \in M_U$, we must have that for every $a \in [\lambda]^{<\omega}$, $j_{E_a}(\kappa) < \lambda$ hence $j_E(\kappa) \leq \lambda$. We conclude that the critical point of the factor map $k_E : M_E \to M_U$ is $\lambda = j_E(\kappa)$. Finally, observe that $j''_E 2^\kappa \in M_U$. To see this, simply note that $j_E \restriction On = (j_E)^{M_U} \restriction On$[14] and therefore $j''_E 2^\kappa = (j_E)^{M_U \prime\prime} 2^\kappa \in M_U$. ∎

6 In the canonical inner models

In this section, we work within the framework of *UA* and "every irreducible is Dodd sound." By results of Goldberg [17] and Schlutzenberg [34], these assumptions hold in the extender models $L[E]$. Our first goal of this section is to prove Main Theorem 1.3 regarding the characterization of σ-complete non-Galvin ultrafilters. To do that, we will need some preparatory results.

Theorem 6.1 (UA) *Suppose κ is either successor or strongly inaccessible and U is a κ-irreducible non-κ-complete ultrafilter on κ. Then $\diamondsuit^-_{\text{thin}}(U)$.*

Proof By [17, Theorems 8.2.22 and 8.2.23], M_U is closed under $<\kappa$-sequences and every $A \in [M_U]^\kappa$ is covered by some $B \in M_U$ such that $|B|^{M_U} = \kappa$. By the assumptions of the theorem, U is not κ-complete and therefore $\mathrm{crit}(j_U) < \kappa$. Let $E^\kappa_\omega = \{\nu < \kappa \mid \mathrm{cf}(\nu) = \omega\}$, define the function $g : E^\kappa_\omega \to \kappa$ by $g(\nu) = \rho$ for the minimal measurable cardinal ρ such that $j_U(\rho) > \nu$. By [17, Lemma 4.2.36], $g(\nu)$ is well defined and $g(\nu) \leq \nu$. Since $\mathrm{cf}(\nu) = \omega$, $g(\nu) < \nu$. By Födor, there is an unbounded $S \subseteq E^\kappa_\omega$ and $\kappa^* < \kappa$ such that for every $\nu \in S$, $g(\nu) = \kappa^*$. In particular, $j_U(\kappa^*) \geq \kappa$. If $j_U(\kappa^*) > \kappa$, let $\gamma = \kappa^*$,

[13] Since by the results of [29], if there is a weakly iterable premouse with a κ-compact cardinal then in that inner model κ is also κ^+-Π^1_1-subcompact cardinal.

[14] This is since M_U is closed under κ-sequences and thus the class of functions from $[\kappa]^{<\omega}$ to the ordinals is the same from the point of view of V and M_U. Now both $j_E \restriction On$ and $(j_E)^{M_U} \restriction On$ are completely determined by those functions.

otherwise κ is a limit of M_U-strongly inaccessible cardinals. Let $\kappa^* < \gamma < \kappa$ be the least strongly inaccessible cardinal. In any case, $j_U(\gamma) > \kappa$ and since M_U is closed under $< \kappa$-sequences, γ is a strongly inaccessible cardinal in V. Therefore, $j''_U P_\kappa(\kappa)$ is covered by a set $B \in M_U$ of cardinality less than $j_U(\gamma)$. Let $A = \{\bigcup S : S \in [B]^\kappa \cap M_U\}$. Then $|A|^{M_U} < j_U(\gamma)$, and for any $S \subseteq \kappa$, $j_U(S) \cap \kappa_* \in A$ where $\kappa_* = \sup j''_U \kappa \geq j_U(\gamma)$. Note that $\kappa_* > j_U(f)(\alpha)$ for any $f : \kappa \to \kappa$ and $\alpha < \kappa_*$ and, in particular, there is no function $f : \kappa \to \kappa$ such that $j_U(f)(|A|^{M_U}) \geq \kappa_*$. We conclude that A witnesses $\diamondsuit^-_{\text{thin}}(U)$. ■

