Canad. J. Math. Vol. 70 (1), 2018 pp. 3–25
http://dx.doi.org/10.4153/CJM-2017-024-8

Fluctuation of Matrix Entries and Application to Outliers of Elliptic Matrices

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Abstract. For any family of $N \times N$ random matrices $(\mathbf{A}_k)_{k\in K}$ that is invariant, in law, under unitary conjugation, we give general sufficient conditions for central limit theorems for random variables of the type $\mathrm{Tr}(\mathbf{A}_k\mathbf{M})$, where the matrix $\mathbf{M}$ is deterministic (such random variables include, for example, the normalized matrix entries of $\mathbf{A}_k$). A consequence is the asymptotic independence of the projection of the matrices $\mathbf{A}_k$ onto the subspace of null trace matrices from their projections onto the orthogonal of this subspace. These results are used to study the asymptotic behavior of the outliers of a spiked elliptic random matrix. More precisely, we show that the fluctuations of these outliers around their limits can have various rates of convergence, depending on the Jordan Canonical Form of the additive perturbation. Also, some correlations can arise between outliers at a macroscopic distance from each other. These phenomena have already been observed with random matrices from the Single Ring Theorem.

1 Introduction

This paper is first concerned with the fluctuations of linear functions of entries of unitarily invariant random matrices when the dimension tends to infinity. Then it deals with the application of such limit theorems to the fluctuations of the outliers of spiked elliptic matrices.

The first problem is to determine conditions under which, for given collections $(\mathbf{A}_k)_{k\in K}$ of random matrices and $(\mathbf{M}_\ell)_{\ell\in L}$ of non-random matrices, the finite marginals of

$$\Big(\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell) - \mathbb{E}\,\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell)\Big)_{k\in K,\ell\in L} \tag{1.1}$$

converge as the dimension N tends to infinity. We shall always suppose that $\mathbf{A}_k$ and $\mathbf{M}_\ell$ have Euclidean norms of order $\sqrt{N}$, *i.e.*, that the random variables

$$\frac{1}{N}\,\mathrm{Tr}\,\mathbf{A}_k\mathbf{A}_k^* \quad \text{and} \quad \frac{1}{N}\,\mathrm{Tr}\,\mathbf{M}_\ell\mathbf{M}_\ell^*$$

are bounded in probability. The case

$$\mathbf{M}_\ell = \sqrt{N} \times (\text{an elementary matrix}) \tag{1.2}$$

Received by the editors February 10, 2016; revised August 16, 2016.
Published electronically November 15, 2017.
Author G. C. was partially supported by the ERC advanced grant “Non-commutative Distributions in Free Probability", held by R. Speicher.
AMS subject classification: **60B20**, 15B52, 60F05, 46L54.
Keywords: random matrix, Gaussian fluctuation, spiked model, elliptic random matrix, Weingarten calculus, Haar measure.

is a classical example. In this framework, the main hypothesis we need for the random vector of (1.1) to be asymptotically Gaussian is the global invariance, in law, of $(\mathbf{A}_k)_{k\in K}$ under unitary conjugation, *i.e.*, for any non-random unitary matrix $\mathbf{U}$,

$$(\mathbf{A}_k)_{k\in K} \overset{\text{law}}{=} (\mathbf{U}\mathbf{A}_k\mathbf{U}^*)_{k\in K}.$$

It then appears that the question decomposes into two independent problems: one associated with the projections of $\mathbf{A}_k$ onto the space of null trace matrices (see Theorem 2.1) and one associated with the convergence of the centered traces of $\mathbf{A}_k$, and both give rise to independent asymptotic fluctuations (see Theorem 2.3 and Corollary 2.4). These results extend an already proved partial result in this direction [8, Theorem 6.4] (see also [42, Theorem 1.2] in the particular case of real symmetric matrices $\mathbf{A}_k$). The main advantages of Theorems 2.1 and 2.3 over the results of [8, 42] is, first, that they do not require the matrices $\mathbf{M}_\ell$ to have finitely many non-zero entries (or to be well approximated by such matrices) and, secondly, that they give the asymptotic independence mentioned above. Besides, the technical hypotheses needed here are weaker than in the existing literature. Our proofs are based on the so-called *Weingarten calculus*, an integration method for the Haar measure on the unitary group developed by Collins and Śniady [22, 24].

