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THE DEPTH OF CENTRES OF MAPS ON DENDRITES

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Abstract

Xiong proved that if $f : I \to I$ is any map of the unit interval I, then the depth of the centre of f is at most 2, and Ye proved that for any map $f : T \to T$ of a finite tree T, the depth of the centre of f is at most 3. It is natural to ask whether the result can be generalized to maps of dendrites. In this note, we show that there is a dendrite D such that for any countable ordinal number λ there is a map $f : D \to D$ such that the depth of centre of f is λ. As a corollary, we show that for any countable ordinal number λ there is a map (respectively a homeomorphism) f of a 2-dimensional ball B^2 (respectively a 3-dimensional ball B^3) such that the depth of centre of f is λ.

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1. Introduction

In [6], Xiong proved that if $f : I \to I$ is any map of the unit interval $I = [0, 1]$, then the depth $d(f)$ of the centre of f is at most 2, and in [7], Ye proved that for any map $f : T \to T$ of a (finite) tree T, the depth $d(f)$ of centre of f is at most 3. It is natural to ask whether the result can be generalized to maps of dendrites. In [5], Neumann proved that for any C^∞ n-manifold M with $n \geq 3$ and any countable ordinal number λ, there is a C^∞ flow ϕ on M such that the depth of centre of the flow ϕ is λ.

In this note, firstly we study the depth of centre of maps of 0-dimensional compacta. As corollaries, we show the following:

(1) There is a dendrite D such that for any countable ordinal number λ there is a map $f : D \to D$ such that the depth $d(f)$ of centre of f is λ.

(2) For any countable ordinal number λ there is a map f of a disk B^2 such that $d(f) = \lambda$.

(3) For any countable ordinal number λ there is a homeomorphism $h : B^3 \to B^3$ of a 3-dimensional ball B^3 such that $h|\partial B^3 = \mathrm{id}$, $d(h) = \lambda$ and $\Omega_\lambda(h) = \partial B^3 \bigcup Z$, where Z is a compact countable set in $B^3 - \partial B^3$.

All spaces considered in this note are assumed to be separable metric spaces. By a *continuum*, we mean a non-empty, compact, connected, metric space. Let I be the unit interval [0, 1]. A *tree* is a 1-dimensional connected compact polyhedron which contains no simple closed curve. A continuum D is a *dendrite* if D is a locally connected continuum and D contains no simple closed curve (see [4] for topological properties of dendrites). A point e of a dendrite D is called an *end point* if there is no subset A of D such that $e \in A$ and A is homeomorphic to the open interval (0, 1). Let $E(D)$ be the set of all end points of D. Note that a compactum X is a dendrite if and only if X is a 1-dimensional compact absolute retract (= AR). The dynamics of maps (=continuous functions) of I and trees are considerably well-understood. Recently, the dynamical behavior of maps of dendrites have often appeared in Julia sets of complex dynamical systems.

Let X be a compact metric space with metric d and $f : X \to X$ a map. A point $x \in X$ is a *periodic point* of f if there is a natural number $n \geq 1$ such that $f^n(x) = x$. A point $x \in X$ is a *recurrent point* of f if for each $\epsilon > 0$ there is a natural number $n \geq 1$ such that $d(f^n(x), x) < \epsilon$. A point $x \in X$ is a *non-wandering point* of f if for any neighborhood U of x in X there is a natural number $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. By $P(f)$, we mean the set of all periodic points of f, and by $R(f)$ the set of all recurrent points of f. Also, the set of non-wandering points of f will be denoted by $\Omega(f)$. The notions of periodic points, recurrent points and non-wandering points are very important in the study of dynamical systems. Note that $P(f) \subset R(f) \subset \Omega(f)$, $\Omega(f)$ is a closed subset of X and $f(\Omega(f)) \subset \Omega(f)$.

