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QUASIPRIMITIVITY AND QUASIGROUPS

J.D. PHILLIPS AND J.D.H. SMITH

It is well known that Q is a simple quasigroup if and only if $\mathrm{Mlt}\,Q$ acts primitively on Q. Here we show that Q is a simple quasigroup if and only if $\mathrm{Mlt}\,Q$ acts quasiprimitively on Q, and that Q is a simple right quasigroup if and only if $\mathrm{RMlt}\,Q$ acts quasiprimitively on Q.

A *quasigroup* is set with a single binary operation, denoted by juxtaposition, such that in $xy = z$, knowledge of any two of x, y and z specifies the third uniquely. A *right quasigroup* is a set with a single binary operation whose right translations biject.

The *multiplication group*, $\mathrm{Mlt}\,Q$, of a quasigroup Q is the subgroup of the group of all bijections on Q generated by right and left translations, that is $\mathrm{Mlt}\,Q := \langle R(x),\ L(x) : x \in Q \rangle$, where $R(x)$ (respectively, $L(x)$) is right (respectively, left) translation by x. The *right multiplication group*, $\mathrm{RMlt}\,Q$, of a right quasigroup Q is the subgroup of the group of all bijections on Q generated by right translations, that is $\mathrm{RMlt}\,Q := \langle R(x) : x \in Q \rangle$. A quasigroup Q is called *type 1* if $\mathrm{RMlt}\,Q = \mathrm{Mlt}\,Q$. For example, commutative quasigroups are type 1; so too are finite simple Moufang loops [**4**].

A permutation group G on a set Q acts *primitively* on Q if the only G-invariant partitions of Q are the two trivial partitions $\{Q\}$ and $\{\{x\} : x \in Q\}$. Of course, if G acts primitively on Q then each nontrivial normal subgroup of G is transitive on Q. In [**5**], Praeger used this fact to generalise the notion of primitivity: a permutation group G on a set Q acts *quasiprimitively* on Q if each non-trivial normal subgroup of G is transitive on Q. This definition is useful because, as Praeger proved [**5**], there is an O'Nan-Scott type theorem classifying all quasiprimitive permutation groups as one of eight types.

Given a quasigroup Q, there exist two binary operations $/$, $\backslash$ on Q such that $(xy)/y = x$, $(x/y)y = x$, $x\backslash(xy) = y$, and $x(x\backslash y) = y$. Conversely, an algebra with three binary operations satisfying these four identities is a quasigroup — as defined at the beginning of this paper — under any one of these operations [**2**]. Similarly, right quasigroups are axiomatised by the first two of the four quasigroup identities above. A

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congruence on a (right) quasigroup Q is an equivalence relation V on Q such that if x_1Vy_1 and x_2Vy_2, then $x_1x_2fVy_1y_2f$, where f is any one of the (two) three binary operations on Q. Q is *simple* if its only congruences are the trivial congruence and the improper congruence. We record the following well known fact [**2**]:

PROPOSITION 1. *A quasigroup Q is simple if and only if* Mlt Q *acts primitively on Q.*

THEOREM 2. *A quasigroup Q is simple if and only if* Mlt Q *acts quasiprimitively on Q.*

PROOF: $(\longrightarrow)$ Quasiprimitivity is a generalisation of primivity.

$(\longleftarrow)$ Let V be a nontrivial congruence on Q. There is a corresponding normal subgroup $\operatorname{Nor}(V) := \{g \in \operatorname{Mlt} Q : \forall q \in Q, (q, qg) \in V\}$ of $\operatorname{Mlt} Q$. Now pick $x \in Q$. The subset $x \operatorname{Nor}(V)$ of Q is contained in the congruence class of x. But since $\operatorname{Mlt} Q$ acts quasiprimitively on Q, $x \operatorname{Nor}(V) = Q$. Thus V is improper and Q is simple. ∎

An important research program from the theory of quasigroups is to determine which permutation group actions are the actions of a multiplication group of a quasigroup, and which permutation groups are not [**2, 4**]. The following corollary to Theorem 2 advances this program.

COROLLARY 3. *An imprimitive, quasiprimitive action is not a multiplication group action.*

For examples of imprimitive, quasiprimitive actions see [**5**]. While every symmetric group, under its natural action, can be realised as a multiplication group of some quasigroup, [**3**, Proposition 4.2] shows that the actions of symmetric groups of odd prime power degree bigger than three on unordered pairs cannot be multiplication group actions. Corollary 3 above expands on this theme: it is the first example of a *general class* of permutation actions — that is, the quasiprimitive, imprimitive actions — that cannot be multiplication group actions. The following result gives a unilateral version of Theorem 2.

THEOREM 4. *Q is a simple right quasigroup if and only if* RMlt Q *acts quasiprimitively on Q.*

PROOF: $(\longrightarrow)$ Let N be a non-trivial normal subgroup of $\operatorname{RMlt} Q$. The sets $xN, x \in Q$, partition Q. Let Q/N denote the set of equivalence classes. Define a binary operation on Q/N by $(xN)(yN) = (xyN)$. It is easy to check that this operation is well defined and that under this operation Q/N is a right quasigroup [**1**]. It is also clearly a proper homomorphic image of Q. But Q is simple, so this image must be trivial. That is, there is only one equivalence class, and N is a transitive on Q.

$(\longleftarrow)$ Given a right quasigroup epimorphism $f : Q \to M$, there is a group epimorphism $F : \operatorname{RMlt} Q \to \operatorname{RMlt} M;\ R(x) \to R(xf)$. Pick $x \in Q$. The subset $x \operatorname{Ker} F$

is in the congruence (ker f) class of x. But since RMlt Q acts quasiprimitively on $Q, x\,\mathrm{Ker}\,F = Q$. Hence, ker f is improper, and Q is simple. ∎

COROLLARY 5. *If Q is a type 1 simple quasigroup, then Q is a simple right quasigroup.*

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Department of Mathematics
Saint Mary's College
Moraga, CA 94575
United States of America

Department of Mathematics
Iowa State University
Ames, IA 50011
United States of America

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