All these results belong to a long list of theorems begun in 1906 with the theorem by Borel [15] stating that any coordinate of a uniformly distributed random vector of the sphere of $\mathbb{R}^N$ with radius $\sqrt{N}$ is asymptotically standard Gaussian as $N \to \infty$, and continued with [2, 7, 13, 19, 21, 25, 28, 34, 42] on central limit theorems on large matrix spaces. Some of the results from these papers can be deduced from this paper (see Remark 2.7).

Second order freeness, a theory developed in the last decade, deals with Gaussian fluctuations (called *second order limits*) of traces of large random matrices. The most emblematic articles on this theory are [23, 35–37]. As explained in Remark 2.5, our results cannot be deduced from this theory, because the test matrices we consider (*i.e.*, the matrices $\mathbf{M}_\ell$) are not supposed to have second order limits. Precisely, in classical applications of our results, *i.e.*, the case of (1.2), the matrices $\mathbf{M}_\ell$ do not have any second order limit. However, we shall see in Section 2.2 that our results extend the consequences of the existence of a second order limit for unitarily invariant matrix ensembles. The general results about asymptotic fluctuations of matrix entries that we prove here are then applied to the fluctuations of the outliers of Gaussian elliptic matrices. From the macroscopic point of view, one can prove [20] that the global behavior of the spectrum of a large random matrix is not altered by a low rank additive perturbation. However, some of the eigenvalues, called *outliers*, can deviate away from the bulk, depending on the strength of the perturbation. This phenomenon known as the *BBP transition*, first brought to light for empirical covariance matrices by Johnstone [30],was proved by Baik, Ben Arous and Péché [6], and then extended to several Hermitian models [8–11, 16–18, 27, 31, 32, 48, 49]. Non-Hermitian models have been also studied: i.i.d. matrices [14, 41, 51], real elliptic matrices [46], matrices from the Single Ring Theorem [12] and also nearly Hermitian matrices [44, 45]. As an application of our main result, we investigate the fluctuations of the outliers and due to the non-Hermitian structure, we prove, as in [12, 14, 41, 44], that the distribution of the fluctuations highly depends on the shape of the Jordan Canonical Form of the perturbation. In particular, the convergence rates depend on the sizes of the Jordan blocks. Also, the outliers tend to locate around their limits at the vertices of regular polygons (see Figure 1). At last, as in [12], we prove the quite surprising fact that outliers at macroscopic distance from each other can have correlation fluctuations (see Remark 2.20), .

The paper is organized as follows. In Section 2, we state our main results (Theorems 2.1, 2.3, 2.12, and 2.18) and their corollaries. These theorems are then proved in the following sections and an appendix is devoted to a technical result needed here.

Notation 1.1. For u, v sequences, $u = o(v)$ means that $u/v \to 0$ and $u = O(v)$ means that u/v is bounded. Also, the dimension N of the matrices is most times an implicit parameter.

2 Main Results

2.1 General Results

Let $\mathbf{A} = (\mathbf{A}_k)_{k\in K}$ be a collection of $N \times N$ random matrices and let $\big(\mathbf{M}_\ell\big)_{\ell\in L}$ be a collection $N \times N$ non random matrices, both implicitly depending on N.

Hypothesis 1 (a) $\mathbf{A}$ is invariant in distribution under unitary conjugation: for any non random unitary matrix $\mathbf{U}$, $(\mathbf{A}_k)_{k\in K} \overset{\text{law}}{=} (\mathbf{U}\mathbf{A}_k\mathbf{U}^*)_{k\in K}$;

(b) for each $k \in K$, and each $p, q \geq 1$, $\frac{1}{N}\operatorname{Tr}(\mathbf{A}_k\mathbf{A}_k^*)^p$ is bounded in L^q independently of N;

(c) for each $k, k' \in K$, we have the following convergences in L^2, to deterministic limits:

$$\lim_{N\to\infty} \frac{1}{N}\operatorname{Tr}\mathbf{A}_k\mathbf{A}_{k'} - \frac{1}{N}\operatorname{Tr}\mathbf{A}_k \cdot \frac{1}{N}\operatorname{Tr}\mathbf{A}_{k'} = \tau(k,k'),$$
$$\lim_{N\to\infty} \frac{1}{N}\operatorname{Tr}\mathbf{A}_k\mathbf{A}_{k'}^* - \frac{1}{N}\operatorname{Tr}\mathbf{A}_k \cdot \frac{1}{N}\operatorname{Tr}\mathbf{A}_{k'}^* = \tau(k,\overline{k'});$$