Let $\Omega_0(f) = X$ and $\Omega_1(f) = \Omega(f)$. For any ordinal number $\lambda \geq 1$, recursively we will define $\Omega_\lambda(f)$ as follows: If $\lambda = \alpha + 1$, then we set $\Omega_\lambda(f) = \Omega(f|\Omega_\alpha(f))$. If λ is a limit ordinal number, we set $\Omega_\lambda(f) = \bigcap_{\alpha<\lambda} \Omega_\alpha(f)$.

Then we see that there is a countable ordinal number γ such that $\Omega_\gamma(f) = \Omega_{\gamma+1}(f)(= \overline{R(f)})$. The minimal such γ is called the *depth of the centre of* f, and it is denoted by $d(f)$. Note that $d(\mathrm{id}) = 0$. In general, it is difficult to determine the centre $\Omega_\gamma(f)$ $(= \overline{R(f)})$ and the depth $d(f)$ of the centre of f. We are interested in the depth $d(f)$ of the centre of a map f.

Let X be a compactum with metric d. Then

$$2^X = \{A \mid A \text{ is a non-empty closed subset of } X\}$$

is the hyperspace with the Hausdorff metric d_H, that is,

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset U(B, \epsilon), B \subset U(A, \epsilon)\}$$

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where $U(A, \epsilon)$ is the ϵ-neighborhood of A in X. Note that 2^X is a compact metric space with the metric d_H.

2. The depth of centres of maps of compact countable sets

In this section, we study the 0-dimensional case. We prove the following.

PROPOSITION 2.1. *For any countable ordinal number λ there is a compact countable set Z_λ and a homeomorphism $f_\lambda : Z_\lambda \to Z_\lambda$ such that $d(f_\lambda) = \lambda$.*

PROOF. Note that $d(\mathrm{id}) = 0$. Recursively, for any countable ordinal number $\lambda > 0$ we will construct a compact countable set Z_λ and a homeomorphism $f_\lambda : Z_\lambda \to Z_\lambda$ such that $d(f_\lambda) = \lambda$. Let ω be the first infinite ordinal number and $\mathbb{Z}$ the set of integers. Firstly, we consider the case that λ is not a limit ordinal.

I(1): Case of $\lambda = 1$. Let

$$Z_1 = \{x_1(n) | n \in \mathbb{Z}\} \oplus \{x(\infty)\},$$

where $x_1(i) \neq x_1(j) (i \neq j)$ and $\oplus$ implies the disjoint union. Then we can define a metric d_1 on Z_1 satisfying $\lim_{n\to\infty} x_1(n) = x(\infty) = \lim_{n\to\infty} x_1(-n)$ (see Figure 1). Define a function $f_1 : Z_1 \to Z_1$ by $f_1(x_1(n)) = x_1(n+1)$, $f_1(x(\infty)) = x(\infty)$. Note that Z_1 is compact and f_1 is continuous. Then $d(f) = 1$.

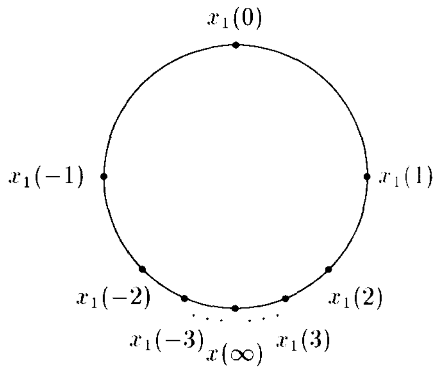

FIGURE 1

I$(m+1)$: Case of $\lambda = m + 1$ $(1 \leq m < \omega)$. We assume that the set Z_m, a metric d_m on Z_m and a homeomorphism f_m of Z_m have been obtained. Let

$$Z_{m+1} = Z_m \oplus \{x_{m+1}(n) | n \in \mathbb{Z}\},$$

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where $x_{m+1}(i) \neq x_{m+1}(j) (i \neq j)$. Define a function $f_{m+1} : Z_{m+1} \to Z_{m+1}$ by $f_{m+1}(x_{m+1}(n)) = x_{m+1}(n+1)$, $f_{m+1}|Z_m = f_m$. Then we can define a metric d_{m+1} on Z_{m+1} such that d_{m+1} is an extension of d_m, $\lim_{n\to\infty} x_{m+1}(-n) = x(\infty) \in Z_1$,

$$\lim_{n\to\infty} (d_{m+1})_H(\mathrm{Cl}(\{x_{m+1}(j) | j \geq n\}), Z_m) = 0$$

and $\Omega(f_{m+1}) = Z_m$ (see Figure 2). Hence $d(f_{m+1}) = m + 1$. Note that $Z_m \subset Z_{m+i}$ for each $i \geq 0$.

