BULL. AUSTRAL. MATH. SOC. 13G05, 13C05
VOL. 60 (1999) [129–135]

QUASI-PROJECTIVITY OVER DOMAINS

DMITRI ALEXEEV

Let R be an integral domain with quotient field Q. We investigate quasi- and Q-projective ideals, and properties of domains all ideals of which are quasi-projective. It is shown that the so-called $1\frac{1}{2}$-generated ideals are quasi-projective, moreover, projective. A module M is quasi-projective if and only if, for a projective ideal P of R, the tensor product $M \otimes_R P$ is quasi-projective. Domains whose all ideals are quasi-projective are characterised as almost maximal Prüfer domains. Q is quasi-projective if and only if every proper submodule of Q is complete in its R-topology.

INTRODUCTION

Quasi-projective (sometimes called "self-projective") modules were introduced by Miyashita [**7**] as a generalisation of projective modules. These modules have been studied by a number of authors over a variety of rings. In particular, Herrmann [**3**] investigated quasi-projective modules over valuation domains. This paper contains generalisations of some of his results to arbitrary integral domains.

Let R be an integral domain with quotient field Q. In the case of valuation domains, quasi-projectivity of an ideal is equivalent to its Q-projectivity. This does not seem to be true over general domains, although it holds if the ideal is finitely generated. Q-projectivity of ideals always implies their quasi-projectivity. We show that a submodule V of Q is quasi-projective if and only if $\operatorname{Ext}^1_R(V, U) = 0$ for each submodule U of Q.

For an ideal I of R, this characterisation of quasi-projectivity reduces to vanishing $\operatorname{Ext}^1_R(I, J)$ for each ideal J of R. This is used to identify the domains all of whose ideals are quasi-projective: these are exactly the almost maximal Prüfer domains. If we only assume the finitely generated ideals of R to be quasi-projective, then R becomes a Prüfer domain (not necessarily almost maximal).

Finally, we study the quasi-projectivity of Q. Q is quasi-projective if and only if every proper submodule of Q is complete in its R-topology.

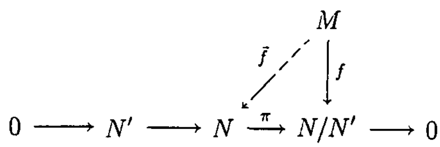
Received 14th January, 1999
This paper is a part of the author's Ph.D. thesis written under supervision of Professor Laszlo Fuchs at Tulane University.

https://doi.org/10.1017/S0004972700033396 Published online by Cambridge University Press

1. Preliminaries

Let R be an integral domain, that is, a commutative ring with no zero divisors. Unless otherwise noted, we assume all modules to be unital R-modules, all ideals to be ideals of R, and all maps to be R-homomorphisms.

DEFINITION. Let M and N be R-modules. M is called *N-projective* or *projective relative to* N if it is projective relative to all exact sequences of the form $0 \to N' \to N \xrightarrow{\pi} N/N' \to 0$, where N' is a submodule of N and π is the canonical projection. That is, for every homomorphism $f\colon M \to N/N'$ there exists a map $\overline{f}\colon M \to N$ such that the following diagram commutes:



Thus, a module M is projective if and only if it is N-projective for all R-modules N. The module M is called *self-* or *quasi-projective* if it is M-projective.

We shall need the following lemma from Anderson and Fuller [**1**, p.188].

LEMMA 1.1. *Let M be an R-module.*

(a) *If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules and M is B-projective, then M is projective relative to A and C as well.*

(b) *If M is projective relative to M_i for each i in a finite index set I, then M is $\bigoplus_{i \in I} M_i$-projective. Moreover, if M is finitely generated, this holds for an arbitrary index set I.*

The following corollary is a direct consequence of Lemma 1.1.

COROLLARY 1.2. *Let I be an ideal of domain R and V a submodule of Q.*

(a) *If V is Q-projective, then V is U-projective for every submodule U of Q.*

(b) *If I is quasi-projective, then I is J-projective for every ideal J of R.*

Unlike projective ideals, quasi-projective ideals need not be finitely generated. In fact, Theorem 4.2 shows that if R is an almost maximal Prüfer domain, then *all* ideals of R are quasi-projective.