Corollary 6.2 (UA) *If U is a σ-complete ultrafilter over κ^+ then $\diamondsuit^-_{\text{thin}}(U)$ and, in particular, U is non-Galvin.*

Proof By [17, Lemma 8.2.24], $U = \sum_D \langle W_\xi \rangle_{\xi < \lambda_D}$, where D is an ultrafilter over $\lambda_D < \kappa^+$, $\langle W_\xi \rangle_{\xi < \lambda_D}$ is discrete and $M_D \vDash W = [\xi \mapsto W_\xi]_D$ is $j_D(\kappa^+)$-irreducible which cannot be $j_D(\kappa^+)$-complete. By the previous theorem, $M_D \vDash \diamondsuit^-_{\text{thin}}(W)$. Therefore, for D-almost all ξ, $\diamondsuit^-_{\text{thin}}(W_\xi)$ which by Lemma 4.11, implies that $\diamondsuit^-_{\text{thin}}(\sum_D \langle W_\xi \rangle_{\xi < \lambda_D})$ holds. ■

Theorem 6.3 (UA) *Assume that every irreducible is Dodd sound. If W is a κ-complete ultrafilter over κ, then the following are equivalent:*

(1) *W has the Galvin property.*
(2) $\neg \diamondsuit^-_{\text{thin}}(W)$.
(3) *W is an n-fold sum of κ-complete p-points over κ*

Proof Let W be κ-complete ultrafilter. If W is an n-fold sum of κ-complete p-points then by Theorem 2.10, W has the Galvin property which by Theorem 4.9 implies $\neg\diamondsuit^-_{\text{thin}}(W)$. Let W be a κ-complete ultrafilter over κ which is not an n-fold sum of κ-complete p-points. Let $U \leq_{RF} W$ be irreducible, which exists since W is nontrivial. If U is not a p-point then by the assumptions of the theorem, U is a non p-point ultrafilter Dodd sound over κ and therefore by Lemma 3.8, $\diamondsuit^*_{\text{thin}}(U)$ holds and thus also $\diamondsuit^-_{\text{thin}}(U)$. Since $U \leq_{RK} W$, Lemma 4.10 applies, so we can conclude that $\diamondsuit^-_{\text{thin}}(W)$. Hence, we may restrict ourselves to the case where there is a p-point RF-below W (and this p-point must be κ-complete). By [17, Theorem 5.3.14], there is a $\leq_{RF}$-maximal $U \leq_{\text{RF}} W$ that is an n-fold sum of κ-complete p-points over κ. Let $\langle W_\xi \rangle_{\xi < \kappa}$ be a discrete sequence with $W = U\text{-}\lim \langle W_\xi \rangle_{\xi < \kappa}$. By the choice of U, the embedding $j_U : V \to M_U$ can be factored as a finite iterated ultrapower

$$V = M_0 \xrightarrow{j_{0,1}} M_1 \xrightarrow{j_{1,2}} \cdots \xrightarrow{j_{n-1,n}} M_n = M_U,$$

where in M_k, $j_{k,k+1}$ is the ultrapower embedding associated with a κ_k-complete p-point U_k over κ_k and $\kappa_k = j_{0,k}(\kappa)$. Also, denote by Z_k the ultrafilter associated with $j_{0,k}$; i.e.,

$$Z_k = U_0^\frown U_1^\frown \cdots {}^\frown U_{k-2}^\frown U_{k-1}.$$

For this notation, see Definition 2.6. Since W_ξ is nonprincipal, there is an irreducible ultrafilter $D_\xi \leq_{\text{RF}} W_\xi$. Suppose that D_ξ is ρ_ξ-complete uniform ultrafilter over δ_ξ for some $\rho_\xi \leq \delta_\xi \leq \kappa$. Note that $\sum_U D_\xi \leq_{RF} W$. Let m be the least such that $\kappa_{m-1} < \delta^* := [\xi \mapsto \delta_\xi]_U \leq \kappa_m$, where κ_{-1} is defined to be 0. Let $D^* = [\xi \mapsto D_\xi]_U$ is an M_U-ultrafilter over δ^*. Note that $D^* \in M_m$ since $M_n \subseteq M_m$ and since $\text{crit}(j_{m,n}) = \kappa_m$ it is