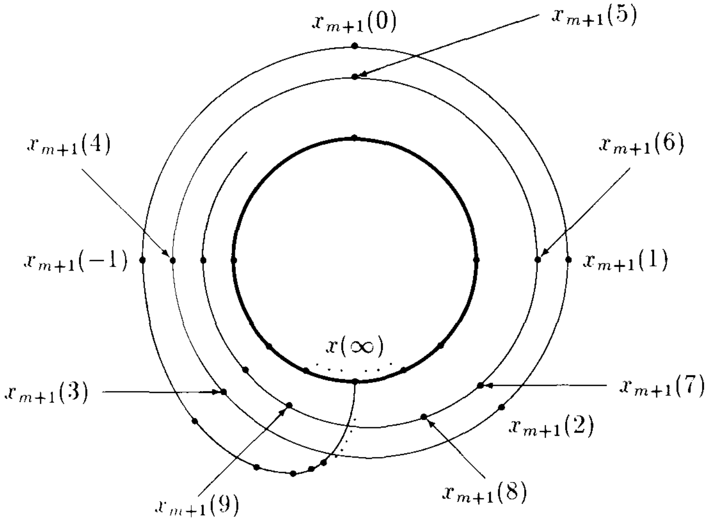

FIGURE 2

Next, we consider the case that λ is a countable ordinal number which is not limit.

I$(\omega + m)$: Case of $\lambda = \omega + m$, where $1 \leq m < \omega$. Consider the set Z_m and a metric d_m on Z_m (see the case I(m)). Take a sequence $Z_{m+1}, Z_{m+2}, \ldots$ of sets and a sequence $d_{m+1}, d_{m+2}, \ldots$ of metrics such that d_{m+i} is a metric on Z_{m+i} and each d_{m+i} is an extension of d_m. Moreover, we can take metrics d_{m+i} on Z_{m+i} satisfying the following condition: Z_{m+i} is contained in the i^{-1}-neighborhood of $Z_m (\subset Z_{m+i})$, that is, $\lim_{i\to\infty} d_{m+i}(Z_{m+i}, Z_m) = 0$ and for any $\epsilon > 0$, there is $\delta > 0$ such that there is some i_0 such that if $i \geq i_0$ and $x \in Z_m \subset Z_{m+i}$, then

$$f_{m+i}(U_{m+i}(x, \delta)) \subset U_{m+i}(f_{m+i}(x), \epsilon),$$

where $U_m(x, \delta)$ (respectively $U_{m+i}(x, \delta)$) denotes the δ-neighborhood of x in Z_m (respectively Z_{m+i}). Intuitively, we may consider that the sets Z_{m+i} and maps f_{m+i} converge to the set $Z_m (\subset Z_{m+i})$ and the map $f_m : Z_m \to Z_m$, respectively.

Set $Z_{\omega+m} = \bigoplus_{i=1}^{\infty} Z_{m+i} \oplus Z_m$. Define a function $f_{\omega+m} : Z_{\omega+m} \to Z_{\omega+m}$ by $f_{\omega+m}|Z_{m+i} = f_{m+i}$, $f_{\omega+m}|Z_m = f_m$. By using the metric d_{m+i} on Z_{m+i} as above ($i \geq 1$), we can define a metric $d_{\omega+m}$ on $Z_{\omega+m}$ so that $d_{\omega+m}$ is an extension of d_{m+i} for each i,

$$\lim_{i\to\infty} (d_{\omega+m})_H(Z_{m+i}, Z_m) = 0,$$

and the following condition is satisfied:

(†) for any $x \in Z_m$ and any $\epsilon > 0$ there is $\delta > 0$ such that

$$f_{\omega+m}(U_{\omega+m}(x, \delta)) \subset U_{\omega+m}(f_{\omega+m}(x), \epsilon).$$

In particular, $f_{\omega+m}$ is continuous (see Figure 3). Note that $\Omega_\omega(f_{\omega+m}) = Z_m$. Hence $d(f_{\omega+m}) = \omega + m$.