(d) for each $\ell, \ell' \in L$, we have the following convergences:

$$\lim_{N\to\infty} \frac{1}{N}\operatorname{Tr}\mathbf{M}_\ell\mathbf{M}_{\ell'} - \frac{1}{N}\operatorname{Tr}\mathbf{M}_\ell \cdot \frac{1}{N}\operatorname{Tr}\mathbf{M}_{\ell'} = \eta_{\ell\ell'}, \tag{2.1}$$

$$\lim_{N\to\infty} \frac{1}{N}\operatorname{Tr}\mathbf{M}_\ell\mathbf{M}_{\ell'}^* - \frac{1}{N}\operatorname{Tr}\mathbf{M}_\ell \cdot \frac{1}{N}\operatorname{Tr}\mathbf{M}_{\ell'}^* = \beta_{\ell\ell'}. \tag{2.2}$$

Under this sole hypothesis, we first have the following result focused on the case where the $\mathbf{M}_\ell$s all have null trace, *i.e.*, focused on the projections of the above $\mathbf{A}_k$s onto the space of such matrices.

Theorem 2.1 *Under Hypothesis 1, if* $\operatorname{Tr}(\mathbf{M}_\ell) = 0$ *for each* ℓ*, then the finite-dimensional marginal distributions of*

$$\big(\operatorname{Tr}(\mathbf{A}_k\mathbf{M}_\ell)\big)_{k\in K,\ell\in L} \tag{2.3}$$

converge to those of a complex centered Gaussian vector $(\mathcal{G}_{k,\ell})_{k\in K,\ell\in L}$ *with covariances* $\mathbb{E}[\mathcal{G}_{k,\ell}\mathcal{G}_{k',\ell'}] = \eta_{\ell\ell'}\tau(k,k')$ *and* $\mathbb{E}[\mathcal{G}_{k,\ell}\overline{\mathcal{G}_{k',\ell'}}] = \beta_{\ell\ell'}\tau(k,\overline{k'})$.

Remark 2.2 Note that by invariance of the distribution of $\mathbf{A}$ under unitary conjugation, we have $\mathbb{E}\,\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell) = \mathbb{E}(\frac{1}{N}\,\mathrm{Tr}\,\mathbf{A}_k)\,\mathrm{Tr}\,\mathbf{M}_\ell$. Hence the random variables of (2.3) are centered and the ones of (2.5) below rewrite $\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell) - \mathbb{E}(\frac{1}{N}\,\mathrm{Tr}\,\mathbf{A}_k)\,\mathrm{Tr}\,\mathbf{M}_\ell$.

The following theorem gives the joint fluctuations of the projections of the $\mathbf{A}_k$s on null trace matrices and of their traces.

Hypothesis 2 The finite-dimensional marginal distributions of the process

$$(\mathrm{Tr}\,\mathbf{A}_k - \mathbb{E}\,\mathrm{Tr}\,\mathbf{A}_k)_{k\in K}$$

converge to those of a random centered vector $(\mathcal{T}_k)_{k\in K}$ and for each $\ell \in L$, there is $\alpha_\ell \in \mathbb{C}$ such that

$$\lim_{N\to\infty} \frac{1}{N}\,\mathrm{Tr}\,\mathbf{M}_\ell = \alpha_\ell. \tag{2.4}$$

Theorem 2.3 *Under Hypotheses 1 and 2, the finite-dimensional marginal distributions of*

$$\big(\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell) - \mathbb{E}\,\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell)\big)_{k\in K,\ell\in L} \tag{2.5}$$

converge to those of $(\mathcal{G}_{k,\ell} + \alpha_\ell\mathcal{T}_k)_{k\in K,\ell\in L}$, *where* $(\mathcal{G}_{k,\ell})_{k\in K,\ell\in L}$ *is a complex centered Gaussian vector independent from* $(\mathcal{T}_k)_{k\in K}$ *and with covariances*

$$\mathbb{E}[\mathcal{G}_{k,\ell}\mathcal{G}_{k',\ell'}] = \eta_{\ell\ell'}\tau(k,k') \quad \textit{and} \quad \mathbb{E}[\mathcal{G}_{k,\ell}\overline{\mathcal{G}_{k',\ell'}}] = \beta_{\ell\ell'}\tau(k,\overline{k'}).$$

A direct consequence of this theorem is the asymptotic independence of the projections of the matrices $\mathbf{A}_k$ onto the subspace of null trace matrices from their projections onto the orthogonal of this subspace.