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an M_m-ultrafilter. Moreover, $M_n^{\kappa_m} \cap M_m = M_n^{\kappa_m} \cap M_n$ and therefore $(j_{D^*})^{M_m} \restriction M_n = (j_{D^*})^{M_n}$. By elementarity of $j_{D^*}^{M_m}$, $j_{D^*}^{M_n} \circ j_{m,n} = j_{D^*}^{M_m}(j_{m,n}) \circ j_{D^*}^{M_m}$ and we have that

$$j_{D^*}^{M_n} \circ j_U = j_{D^*}^{M_m}(j_{m,n}) \circ j_{D^*}^{M_m} \circ j_{0,m}. \tag{6.1}$$

Claim 6.4 *If $M_m \models D^*$ is not κ_m-complete, then $\diamondsuit^-_{\text{thin}}(W)$ holds.* ■

Proof of claim. Since D_ξ is irreducible, by our assumption, it is a non κ-complete Dodd sound ultrafilter. Note that in this case $m > 0$, since if $m = 0$, the D^* must be κ-complete. Let us split unto cases:

Case 1: If $\delta^* = \kappa_m$, then D^* is a uniform ultrafilter on κ_m and it must be κ_m-irreducible. By Theorem 6.1, $M_m \models \diamondsuit^-_{\text{thin}}(D^*)$ holds. By Lemma 3.15, we conclude that $\diamondsuit^-_{\text{thin}}(Z_m^\frown D^*)$ holds in V (see Definition 2.6 for this notation), and hence by Lemma 4.10 $\diamondsuit^-_{\text{thin}}(W)$ follows as well.

Case 2: Assume that $\delta^* < \kappa_m$.

Case 2(b): Assume $\text{crit}(j_{D^*}^{M_m}) > \kappa_{m-1}$. Note that the two-step iteration ultrapower $j_{D^*}^{M_m} \circ j_{U_{m-1}}$ is given by a κ_{m-1}-complete p-point on κ_{m-1} in M_m (see [1, Lemma 1.11]), which contradicts the maximality of U.

Case 2(c): Assume $\text{crit}(j_{D^*}^{M_m}) \le \kappa_{m-1} < \delta^* < \kappa_m$. Since D^* is an irreducible uniform ultrafilter over $\lambda_{D^*} \ge \kappa_{m-1}^+$, D^* is κ_{m-1}^+-irreducible and therefore by [17, Theorem 8.2.22], M_{D^*} is closed under κ_{m-1}-sequences which in turn implies that $P(\kappa_{m-1}) \subseteq M_{D^*}$. By Lemma [17, Lemma 4.2.36], $j_{D^*}^{M_m}(\kappa_{m-1}) > \kappa_{m-1}$. Let $\lambda = j_{D^*}^{M_m}(\kappa_{m-1})$. We claim that $U_{m-1}{}^\frown D^*$ is λ-sound and that for every function $f : \kappa_{m-1} \to \kappa_{m-1}$, $j_{U_{m-1}{}^\frown D^*}(f)(\kappa_{m-1}) < \lambda$ which by Corollary 4.8 implies that $\diamondsuit^-_{\text{thin}}(U_{m-1}{}^\frown D^*)$. Indeed, for any function $f : \kappa_{m-1} \to \kappa_{m-1}$, since $j_{D^*}^{M_m}(\kappa_{m-1}) > \kappa_{m-1}$, $j_{D^*}^{M_m}(j_{U_{m-1}}(f))(\kappa_{m-1}) = j_{D^*}^{M_m}(j_{U_{m-1}}(f) \restriction \kappa_{m-1})(\kappa_{m-1}) = j_{D^*}^{M_m}(f)(\kappa_{m-1})$, and $j_{D^*}^{M_m}(f) : j_{D^*}^{M_m}(\kappa_{m-1}) \to j_{D^*}^{M_m}(\kappa_{m-1})$. Hence $j_{D^*}^{M_m}(f)(\kappa_{m-1}) < j_{D^*}^{M_m}(\kappa_{m-1})$.