2. Q-projectivity

We now turn our attention to Q-projectivity.

LEMMA 2.1. *Let R be an integral domain. For a submodule V of Q the following statements are equivalent.*

(a) *V is Q-projective.*

https://doi.org/10.1017/S0004972700033396 Published online by Cambridge University Press

(b) $\mathrm{Ext}^1_R(V,U)=0$ *for all submodules* U *of* Q.

(c) V *is* U*-projective for all submodules* U *of* Q.

Moreover, if $V=I$ *is an ideal of* R*, then these conditions are equivalent to:*

(d) *For every submodule* U *of* Q*,* Q/U *has the injective property relative to the exact sequence* $0\to I\to R\to R/I\to 0$.

PROOF: [(a) $\Leftrightarrow$ (b)] By applying functor $\mathrm{Hom}_R(V,*)$ to the exact sequence

$$0\to U\to Q\to Q/U\to 0,$$

we obtain the exact sequence

$$0\to \mathrm{Hom}_R(V,U)\to \mathrm{Hom}_R(V,Q)\xrightarrow{p}\mathrm{Hom}_R(V,Q/U)\to \mathrm{Ext}^1_R(V,U)\to \mathrm{Ext}^1_R(V,Q)=0.$$

Here $\mathrm{Ext}^1_R(V,U)=0$ if and only if p is an epimorphism.

[(a) $\Leftrightarrow$ (c)] This is Corollary 1.2 (a).

[(b) $\Leftrightarrow$ (d)] By applying the functors $\mathrm{Hom}_R(*,U)$ and $\mathrm{Hom}_R(R/I,*)$, respectively, to the exact sequences

$$0\to I\to R\to R/I\to 0 \quad\text{and}\quad 0\to U\to Q\to Q/U\to 0,$$

we obtain the exact sequences

$$0=\mathrm{Ext}^1_R(R,U)\to \mathrm{Ext}^1_R(I,U)\to \mathrm{Ext}^2_R(R/I,U)\to \mathrm{Ext}^2_R(R,U)=0,$$

$$0=\mathrm{Ext}^1_R(R/I,Q)\to \mathrm{Ext}^1_R(R/I,Q/U)\to \mathrm{Ext}^2_R(R/I,U)\to \mathrm{Ext}^2_R(R/I,Q)=0.$$

It follows that $\mathrm{Ext}^1_R(I,U)\cong \mathrm{Ext}^1_R(R/I,Q/U)$. From the exact sequence

$$0\to \mathrm{Hom}_R(R/I,Q/U)\to \mathrm{Hom}_R(R,Q/U)\xrightarrow{p}\mathrm{Hom}_R(I,Q/U)$$
$$\to \mathrm{Ext}^1_R(R/I,Q/U)\to \mathrm{Ext}^1_R(R,Q/U)=0$$

we conclude that p is epic if and only if $\mathrm{Ext}^1_R(R/I,Q/U)=0$. ∎

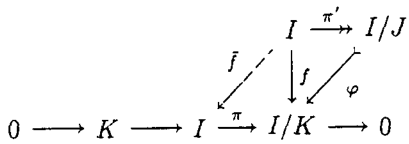
3. QUASI-PROJECTIVITY

We begin this section by introducing a class of quasi-projective ideals.

DEFINITION. An ideal I of R is called $1\frac{1}{2}$*-generated* if it is two-generated, but one of the generators can be chosen to be any non-zero element of I. In other words, every proper homomorphic image of I is a cyclic module. This is equivalent to saying that for any non-zero ideal $J<I$ we can find an $a\in I$ such that $I=Ra+J$.