Corollary 2.4 *Under Hypotheses 1 and 2, suppose that for any* $\ell \in L$, *we have* $\mathrm{Tr}(\mathbf{M}_\ell) = 0$. *Then the processes* $\big(\mathrm{Tr}(\mathbf{A}_k\mathbf{M}_\ell)\big)_{k\in K,\ell\in L}$ *and* $(\mathrm{Tr}\,\mathbf{A}_k - \mathbb{E}\,\mathrm{Tr}\,\mathbf{A}_k)_{k\in K}$ *are asymptotically independent.*

Remark 2.5 It has been proved [36] that unitary invariance implies second order freeness in many cases. However, Theorems 2.1 and 2.3, as well as their corollaries, cannot be deduced from the theory of second order freeness. The reason is that neither the random matrices $\mathbf{A}_k$ nor the matrices $\mathbf{M}_\ell$ are supposed to have a second order limit. Even in the case where the random matrices $\mathbf{A}_k$ have a second order limit (see §2.2), the test matrices that we consider, *i.e.*, the matrices $\mathbf{M}_\ell$, are not supposed to have a second order limit, nor to be well approximated by matrices having a second order limit. For example, if $\mathbf{M}_\ell = \sqrt{N}\times$(an elementary matrix) (a typical case of application of our results), then for any $p \geq 2$, $\frac{1}{N}\,\mathrm{Tr}(\mathbf{M}_\ell\mathbf{M}_\ell^*)^p = N^{p-1}$, so that the sequence does not have any finite limit as $N \to \infty$, nor is it bounded, which would be required to prove our results as application of second order freeness.

2.2 Second-order Freeness Implies Fluctuations of Matrix Elements

As explained in Remark 2.5, our results do not follow from second order freeness theory. However, we shall see in the following corollary that they extend the consequences of the existence of a second order limit for unitarily invariant matrix ensembles. Let $\mathbb{C}\langle x_k, x_k^*, k \in K\rangle$ denote the space of polynomials in the non-commutative variables x_k, x_k^*, indexed by $k \in K$. Corollary 2.6 follows directly from Theorem 2.3.

Corollary 2.6 *Let $(\mathbf{A}_k)_{k\in K}$ be a collection of $N \times N$ random matrices that is invariant by unitary conjugation and that converges in a second order $*$-distribution to some family $a = (a_k)_{k\in K}$ in $(\mathcal{A}, \tau_1, \tau_2)$ as $N \to \infty$. Let $(\mathbf{M}_\ell)_{\ell\in L}$ be a collection of non-random matrices satisfying (2.1), (2.2), and (2.4). Then the finite-dimensional marginal distributions of*

$$\Big(\operatorname{Tr}(P(\mathbf{A})\mathbf{M}_\ell) - \mathbb{E}\operatorname{Tr}(P(\mathbf{A})\mathbf{M}_\ell)\Big)_{P\in\mathbb{C}\langle x_k, x_k^*, k\in K\rangle, \ell\in L}$$

converge to those of a complex centered Gaussian vector $\big(\mathcal{H}_{P,\ell}\big)_{P\in\mathbb{C}\langle x_k, x_k^*, k\in K\rangle, \ell\in L}$ such that, for all $P, Q \in \mathbb{C}\langle x_k, x_k^*, k \in K\rangle$ and $\ell, \ell' \in L$,

$$\mathbb{E}\,\mathcal{H}_{P,\ell}\mathcal{H}_{Q,\ell'} = \alpha_\ell\alpha_{\ell'}\tau_2(P(a), Q(a)) + \eta_{\ell\ell'}\Big(\tau_1(P(a)Q(a)) - \tau_1(P(a))\tau_1(Q(a))\Big),$$

$$\begin{aligned}\mathbb{E}\,\mathcal{H}_{P,\ell}\overline{\mathcal{H}_{Q,\ell'}} &= \alpha_\ell\overline{\alpha_{\ell'}}\tau_2(P(a), Q(a)^*) \\ &\quad + \beta_{\ell\ell'}\Big(\tau_1(P(a)Q(a)^*) - \tau_1(P(a))\tau_1(Q(a)^*)\Big).\end{aligned}$$

Remark 2.7 The following matrices have been shown to converge in second order $*$-distribution.