To see that $U_{m-1}{}^\frown D^*$ is λ-sound, derive the (κ_{m-1}, λ)-extender E from $j_{D^*}^{M_m}$ inside M_m. Note that E is also the (κ_{m-1}, λ)-extender derived from $j_{D^*} \circ j_{m-1,m}$ since for $\alpha < j_{D^*}^{M_m}(\kappa_{m-1})$, we have that $\alpha \in j_{D^*}^{M_m}(j_{m-1,m}(X)) \cap j_{D^*}^{M_m}(\kappa_{m-1})$ iff $\alpha \in j_{D^*}^{M_m}(j_{m-1,m}(X) \cap \kappa_{m-1})$ iff $\alpha \in j_{D^*}^{M_m}(X)$.

Now, D^* is a uniform ultrafilter over $\delta^* > \kappa_{m-1}$, hence we have that $j_{D^*}^{M_m}(\kappa) < [id]_{D^*}$ and since D^* is Dodd sound we have that $E \in (M_{D^*})^{M_m}$. In particular, $\{j_E(X) \mid X \subseteq \kappa_{m-1}\} \in (M_{D^*})^{M_m}$, where $j_E : M_{m-1} \to M_E$. Let $k_E : M_E \to (M_{D^*})^{M_m}$ be the factor map. It follows that $\text{crit}(k_E) = j_{D^*}^{M_m}(\kappa_{m-1})$. Finally, note that $j_{U_{m-1}{}^\frown D^*}(X) \cap j_{D^*}^{M_m}(\kappa_{m-1}) = j_E(X)$, hence

$$\{j_{U_{m-1}{}^\frown D^*}(X) \cap j_{D^*}^{M_m}(\kappa_{m-1}) \mid X \subseteq \kappa_{m-1}\} \in (M_{D^*})^{M_m}$$

as desired. We conclude that $M_{m-1} \models \diamondsuit^-_{\text{thin}}(U_{m-1}{}^\frown D^*)$. By Lemma 4.12 $\diamondsuit^-_{\text{thin}}(Z_{m-1}{}^\frown U_{m-1}{}^\frown D^*)$, and this ultrafilter is Rudin–Keisler below W. ■

By the claim, we may assume that for $M_m \vDash D^*$ is κ_m-complete over κ_m. It follows again that in M_m, D^* cannot be a p-point, as this would contradict the maximality of U, recalling that $\sum_U D_\xi \leq_{RF} W$ and that this ultrafilter $\sum_U D_\xi$ can be represented as an $(n+1)$-fold sum of κ-complete p-points by (6.1). Since D^* is irreducible in M_m, $M_m \vDash D^*$ is Dodd-sound and non p-point. By Lemma 3.8 $M_m \vDash \diamondsuit^*_{\text{thin}}(D^*)$ holds. In particular, $\diamondsuit^-_{\text{thin}}(D^*)$ holds. In any case, Lemma 3.15 applies to conclude that $\diamondsuit^-_{\text{thin}}(Z_m {}^\frown D^*)$ holds, and since this ultrafilter is RK-below W, Lemma 4.10 ensures that $\diamondsuit^-_{\text{thin}}(W)$ holds. ∎

Theorem 6.5 (UA) *Assume that every irreducible ultrafilter is Dodd sound. For every σ-complete ultrafilter W over κ, the following are equivalent:*

(1) *W has the Galvin property.*
(2) $\neg\diamondsuit^-_{\text{thin}}(W)$.
(3) *W is the D-sum of n-fold sums of κ-complete p-points over κ and D is a σ-complete ultrafilter on $\lambda < \kappa$.*