LEMMA 3.1. $1\frac{1}{2}$*-generated ideals of a domain* R *are quasi-projective.*

PROOF: Let I be a $1\frac{1}{2}$-generated ideal. Suppose we are given a subideal $0 \neq K < I$, the canonical projection $\pi\colon I \to I/K$, and a homomorphism $f\colon I \to I/K$. To prove quasi-projectivity of I we find a map $\overline{f}\colon I \to I$ such that $\pi\overline{f} = f$. The map f factors as the canonical projection $\pi'\colon I \to I/J$, where $J = \operatorname{Ker} f$, followed by an embedding $\varphi\colon I/J \to I/K$. The cyclic module I/J is isomorphic to a submodule of cyclic module I/K. Therefore, there exists an $r \in R$ such that $r(I/K) = f(I) = \varphi(I/J)$. We claim that $\overline{f}\colon I \to I$ defined by $\overline{f}(t) = rt$ $(t \in I)$ is a lifting of f making the diagram



commutative.

To prove this claim, choose an $a \in R$ such that $I = Ra + (K \cap J)$. Then $I = Ra + J = Ra + K$. For the diagram to commute, it suffices to have $rJ \subset K$. We have the following isomorphisms:

$$\frac{I}{K} = \frac{Ra+K}{K} \cong \frac{Ra}{Ra\cap K}, \quad f(I) \cong \frac{I}{J} = \frac{Ra+J}{J} \cong \frac{Ra}{Ra\cap J}.$$

Now $r(I/K) = f(I)$ implies that

$$\frac{rRa}{rRa\cap K} \cong \frac{Ra}{Ra\cap J} \cong \frac{rRa}{rRa\cap rJ}.$$

We conclude that $rRa \cap K = rRa \cap rJ$. It remains to note that

$$J = (Ra\cap J) + (J\cap K) \quad \text{and} \quad K = (Ra\cap K) + (J \cap K)$$

to conclude $rJ = (rRa \cap rJ) + r(J\cap K) = (rRa\cap K) + r(J\cap K) \subset K$. This completes the proof. ∎

Rangaswamy and Vanaja [9] show that finitely generated torsion-free quasi-projective modules over a domain are projective. This immediately implies the following theorem.

THEOREM 3.2. *$1\frac{1}{2}$-generated ideals of any domain are projective.*

The result also appears in [4] in a more general setting. Observe that over Prüfer domains of finite character, projective ideals are $1\frac{1}{2}$-generated.

THEOREM 3.3. *Let P be a non-zero projective ideal of a domain R and M an R-module. M is quasi-projective if and only if $P\otimes_R M$ is quasi-projective.*

PROOF: Suppose that $P\otimes_R M$ is quasi-projective. For a given map $f\colon M \to M/N$ $(N \subset M)$, we need to find a lifting $f'\colon M \to M$. Tensoring with P preserves exact sequences. The quasi-projectivity of $P\otimes_R M$ implies that there exists a map $\varphi\colon P\otimes_R M \to P\otimes_R M$ such that the following diagram is commutative.

https://doi.org/10.1017/S0004972700033396 Published online by Cambridge University Press

$$\begin{array}{ccccccccc} & & & & & & P \otimes_R M & & \\ & & & & & \overset{\varphi}{\swarrow} & \downarrow {\scriptstyle 1 \otimes_R f} & & \\ 0 & \longrightarrow & P \otimes_R N & \xrightarrow{1 \otimes_R i} & P \otimes_R M & \xrightarrow{1 \otimes_R p} & P \otimes_R M/N & \longrightarrow & 0 \end{array}$$

To show that φ induces a map $f'\colon M \to M$, we tensor the above diagram with the inverse P^{-1} of P. Since $P^{-1} \otimes_R P \cong R$ and $R \otimes_R M \cong M$ for any R-module M naturally, the map $\overline{f} = 1_{P^{-1}} \otimes_R \varphi\colon M \to M$ is a desired lifting for f.

Conversely, suppose that M is a quasi-projective module. We have

$$P^{-1} \otimes_R (P \otimes_R M) \cong M,$$

where P^{-1} is projective. By the first part of the proof, this implies that $P \otimes_R M$ is quasi-projective. □

It is worth while mentioning a consequence of Theorem 3.3 to the *class semigroup* of R. This is the semigroup of isomorphy classes of fractional ideals where identity is represented by R and whose operation is ideal multiplication. Isomorphy classes of invertible ideals form a subgroup, the *class group*, of this semigroup. Theorem 3.3 asserts that quasi-projective ideals form complete orbits in the ideal semigroup of R under the action of the class group.