- Wishart matrices and matrices of the type $\mathbf{UAV}$ or $\mathbf{UAU}^*$, with $\mathbf{U}, \mathbf{V}$ independent and Haar distributed on $U(N)$ and $\mathbf{A}$ deterministic with a limit spectral distribution [23, 35–37].
- GUE matrices or more generally matrix models where the entries interact via a potential [29].
- Ginibre matrices [43].
- random unitary matrices distributed according to the Haar measure on the unitary group $U(N)$ [26].
- matrices arising from the heat kernel measure on $U(N)$ [33] and on $\mathrm{GL}_N(\mathbb{C})$ [19].

A consequence of Corollary 2.6 is that any non-commutative polynomial in independent random matrices taken from the list above has asymptotically Gaussian entries that are independent modulo a possible symmetry.

2.3 Left and Right Unitary Invariant Matrices

Here is another corollary on random matrices invariant by left and right unitary multiplication.

Corollary 2.8 *Let $\mathbf{A} = (\mathbf{A}_k)_{k\in K}$ be a collection of $N \times N$ random matrices such that*

(a′) $\mathbf{A}$ *is invariant, in law, by left and right multiplication by unitary matrices: for any non random unitary matrix* $\mathbf{U}$, $\mathbf{A} \overset{law}{=} \left(\mathbf{U}\mathbf{A}_k\right)_{k\in K} \overset{law}{=} \left(\mathbf{A}_k\mathbf{U}\right)_{k\in K}$;
(b′) *for each* k *and each* p, q, $\frac{1}{N}\operatorname{Tr}|\mathbf{A}_k|^{2p}$ *is bounded in* L^q *independently of* N;
(c′) *for each* k, k', *the sequence* $\frac{1}{N}\operatorname{Tr}\mathbf{A}_k\mathbf{A}_{k'}^*$ *converges in* L^2 *to some non random limits denoted* $\tau(k,\overline{k'})$.

Let $(\mathbf{M}_\ell)_{\ell\in L}$ *be a collection of non-random matrices satisfying* (2.1), (2.2), *and* (2.4). *Then the finite-dimensional marginal distributions of* $(\operatorname{Tr}(\mathbf{A}_k\mathbf{M}_\ell))_{k\in K,\ell\in L}$ *converge to those of a complex centered Gaussian vector* $(\mathcal{G}_{k,\ell})_{k\in K,\ell\in L}$ *with covariances*

$$\mathbb{E}\,\mathcal{G}_{k,\ell}\mathcal{G}_{k',\ell'} = 0 \quad \text{and} \quad \mathbb{E}\,\mathcal{G}_{k,\ell}\overline{\mathcal{G}_{k',\ell'}} = \beta_{\ell,\ell'}\tau(k,\overline{k'}).$$

The proof of this corollary is postponed to Section 3.2: we show that the hypotheses of the corollary imply Hypotheses 1 and 2.

2.4 Permutation Matrix Entries Under Randomized Basis

Let S be a uniform random $N\times N$ permutation matrix. For T_d the number of d-cycles of the underlying permutation, the distribution of $(T_d)_{d\geq 1}$ converges as $N\to\infty$ to a Poisson process $(\mathcal{Z}_d)_{d\geq 1}$ on the set of positive integers with intensity $1/d$ (see [3]). It implies that each trace $\operatorname{Tr}(S^k)$ $(k\geq 1)$ converges in distribution to $\sum_{d|k} d\mathcal{Z}_d$. Thanks to Theorem 2.3 and Remark 2.2, we deduce directly the following result about the matrix entries of a uniform permutation matrix S conjugated by a uniform unitary matrix.