Proof The proof that (3) ⇒ (1) ⇒ (2) is in the previous theorem. It remains to prove that $\neg\diamondsuit^-_{\text{thin}}(W)$ implies that W is a D-sum of n-fold sums of κ-complete p-points over κ. Equivalently, let us prove the contrapositive, suppose that W is a σ-complete ultrafilter over κ which is not an n-fold sum of p-points. Now, let us move to the general case, suppose that W is just σ-complete. By [17, Lemma 8.2.24], there is a countably complete ultrafilter $D \leq_{RF} W$ on $\lambda < \kappa$ such that if $W = D\text{-}\lim\langle W_\xi\rangle_{\xi<\lambda}$, then $M_D \vDash Z = [\xi \mapsto W_\xi]_D$ is $j_D(\kappa)$-irreducible. If Z is not $j_D(\kappa)$-complete, then by Theorem 6.1. $M_D \vDash \diamondsuit^-_{\text{thin}}(Z)$ and therefore $W = D\text{-}\lim\langle W_\xi\rangle_{\xi<\lambda}$ will also satisfy $\diamondsuit^-_{\text{thin}}$ by Lemma 4.11. If Z is $j_D(\kappa)$-complete, then Z is a $j_D(\kappa)$-complete ultrafilter which is not a D'-sum of n-fold sums of p-points and we fall into the first case where we assumed that W was κ-complete (inside M_D and replacing κ by $j_D(\kappa)$). We conclude that $\diamondsuit^-_{\text{thin}}(Z)$ holds and again, it follows from that $\diamondsuit^-_{\text{thin}}(W)$ holds. ∎

Next, we turn to the proof of Main Theorem 1.5.

Theorem 6.6 (UA) *Assume that every irreducible ultrafilter is Dodd sound. Suppose κ is an uncountable cardinal that carries a κ-complete non-Galvin ultrafilter. Then the Ketonen least non-Galvin κ-complete ultrafilter on κ extends the closed unbounded filter.*

Proof We claim that in this context, the Ketonen least non-Galvin ultrafilter U is equal to the Ketonen least ultrafilter W on a regular cardinal δ extending the closed unbounded filter and concentrating on singular cardinals. First, note that W is irreducible by [17, Corollary 8.2.12].

Claim 6.7 *W is δ-complete* ∎

Proof of Claim 6.7. Suppose toward a contradiction that W is not δ-complete, and let $\mu = crit(j_W) < \delta$. Since W is Dodd sound, j_W is a $2^{<\delta}$-supercompact embedding (see [17, Lemma 4.3.4]), and so j_W witnesses that μ is $2^{<\delta}$-supercompact. In particular, μ is 2^μ-supercompact, and therefore every μ-complete filter on μ extends to a μ-complete ultrafilter. This yields a μ-complete ultrafilter W' on μ extending the closed unbounded filter on μ adjoined with the set of singular cardinals less than μ. Since $\mu < \delta$, it follows that $W' <_{\Bbbk} W$ (see [17, Lemma 3.3.15]) contradicting the minimality of W. ∎

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End of proof of Theorem 6.6. Note that W is not a p-point since W extends the closed unbounded filter but is not normal; therefore by Corollary 3.11, W is non-Galvin, and hence U is below W in the Ketonen order.

Conversely, since U is the Ketonen least non-Galvin ultrafilter, by Theorem 1.2, U is irreducible and not a p-point. Without loss of generality, we can assume that U is Dodd sound. Moreover, U is a γ-complete ultrafilter on γ for some measurable cardinal γ.

Let $\lambda = \sup\{j_U(f)(\gamma) \mid f\colon \gamma \to \gamma\}$. Since U is not a p-point, $\lambda \leq [\mathrm{id}]_U$. Since U is Dodd sound, $\{j_U(A) \cap \lambda : A \subseteq \gamma\} \in M_U$, which implies

$$\{j_U(f) \cap (\lambda \times \lambda) \mid f : \gamma \to \gamma\} \in M_U$$

and hence $\{j_U(f)(\gamma) \mid f\colon \gamma \to \gamma\} \in M_U$, which implies that M_U satisfies $\mathrm{cf}(\lambda) \leq 2^\gamma$.