4. Prüfer Domains

We formulate a close analogue of Lemma 2.1. The proof is similar to the original and is omitted.

LEMMA 4.1. *The following are equivalent for an ideal I of R.*

(a) *I is quasi-projective.*

(b) $\mathrm{Ext}^1_R(I, J) = 0$ *for every ideal J of R.*

(c) *For every ideal J of R, Q/J has the injective property relative to the exact sequence $0 \to I \to R \to R/I \to 0$.*

Lemmas 2.1 and 4.1, allow us to obtain a characterisation of domains all of whose ideals are quasi-projective; this characterisation is similar to that of valuation domains obtained by Herrmann in [**3**]. He has shown that a valuation domain each of whose ideals is quasi-projective is almost maximal. We precede the results with necessary definitions.

DEFINITION. Let M be a module over a commutative ring R. M is said to be *semi-compact* if every finitely solvable set of congruences

$$x \equiv x_\alpha \pmod{M_\alpha},$$

where $x_\alpha \in M$ and M_α are submodules of M which are annihilators of ideals of R, has a simultaneous solution in M. M is called *linearly compact* if every family of cosets of submodules of M that has the finite intersection property has a nonempty intersection.

https://doi.org/10.1017/S0004972700033396 Published online by Cambridge University Press

Following Brandal [2], we call a domain R *maximal* if every homomorphic image of R is linearly compact, and *almost maximal* if every proper homomorphic image of R is linearly compact.

THEOREM 4.2. *For an integral domain R, all of the following are equivalent.*

(a) $\mathrm{Ext}^1_R(I, U) = 0$ *for every ideal I of R and submodule U of Q.*

(b) $\mathrm{Ext}^1_R(I, J) = 0$ *for every pair I, J of ideals of R.*

(c) *Every ideal of R is Q-projective.*

(d) *Every ideal of R is quasi-projective.*

(e) *Every epimorphic image of Q is injective.*

(f) *R is a Prüfer domain and every homomorphic image of Q is semi-compact.*

(g) *R is an almost maximal Prüfer domain.*

PROOF: First, (a) implies (b) and (c) implies (d) trivially. (a) $\Leftrightarrow$ (c) is the statement of Lemma 2.1. (b) $\Leftrightarrow$ (d) is Lemma 4.1. Lemma 2.1 and Baer's criterion prove (c) $\Leftrightarrow$ (e). Matlis [5, Theorem 5] implies (e) $\Leftrightarrow$ (f). Finally, Olberding [8] shows the equivalence of (b), (e), and (g). ∎

5. QUOTIENT FIELDS

In this section we generalise a characterisation of valuation domains with quasi-projective quotient fields to the case of arbitrary integral domains. Herrmann [3] shows that a valuation domain R is complete in its R-topology if and only if Q is quasi-projective. We show that the same holds in general, almost unchanged.

We observe that Q is projective relative to any ideal I of R, or, more generally, to any proper submodule of itself. In fact, Q is divisible and I/J is bounded for ideals $J < I$. The only map $f\colon Q \to I/J$ is the trivial one, which can obviously be lifted. Of course, this property is far from being a sufficient condition for Q to be quasi-projective.

EXAMPLE. Let R be the ring of integers $\mathbb{Z}$. Then $Q = \mathbb{Q}$, the field of rationals. For the localisation $\mathbb{Z}_p$ of $\mathbb{Z}$ at a prime $p \neq 0$, we have $\mathbb{Q}/\mathbb{Z}_p \cong \mathbb{Z}(p^\infty)$. By the above remark, $\mathbb{Q}$ is I-projective for each ideal I of $\mathbb{Z}$. Since there are only countably many maps $\overline{f}\colon \mathbb{Q} \to \mathbb{Q}$, but uncountably many endomorphisms $\varphi\colon \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$, we conclude that it is impossible to lift every $f\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_p$ to a map $\overline{f}\colon \mathbb{Q} \to \mathbb{Q}$ such that $\pi\overline{f} = f$. Hence, $\mathbb{Q}$ is not quasi-projective.