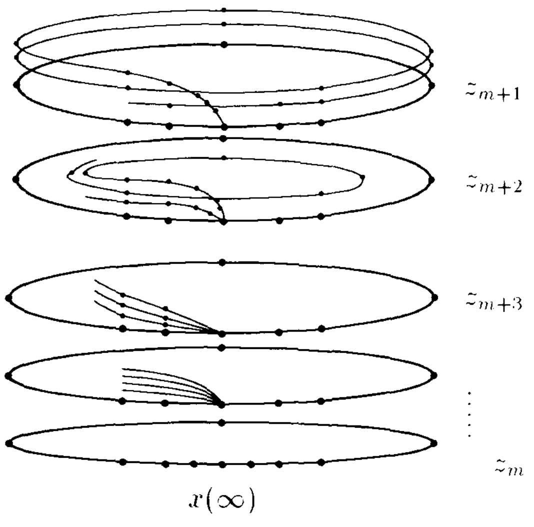

FIGURE 3

I($\alpha + m$): Case that $\lambda > \omega$ is a countable ordinal number which is not limit. Then we can choose the limit ordinal number $\alpha < \lambda$ such that $\lambda = \alpha + m$, where $1 \leq m < \omega$. Take a sequence $\alpha_1 < \alpha_2 < \alpha_3 < \dots$, of ordinal numbers such that $\lim_{i\to\infty} \alpha_i = \alpha$. Note that $\lim_{i\to\infty}(\alpha_i + m) = \alpha$. In this case, by induction we may assume that Z_{α_i+m} can be presented by the form $Z_{\alpha_i+m} = \bigoplus_{j=1}^{\infty} Z_{\beta_{i,j}+m} \oplus Z_m$ and the metric d_{α_i+m} satisfies $\lim_{j\to\infty}(d_{\alpha_i+m})_H(Z_{\beta_{i,j}+m}, Z_m) = 0$, where $\lim_{j\to\infty} \beta_{i,j} = \alpha_i$ (see the case I($\omega + m$)). Set

$$Z_\lambda = \bigoplus_{i=1}^{\infty} Z_{\alpha_i+m} \oplus Z_m.$$

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Define a function $f_\lambda : Z_\lambda \to Z_\lambda$ by $f_\lambda | Z_{\alpha_i+m} = f_{\alpha_i+m}$, $f_\lambda | Z_m = f_m$. In this case we may assume that the metrics d_{α_i+m}, $(i = 1, 2, \ldots)$ on Z_{α_i+m} satisfy the condition (†). By using these metrics, we can define a metric d_λ on Z_λ such that d_λ is an extension of the metric d_{α_i+m} on Z_{α_i+m}, Z_λ is a compactum and moreover d_λ satisfies the condition (†). In particular, f_λ is continuous (see the case I($\omega + m$)). Note that $\Omega_\alpha(f_{\alpha+m}) = Z_m$. Hence $d(f_{\alpha+m}) = \alpha + m$.

Next, we consider the case that λ is a limit ordinal number. In this case, take a sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$, of ordinal numbers such that α_i is not limit for each i, and $\lim_{i\to\infty} \alpha_i = \lambda$. For each α_i we assume that Z_{α_i} and f_{α_i} have been obtained. Set

$$Z_\lambda = \bigoplus_{i=1}^{\infty} Z_{\alpha_i} \oplus \{\infty\}.$$

We can define a metric d_λ on Z_λ such that $\lim_{i\to\infty}(d_\lambda)_H(Z_{\alpha_i}, \{\infty\}) = 0$ and each $Z_{\alpha_i} (\subset Z_\lambda)$ is homeomorphic to Z_{α_i}. Define a function $f_\lambda : Z_\lambda \to Z_\lambda$ by $f_\lambda | Z_{\alpha_i} = f_{\alpha_i}$, and $f_\lambda(\infty) = \infty$. Clearly f_λ is continuous. Then we see that $d(f_\lambda) = \lambda$.