Let D be the ultrafilter on γ derived from j_U using λ. Then D is below U in the Ketonen order. Since $\mathrm{cf}^{M_U}(\lambda) \leq 2^\gamma$, D concentrates on singular cardinals. Moreover, for any $f \in \gamma^\gamma$, λ is closed under $j_U(f)$ – that is, $j_U(f)[\lambda] \subseteq \lambda$ – so D concentrates on the set of closure points of f. It follows that D extends the closed unbounded filter. Therefore, W is below D in the Ketonen order, so by the transitivity of the Ketonen order, W is below U in the Ketonen order. It follows that $U = W$ as claimed. This implies that U extends the club filter, which proves the theorem.

Let us turn our attention to the non-Galvin cardinals. Main Theorem 1.4, which we now prove, shows that the existence of a non-Galvin cardinal is exactly the large cardinal assumption needed to conclude the existence of non-Galvin ultrafilters in an inner model.

Theorem 6.8 (UA) *Assume that every irreducible ultrafilter is Dodd sound. If there is a κ-complete non-Galvin ultrafilter on an uncountable cardinal κ, then there is a non-Galvin cardinal.*

Proof Let W be a non-Galvin ultrafilter on κ. By Theorem 1.5, W is Rudin–Keisler equivalent to an n-fold sum of irreducible ultrafilters. By Theorem 1.2, it is impossible that all these ultrafilters are p-points (even on measure one sets) so κ must carry an irreducible ultrafilter U which is not a p-point. By our assumption, every irreducible is Dodd sound. Since U is a κ-complete, non p-point, Dodd sound ultrafilter, Lemma 5.10 applies, and we conclude that κ is a non-Galvin cardinal. ■

Proposition 6.9 (UA) *If κ is κ-compact and no cardinal $\nu < \kappa$ is κ-supercompact, then κ a limit of non-Galvin cardinals.*

Proof Since κ is κ-compact, a theorem of Kunen [26, Lemma 3] implies that for every $\xi < (2^\kappa)^+$, there is a countably complete ultrafilter U on κ such that $j_U(\xi) > \xi$. Let U_ξ denote the Ketonen least such ultrafilter. By [17, Lemma 7.4.34] and [17, Proposition 8.3.39], U_ξ is a *Mitchell point*: for any ultrafilter $W <_{\Bbbk} U$, W lies below U in the Mitchell order.

Since κ is strongly inaccessible, there is an ω-club $C \subseteq (2^\kappa)^+$ such that for all $\xi \in C$, for all countably complete ultrafilters D of rank less than ξ in the Ketonen order, $j_D(\xi) = \xi$. For $\xi \in C$, U_ξ is a uniform irreducible ultrafilter on κ, and so it follows from [17, Theorem 8.2.23] that U_ξ witnesses $\mathrm{crit}(j_{U_\xi})$ is $<\kappa$-supercompact. Since κ is measurable, it follows that $\mathrm{crit}(j_{U_\xi})$ is κ-supercompact, and so by the assumptions of the proposition, $\mathrm{crit}(j_{U_\xi}) = \kappa$. In other words, U_ξ is κ-complete.

Now, let W witness that κ is a non-Galvin cardinal. Fix $\xi \in C$ larger than the Ketonen rank of W. Then W is below U_ξ in the Mitchell order, and so κ is non-Galvin in M_{U_ξ}. It follows that κ is a limit of non-Galvin cardinals. ∎

In particular, the least cardinal κ that is κ-compact is larger than the least non-Galvin cardinal assuming UA.[15]

7 Open problems

Question 7.1. *Is it consistent that there is a κ-complete uniform ultrafilter U over κ satisfying the Galvin property that is not an n-fold sum of κ-complete p-points over κ?*

Recently, Gitik gave a positive answer to this question, thus our characterization of ultrafilters with the Galvin property cannot be proved in ZFC. The following question seems more plausible for a positive answer in ZFC.

Question 7.2. *Is every uniform κ-complete ultrafilter U over κ^+ non-Galvin, i.e., $\neg\mathrm{Gal}(U, \kappa^+, \kappa^{++})$ holds?*

Under UA, the answer is positive by Corollary 6.2.