Corollary 2.9 *Let* S *be an* $N\times N$ *random permutation matrix which is uniformly distributed,* U *an* $N\times N$ *random unitary matrix that is Haar distributed, and* $(\mathbf{M}_\ell)_{\ell\in L}$ *a collection of non-random matrices satisfying* (2.1), (2.2), *and* (2.4). *Then the finite-dimensional marginal distributions of* $(\operatorname{Tr}(US^kU^*\mathbf{M}_\ell))_{k\geq 1,\ell\in L}$ *converge to those of* $(\mathcal{G}_{k,\ell} + \alpha_\ell\sum_{d|k} d\mathcal{Z}_d)_{k\geq 1,\ell\in L}$, *where* $(\mathcal{G}_{k,\ell})_{k\geq 1,\ell\in L}$ *is a complex centered Gaussian vector with covariances*

$$\mathbb{E}\,\mathcal{G}_{k,\ell}\mathcal{G}_{k',\ell'} = 0 \quad \text{and} \quad \mathbb{E}\,\mathcal{G}_{k,\ell}\overline{\mathcal{G}_{k',\ell'}} = \mathbb{1}_{k=k'}\beta_{\ell,\ell'},$$

and $(\mathcal{Z}_d)_{d\geq 1}$ *is a Poisson process on the set of positive integers with intensity* $1/d$ *which is independent from* $(\mathcal{G}_{k,\ell})_{k\in\mathbb{N},\ell\in L}$.

This is to be compared with the results of [52], where the entries of the matrix S conjugated by a uniform random orthogonal matrix are studied.

2.5 Low Rank Perturbation for Gaussian Elliptic Matrices

Matrices from the *Gaussian elliptic ensemble*, first introduced in [50], can be defined as follows.

Definition 2.10 A *Gaussian elliptic matrix of parameter* $\rho\in[-1,1]$ is a random matrix $\mathbf{Y} = [y_{ij}]_{i,j=1}^N$ such that the following hold:

(i) $\{(y_{ij}, y_{ji}), 1\leq i<j\leq N\}\cup\{y_{ii}, 1\leq i\leq N\}$ is a family of independent random vectors;

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(ii) $\{(y_{ij}, y_{ji}), 1 \le i < j \le N\}$ are i.i.d. Gaussian, centered, such that

$$\mathbb{E}\, y_{ij}^2 = \mathbb{E}\, y_{ji}^2 = \mathbb{E}\, y_{ij}\overline{y_{ji}} = 0, \quad \mathbb{E}\,|y_{ij}|^2 = \mathbb{E}\,|y_{ji}|^2 = 1, \quad \text{and} \quad \mathbb{E}\, y_{ij} y_{ji} = \rho;$$

(iii) $\{y_{ii}, 1 \le i \le N\}$ are i.i.d. Gaussian, centered, such that $\mathbb{E}\, y_{ii}^2 = \rho$ and $\mathbb{E}\,|y_{ii}|^2 = 1$.

Remark 2.11 Gaussian elliptic matrices can be seen as an interpolation between *GUE matrices* and *Ginibre matrices*. Indeed, a Gaussian elliptic matrix $\mathbf{Y}$ of parameter ρ can be realized as

$$\mathbf{Y} = \sqrt{\frac{1+\rho}{2}}\mathbf{H}_1 + \mathrm{i}\sqrt{\frac{1-\rho}{2}}\mathbf{H}_2,$$

where $\mathbf{H}_1$ and $\mathbf{H}_2$ are two independent GUE matrices from the *GUE*. Hence GUE matrices (resp. Ginibre matrices) are Gaussian elliptic matrices of parameter 1 (resp. 0).

One can also define more general *elliptic random matrices* (see [38, 39, 46, 47] for more details). In our case, it is easy to see (using Remark 2.11) that the Gaussian elliptic ensemble is invariant in distribution by unitary conjugation, which allows us to use our Theorem 2.3 for this model. In this section, we are interested in the outliers in the spectrum of these matrices. It is known [50] that when you renormalize the matrix $\mathbf{Y}$ by $\sqrt{N}$, its limiting eigenvalue distribution is the uniform measure μ_ρ on the ellipse

$$\mathcal{E}_\rho := \left\{ z \in \mathbb{C}\,;\, \frac{(\operatorname{Re} z)^2}{(1+\rho)^2} + \frac{(\operatorname{Im} z)^2}{(1-\rho)^2} \le 1 \right\}. \tag{2.6}$$

Also, we know that adding a finite rank matrix $\mathbf{P}$ to such a matrix $\mathbf{Y}$ barely alters its spectrum from the global point of view [39, Theorem 1.8], but may give rise to outliers. The generic location of the outliers has already been studied, but the authors did not consider the fluctuations [46].