Therefore, for any countable ordinal number λ we obtained a compact countable set Z_λ and a homeomorphism $f_\lambda : Z_\lambda \to Z_\lambda$ such that $d(f_\lambda) = \lambda$. This completes the proof.

LEMMA 2.2. *Suppose that A is a closed subset of a space X. Let $g : A \to A$ be a map of A and $r : X \to A$ a retraction, that is, $r|A = \mathrm{id}$. If $f = g \cdot r : X \to X$, then $\Omega(f) = \Omega(g)$.*

PROOF. Since $f|A = g$, $\Omega(g) \subset \Omega(f)$. Let $x \in \Omega(f)$. Note that $x \in A$. Suppose, on the contrary, that $x \notin \Omega(g)$. There is a neighborhood U of x in A such that $g^n(U) \cap U = \phi$ for all $n \geq 1$. Since $r(x) = x$, we choose a neighborhood V of x in X such that $r(V) \subset U$. Then

$$\begin{aligned} f^n(V) \cap V &= f^{n-1}(g \cdot r(V)) \cap V \subset f^{n-1}(g(U)) \cap V \\ &= g^n(U) \cap V \\ &= g^n(U) \cap (A \cap V) \subset g^n(U) \cap U = \phi. \end{aligned}$$

Therefore $x \notin \Omega(f)$. Hence $\Omega(f) = \Omega(g)$. If $d(g) > 0$, then we see that $d(f) = d(g)$.

COROLLARY 2.3. *Let C be a Cantor set. If λ is any countable ordinal number, there is a map $f : C \to C$ such that $d(f) = \lambda$.*

PROOF. We may assume $\lambda > 0$. Let Z_λ and $f_\lambda : Z_\lambda \to Z_\lambda$ be as in (2.1). We may assume that $Z_\lambda \subset C$. Then there is a retraction $r : C \to Z_\lambda$. Put $f = f_\lambda \cdot r : C \to C$. By (2.2), $\Omega(f_\lambda) = \Omega(f)$. Clearly $d(f) = \lambda$.

COROLLARY 2.4. *For any countable ordinal number λ there is a 1-dimensional compactum Y and a flow $\phi : Y \times R \to Y$ such that $d(\phi) = \lambda$.*

PROOF. Let Z_λ, f_λ be as in (2.1). Let $Y = T(f_\lambda)$ be the mapping torus, that is, the space obtained from $Z_\lambda \times I$ by identifying the points $(y, 0)$ and $(f_\lambda(y), 1)$. Naturally we obtain the flow ϕ on Y from f_λ. Then $d(\phi) = \lambda$.

3. The depth of centres of maps of dendrites

In this section, we study the depth of centres of maps of some continua. The following is the main theorem of this note.

THEOREM 3.1. *For any countable ordinal number λ, there is a dendrite D and a map $f : D \to D$ such that the set $E(D)$ of endpoints of D is a compact countable set and $d(f) = \lambda$.*