DEFINITION. We may consider an R-module M as a topological module equipped with the *R-topology*, in which a subbase for the open neighbourhoods of zero is formed by submodules rM $(0 \neq r \in R)$. The R-topology is a uniform topology. An R-module M is said to be *R-complete* if it is Hausdorff and complete in its R-topology.

THEOREM 5.1. *For any domain R, the following statements are equivalent:*

(a) *Q is quasi-projective.*

(b) $\mathrm{Ext}^1_R(Q,U) = 0$ *for every submodule* U *of* Q.

(c) *Every proper submodule of* Q *is complete in its* R*-topology.*

[(a) $\Leftrightarrow$ (b)] We may assume $U \neq 0, Q$. By applying the functor $\mathrm{Hom}_R(Q,*)$ to the exact sequence $0 \to U \to Q \to Q/U \to 0$, we obtain the exact sequence

$$\mathrm{Hom}_R(Q,Q) \xrightarrow{p} \mathrm{Hom}_R(Q,Q/U) \to \mathrm{Ext}^1_R(Q,U) \to \mathrm{Ext}^1_R(Q,Q) = 0.$$

Thus we see that p is epic if and only if $\mathrm{Ext}^1_R(Q,U) = 0$.

[(b) $\Leftrightarrow$ (c)] Matlis [**6**] shows that a reduced torsion-free R-module M is R-complete if and only if $\mathrm{Ext}^1_R(Q,M) = 0$. Letting $M = U$ completes the proof.

We conclude this section with the following theorem.

THEOREM 5.2. *Let* R *be an integral domain. If* Q *is quasi-projective, then* Q/R *is indecomposable.*

PROOF: Let Q be quasi-projective such that Q/R is decomposable. Then, there are submodules A and B of Q properly containing R such that $Q/R = A/R \oplus B/R$. Consider the projection $f\colon Q \to A/R \oplus B/R$ on the first summand. By assumption, there exists a lifting $f'\colon Q \to Q$ of f; this must be a multiplication by some non-zero element $q \in Q$. If $\pi\colon Q \to A/R \oplus B/R$ denotes the canonical projection, then $f = \pi f'$ means that $qB \subseteq R$. Thus, B is a fractional ideal, and so is A. But then $Q = A + B$ is impossible, a contradiction. ∎

REFERENCES

[1] F. Anderson and K. Fuller, *Rings and categories of modules*, (second edition) (Springer-Verlag, Berlin, Heidelberg, New York, 1992).

[2] W. Brandal, 'Almost maximal integral domains and finitely generated modules', *Trans. Amer. Math. Soc.* **183** (1973), 203–222.

[3] P. Herrmann, 'Self-projective modules over valuation rings', *Arch. Math.* **43** (1984), 332–339.

[4] D.C. Lantz and M.B. Martin, 'Strongly two-generated ideals', *Comm. Alg.* **16** (1988), 1759–1777.

[5] E. Matlis, 'Injective modules over Prüfer rings', *Nagoya Math.* **15** (1959), 57–69.

[6] E. Matlis, 'Cotorsion modules', *Mem. Amer. Math. Soc.* **49** (1964).

[7] Y. Miyashita, 'Quasi-projective modules, perfect modules and a theorem for modular lattices', *J. Fac. Sci. Hokkaido Univ. Ser. I* **19** (1966), 86–110.

[8] B. Olberding, 'Almost maximal Prüfer domains' (to appear).

[9] K. M. Rangaswamy and N. Vanaja, 'Quasi projectives in Abelian and module categories', *Pacific J. Math.* **43** (1972), 221–238.

Department of Mathematics
Tulane University
New Orleans LA 70118
United States of America
e-mail: alexeev@math.tulane.edu