For all $N \ge 1$, let $\mathbf{X}_N := \frac{1}{\sqrt{N}}\mathbf{Y}_N$, where $\mathbf{Y}_N$ is an $N \times N$ Gaussian elliptic matrix of parameter ρ and let $\mathbf{P}_N$ be a $N \times N$ random matrix independent from $\mathbf{X}_N$ whose rank is bounded by an integer r (independent of N). We consider the additive perturbation $\widetilde{\mathbf{X}}_N := \mathbf{X}_N + \mathbf{P}_N$. Since, for any unitary matrix $\mathbf{U}$ that is independent of $\mathbf{X}_N$ we have $\mathbf{X}_N \overset{(\mathrm{d})}{=} \mathbf{U}\mathbf{X}_N\mathbf{U}^*$, we can assume that $\mathbf{P}_N$ has the block structure $\mathbf{P}_N = \left(\begin{smallmatrix} \mathbf{P} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $\mathbf{P}$ is a $2r \times 2r$ matrix (indeed, any complex matrix is unitarily similar to an upper triangular matrix, and since the rank of $\mathbf{P}_N$ is lower than r, we have $\dim(\operatorname{Im}\mathbf{P}_N + (\operatorname{Ker}\mathbf{P}_N)^\perp) \le 2r$).

Theorem 2.12 (Outliers for finite rank perturbations of a Gaussian elliptic matrix) *Let $\varepsilon > 0$. Suppose that $\mathbf{P}_N$ does not have any eigenvalue λ such that*

$$|\lambda| > 1 \quad \textit{and} \quad 1 + |\rho| + \varepsilon < \left|\lambda + \frac{\rho}{\lambda}\right| < 1 + |\rho| + 3\varepsilon, \tag{2.7}$$

and has exactly $j \le r$ eigenvalues $\lambda_1(\mathbf{P}_N), \dots, \lambda_j(\mathbf{P}_N)$ (counted with multiplicity) such that, for each $i = 1, \dots, j$,

$$|\lambda_i(\mathbf{P}_N)| > 1 \quad \textit{and} \quad \left|\lambda_i(\mathbf{P}_N) + \frac{\rho}{\lambda_i(\mathbf{P}_N)}\right| > 1 + |\rho| + 3\varepsilon. \tag{2.8}$$

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Then, with probability tending to one, $\widetilde{\mathbf{X}}_N := \mathbf{X}_N + \mathbf{P}_N$ possesses exactly j eigenvalues $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_j$ in $\{z \in \mathbb{C}\,;\, |z| > 1 + |\rho| + 2\varepsilon\}$ and after a proper labeling

$$\widetilde{\lambda}_i = \lambda_i(\mathbf{P}_N) + \frac{\rho}{\lambda_i(\mathbf{P}_N)} + o(1), \tag{2.9}$$

for each $1 \le i \le j$.

Remark 2.13 In [46], the authors proved this result for *real elliptic random matrices* and provided a more precise statement. Indeed, in our conditions (2.7) and (2.8) they replaced the annulus $\{z \in \mathbb{C}\,;\, 1+|\rho|+\varepsilon < |z| < 1+|\rho|+3\varepsilon\}$ (resp. $\{z \in \mathbb{C}, |z| > 1+|\rho|+3\varepsilon\}$) with $\mathcal{E}_{\rho,3\varepsilon}\backslash\mathcal{E}_{\rho,\varepsilon}$ (resp. $\mathcal{E}^c_{\rho,3\varepsilon}$), where $\mathcal{E}_{\rho,\varepsilon}$ is an ε-neighborhood of the ellipse $\mathcal{E}_\rho$ (see (2.6)). Our proof relies on the identity $\operatorname{Tr}(z - \mathbf{X})^{-1} = \sum_{k\ge 0} z^{-k-1} \operatorname{Tr} \mathbf{X}^k$ which is true only when $|z|$ is larger than the spectral radius of X. This is why (2.7) and (2.8) are circular, rather than elliptic, conditions.