PROOF. Firstly, we define some kinds of dendrites by the following general method (see [2]): Let X be a 0-dimensional compact metric space and let $g : X \to X$ be any map of X. Choose an inverse sequence $\boldsymbol{X} = \{X_n, p_{n,n+1} | n = 1, 2, \dots, \}$ of finite sets X_n such that $X_1 = \{*\}$ is a one point set, $p_{n,n+1} : X_{n+1} \to X_n$ is an onto bonding map $(n \geq 1)$ and $X = \operatorname{invlim}\boldsymbol{X}$. For $1 \leq m < n$, let $p_{m,n} = p_{m,m+1} \cdots p_{n-1,n}$ and let $p_n : X \to X_n$ be the natural projection. Now, consider the infinite telescope $T(\boldsymbol{X}) = \cup_{n=1}^{\infty} M(p_{n,n+1})$, where $M(p_{n,n+1})$ denotes the mapping cylinder of $p_{n,n+1} : X_{n+1} \to X_n$, that is, in a topological sum $X_n \cup (X_{n+1} \times [1/(n+1), 1/n])$, $M(p_{n,n+1})$ is obtained by identifying points $(x, 1/n) \in X_{n+1} \times \{1/n\}$ and $p_{n,n+1}(x) \in X_n$ for $x \in X_{n+1}$ and $T(\boldsymbol{X})$ is obtained by identifying each point of $X_n \times \{1/n\}$ in $M(p_{n-1,n})$ and the corresponding point of X_n in $M(p_{n,n+1})$. Put $Y(\boldsymbol{X}) = X \cup T(\boldsymbol{X})$. Define a function $\mu : Y(\boldsymbol{X}) \to I = [0, 1]$ by $\mu([x, t]) = t$ if $[x, t] \in T(\boldsymbol{X})$ and $\mu(x) = 0$ if $x \in X$. Also, define a retraction $\psi_t : Y(\boldsymbol{X}) \to \mu^{-1}([t, 1])(t \in I)$ by $\psi_t(y) = [p_{q(t)}(x), t]$ for $y = x \in X$, $\psi_t(y) = [p_{q(t),n}(x), t]$ for $y = [x, s] \in \mu^{-1}((0, t])$ and $x \in X_n$, and $\psi_t(y) = y$ for $y \in \mu^{-1}([t, 1])$, where $q(t)$ is the natural number such that $1/q(t) \leq t < 1/(q(t) - 1)$. The topology of $Y(\boldsymbol{X})$ is defined by assuming that the totality of the following sets — open sets of $T(\boldsymbol{X})$ and the sets of the form $\psi_{1/n}^{-1}(U) \cap \mu^{-1}([0, 1/n))$, where U is an open set of $X_n (\subset Y(\boldsymbol{X}))$, $n \geq 1$ — is an open base of $Y(\boldsymbol{X})$. Then $Y(\boldsymbol{X})$ is a compact absolute retract, and μ and ψ_t are continuous (see [2]).

Next, for any map $g : X \to X$ we shall construct a map $f : Y(\boldsymbol{X}) \to Y(\boldsymbol{X})$ such that f is an extension of g and $\Omega(f) = \Omega(g) \cup \{p\}$ where $p = * \in X_1 \subset Y(\boldsymbol{X})$.

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For each closed subset A of X, consider the minimal subcontinuum $c(A)$ of $Y(X)$ containing A, that is,

$$c(A) = \mathrm{Cl}(\cup\{[a,b] : a, b \in A\}),$$

where $[a, b]$ is the arc from a to b in $Y(X)$. If an arc $[a, b]$ from a to b is not decreasing with respect to μ (that is, if $x, y \in [a, b]$ and $a \le x \le y \le b$, then $\mu(x) \le \mu(y)$), we call $[a, b]$ an *order arc* from a to b.

Let $\kappa(A)$ be the unique point of $c(A)$ such that $\mu(\kappa(A)) = \min\{\mu(y); y \in c(A)\}$. Define a map $g_1 : \cup_{n=1}^{\infty} X_n \to Y(X)$ such that $g_1(x) = \kappa(g(p_n^{-1}(x)))$. By using this map g_1, we can naturally define a map $g_2 : Y(X) \to Y(X)$ such that g_2 is an extension of g_1 and g, and if $A = [a, b]$ is an order arc from a to b, then $g_2(A)$ is also an order arc from $g_2(a)$ to $g_2(b)$. In this case, we say that g_2 is order-arc preserving. Choose a homeomorphism $h : I \to I$ such that $h(0) = 0$, $h(1) = 1$, and $h(t) > t$ for $0 < t < 1$; for example, $h(t) = \sqrt{t}$. Define a function $f : Y(X) \to Y(X)$ by

$$f(y) = \psi_{h \cdot \mu(y)}(g_2(y)).$$

Then f is continuous and $f(p) = p$, $f|X = g$ and f is *order-arc preserving*. Also, note that if $y \in Y(X) - (X \cup \{p\})$, then $\mu(y) < \mu(f(y))$.