Question 7.3. *Does a non-Galvin cardinal entail the existence of a non-Galvin ultrafilter which extends the club filter?*

By Main Theorem 1.2, a non-Galvin cardinal entails the existence of a non-Galvin ultrafilter. Assuming UA and that every irreducible is Dodd sound, Main Theorem 1.5 shows that a non-Galvin cardinal also entails the existence of κ-complete non-Galvin ultrafilter which extends the club filter.

Question 7.4. *Does every fine normal ultrafilter over $P_\kappa(\kappa^+)$ satisfy $\mathrm{Gal}(U, \kappa, 2^{\kappa^+})$?*

The answer would be interesting even under *UA*. This is the first step toward answering the more general problem.

Question 7.5. *Characterize the Tukey-top ultrafilters on κ with respect to $\lambda < \kappa$ assuming UA plus every irreducible is Dodd sound.*

Question 7.6. *Is there a similar characterization under UA for σ-complete ultrafilters with the Galvin property over singular cardinals?*

We believe that such a characterization exists and that similar methods to those appearing in this paper should be useful.

In the absence of GCH, we have the following questions which are open.

Question 7.7. *If we replace $i''\kappa^+$ by $i''2^\kappa$ in the definition of non-Galvin cardinal, do we get a κ-complete ultrafilter such that $\neg\mathrm{Gal}(U, \kappa, 2^\kappa)$?*

[15] It should be provable from UA that any cardinal κ that is κ-compact is a limit of non-Galvin cardinals. Here, there are two cases. If κ is a limit of cardinals γ that are κ-compact, then each of these cardinals γ is γ-compact, so κ is a limit of non-Galvin cardinals. If κ is not a limit of κ-compact cardinals, one would like to show, as above, that there is a non-Galvin ultrafilter W on κ that is below some κ-complete ultrafilter on κ in the Mitchell order. The issue is that it is unclear how to show that the Mitchell order on κ-complete ultrafilters has rank $(2^\kappa)^+$ if some $\nu < \kappa$ is κ-supercompact.

More generally:

Question 7.8. *Is it consistent that there is a κ-complete ultrafilter U such that $\neg\mathrm{Gal}(U,\kappa,\kappa^+)$ but $\mathrm{Gal}(U,\kappa,2^\kappa)$?*

The result of this paper resolves these two questions under UA plus every irreducible is Dodd sound.

The following two questions address the assumptions in the main theorems of this paper.

Question 7.9. *Is it consistent that there is a cardinal κ which is κ^+-supercompact and that every irreducible ultrafilter is Dodd sound?*

Question 7.10. *Does UA imply that every irreducible ultrafilter is Rudin–Keisler equivalent to a Dodd sound ultrafilter?*

Let us conclude this paper with a diamond-like principle which is a reasonable candidate to be equivalent to non-Galvin ultrafilters. Such a principle would be valuable as there is no known formulation of the Galvin property in terms of the ultrapower. This would be also interesting from the point of view of the Tukey order since this order involves functions which typically have domains of size 2^κ, and thus not available in the ultrapower.

Definition 7.11. We say that $\diamondsuit^-_{\mathrm{par}}(U)$ holds if and only if there is $A \in M_U$, λ and $\langle X_i \rangle_{i<2^\kappa} \subseteq P(\kappa)$ such that:

(1) $\{j_U(X_i) \cap \lambda \mid i < 2^\kappa\} \subseteq A$.
(2) There is no function $f : \kappa \to \kappa$ such that $j_U(f)(|A|^{M_U}) \geq \lambda$.

The argument of Theorem 4.9 can be adjusted to conclude that $\diamondsuit^-_{\mathrm{par}}(U)$ implies that U is non-Galvin.

Question 7.12. *Is $\diamondsuit^-_{\mathrm{par}}(U)$ equivalent to U being non-Galvin?*

The next question seeks an analogous result on ω to the one of this paper.

Question 7.13. *Is it consistent that every ultrafilter on ω which is not Tukey-top is an n-fold sum of p-points?*

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