Next, we show that $\Omega(f) = \Omega(g) \cup \{p\}$. Let $x \in \Omega(f)$. Since $y < \mu(y)$ for any $y \in Y(X) - (X \cup \{p\})$, we see that $x \in X \cup \{p\}$. Suppose, on the contrary, that $x \notin \Omega(g) \cup \{p\}$. Then there is a neighborhood U of x in X such that $g^n(U) \cap U = \phi$ for all $n \ge 1$. Take a point $x_n \in X_n$ such that $x \in \psi_{1/n}^{-1}(x_n) \cap X \subset U$. Set $V = \psi_{1/n}^{-1}(x_n)$. Then $f^n(V) \cap V = \phi$ for all $n \ge 1$. In fact, suppose, on the contrary, that there is $y \in V$ such that $f^n(y) \in V$ for some $n \ge 1$. Choose a point $y' \in X \cap V$ such that $[y', y]$ is an order arc. Then $[f^n(y'), f^n(y)]$ is an order arc. Since $f^n(y) \in V$, $f^n(y') \in V \cap X \subset U$, which implies that $g^n(U) \cap U \ne \phi$. This is a contradiction. Hence $x \in \Omega(g) \cup \{p\}$.

Suppose that $\lambda > 0$ is any countable ordinal number. Choose a compact countable set $X = Z_\lambda$ and a homeomorphism $g = f_\lambda : X = Z_\lambda \to X = Z_\lambda$ such that $d(f_\lambda) = \lambda$. In this case, we may assume that for each $n \ge 2$ $|X_n| \ge 2$, where $|X_n|$ denotes the cardinality of X_n. Then we obtain a map $f : D = Y(X) \to D$ such that $\Omega(f) = \Omega(f_\lambda) \cup \{p\}$ and $E(D) = Z_\lambda$. Hence $d(f) = \lambda$.

By (2.2), we obtain the following.

COROLLARY 3.2. *There is a dendrite D such that for any countable ordinal number λ, there is a map $f : D \to D$ such that $d(f) = \lambda$.*

COROLLARY 3.3. *For any countable ordinal number λ, there is a map $f : B^2 \to B^2$ of a disk (= 2-dimensional ball) B^2 such that $d(f) = \lambda$.*

PROOF. Let $\lambda > 0$ be any countable ordinal number. By (3.1), we can choose a map $g : D \to D$ of a dendrite D such that $d(g) = \lambda$. Since D is a dendrite, we may assume that $D \subset B^2$. Since D is an AR, there is a retraction $r : B^2 \to D$. Put $f = g \cdot r : B^2 \to B^2$. Then $d(f) = \lambda$.

In [5], Neumann proved that for any C^∞ n-manifold M with $n \geq 3$ and any countable ordinal number λ, there is a C^∞ flow ϕ on M such that the depth of the centre of ϕ is λ.

Here, we prove the following.

COROLLARY 3.4. *For any countable ordinal number λ, there is a homeomorphism $h : B^3 \to B^3$ of a 3-dimensional ball B^3 such that $h|\partial B^3 = \mathrm{id}$, $d(h) = \lambda$ and $\Omega_\lambda(h) = \partial B^3 \cup Z$, where Z is a countable compactum in $B^3 - \partial B^3$.*

PROOF. We may assume that $B^3 = B^2 \times [-1, 1]$. Choose a compact countable set Z_λ and a homeomorphism $f_\lambda : Z_\lambda \to Z_\lambda$ such that $d(f_\lambda) = \lambda$. We may assume that $X = Z_\lambda \subset (B^2 - \partial B^2) \times \{0\}$. By [3, Chapter 13], we can choose a homeomorphism $g : B^2 \to B^2$ such that g is an extension of f and $g|\partial B = \mathrm{id}$. We can choose a map $\psi : B^2 \times [-1, 1] \to [-1, 1]$ satisfying the following conditions:

(1) $\psi(x, t) = t$ for $x \in \partial B^2 \times [-1, 1]$,
(2) $\psi(x, -1) = -1$, $\psi(x, 1) = 1$ for each $x \in B^2$,
(3) $\psi(x, t) > t$ if $x \notin X$, $-1 < t < 1$,
(4) $\psi(x, 0) = 0$ if $x \in X$, and
(5) $\psi(x, t) > t$ if $x \in X$, $t \neq -1, 0, 1$.

Consider the suspension $S(B^2)$ of B^2, that is, $S(B^2)$ is the quotient space of $B^2 \times [-1, 1]$ in which $B \times \{-1\}$ and $B^2 \times \{1\}$ are identified to two different points. If $(x, t) \in B^2 \times [-1, 1]$, we use $[x, t]$ to denote the corresponding point of $S(B^2)$ under the quotient map $q : B^2 \times [-1, 1] \to S(B^2)$. Note that $S(B^2) = B^3$ is a 3-dimensional ball. Define a homeomorphism $h : B^3 \to B^3$ by $h([x, t]) = [g(x), \psi(x, t)]$.

Suppose that $(x, t) \in (B^2 \times [-1, 1]) - (X \cup \partial(B^2 \times [-1, 1]))$. Since $\psi(x, t) > t$, we can choose a neighborhood U of (x, t) such that $\psi(U) \cap p(U) = \phi$, where $p : B^2 \times [-1, 1] \to [-1, 1]$ is the natural projection. Since $\psi(x, t)$ is not decreasing with respect to t, we see that $h^n(q(U)) \cap q(U) = \phi$ for all $n \geq 1$. Hence $\Omega(h) \subset X \cup \partial B^3$. If λ is any ordinal number with $\lambda \geq \omega$, then we see that $d(h) = \lambda$. Suppose that $0 < \lambda = m < \omega$. In this case, moreover, we can choose a homeomorphism $g : B^2 \to B^2$ such that there is a small disk D' which is a neighborhood of $x_m(0) \in Z_m - Z_{m-1}$ satisfying $g^i(D') \cap g^j(D') = \phi (i \neq j)$, $g^i(D') \cap Z_{m-1} = \phi$ for each i, and $\lim_{n \to \pm\infty} \mathrm{diam}(g^i(D')) = 0$ (see Figure 4 and the proof of [3, Theorem 1, p. 91]), where $Z_0 = \{x(\infty)\}$. Then $\Omega(h) = \partial(B^3) \cup \Omega(f_\lambda) = \partial B^3 \cup Z_{m-1}$. Hence we see that $d(h) = m$.

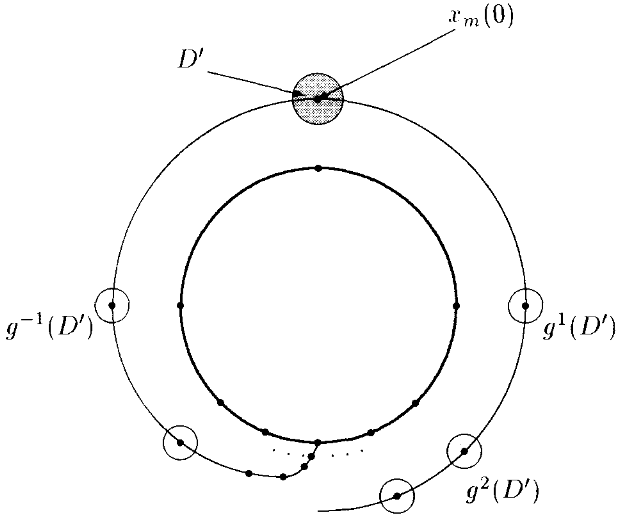

FIGURE 4

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