To study the fluctuations of the outliers $\widetilde{\lambda}_i$ around their generic locations as given by (2.9), we need to specify the shape of the matrix $\mathbf{P}$ as it is done in [12]. Indeed, since $\mathbf{P}$ is not Hermitian, we need to introduce its Jordan Canonical Form (JCF,) that which is supposed to be independent of N, except for its kernel part. We know that in a proper basis, $\mathbf{P}$ is a direct sum of *Jordan blocks*, *i.e.*, blocks of the form

$$\mathbf{R}_p(\theta) = \begin{pmatrix} \theta & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \theta \end{pmatrix} \in \mathbb{C}^{p\times p}, \quad \theta \in \mathbb{C}, p \ge 1.$$

Let us denote by $\theta_1, \dots, \theta_q$ the distinct eigenvalues of $\mathbf{P}$ satisfying condition (2.8). For convenience, henceforth we shall write $\widehat{\theta}_i := \theta_i + \frac{\rho}{\theta_i}$. We introduce a positive integer α_i, some positive integers $p_{i,1} > \cdots > p_{i,\alpha_i}$ corresponding to the distinct sizes of the blocks relative to the eigenvalue θ_i, and $\beta_{i,1}, \dots, \beta_{i,\alpha_i}$ such that for all j, $\mathbf{R}_{p_{i,j}}(\theta_i)$ appears $\beta_{i,j}$ times, so that, for a certain $2r \times 2r$ invertible matrix $\mathbf{Q}$, we have

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{P}\mathbf{Q} = \left(\bigoplus_{i=1}^{q} \bigoplus_{j=1}^{\alpha_i} \underbrace{\begin{pmatrix} \mathbf{R}_{p_{i,j}}(\theta_i) & & \\ & \ddots & \\ & & \mathbf{R}_{p_{i,j}}(\theta_i) \end{pmatrix}}_{\beta_{i,j} \text{ blocks}} \right) \oplus \widehat{\mathbf{P}}, \tag{2.10}$$

where $\oplus$ is defined for square block matrices by $\mathbf{M} \oplus \mathbf{N} := \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{pmatrix}$ and $\widehat{\mathbf{P}}$ is a matrix whose eigenvalues θ are such that $|\theta| < 1$ or $|\theta + \rho\theta^{-1}| < 1 + \rho + \varepsilon$.

The asymptotic orders of the fluctuations of the eigenvalues of $\mathbf{X}_N + \mathbf{P}_N$ depend on the sizes $p_{i,j}$ of the blocks. We know by Theorem 2.12 that there are $\sum_{j=1}^{\alpha_i} p_{ij} \times \beta_{i,j}$ eigenvalues $\widetilde{\lambda}$ of $\mathbf{X}_N + \mathbf{P}_N$ that tend to $\widehat{\theta}_i = \theta_i + \rho\theta_i^{-1}$; we shall write them with a $\widehat{\theta}_i$ on the top left corner, as follows

$${}^{\widehat{\theta}_i}\widetilde{\lambda}.$$

Theorem 2.18 will state that, for each block with size $p_{i,j}$ corresponding to θ_i in the JCF of $\mathbf{P}$, there are $p_{i,j}$ eigenvalues which we write as

$$ {}^{\widehat{\theta}_i}_{p_{i,j}}\widetilde{\lambda}, $$

whose convergence rate will be $N^{-1/(2p_{i,j})}$. As there are $\beta_{i,j}$ blocks of size $p_{i,j}$, there are actually $p_{i,j} \times \beta_{i,j}$ eigenvalues tending to $\widehat{\theta}_i$ with convergence rate $N^{-1/(2p_{i,j})}$ (we shall write them ${}^{\widehat{\theta}_i}_{p_{i,j}}\widetilde{\lambda}_{s,t}$ with $s \in \{1,\dots,p_{i,j}\}$ and $t \in \{1,\dots,\beta_{i,j}\}$). It would be convenient to denote by $\Lambda_{i,j}$ the vector with size $p_{i,j} \times \beta_{i,j}$ defined by

$$ \Lambda_{i,j} := \Big(N^{1/(2p_{i,j})} \cdot \Big({}^{\widehat{\theta}_i}_{p_{i,j}}\widetilde{\lambda}_{s,t} - \widehat{\theta}_i \Big) \Big)_{\substack{1\le s\le p_{i,j} \\ 1\le t\le \beta_{i,j}}}. \tag{2.11} $$

As in [12], we now define the family of random matrices that we shall use to characterize the limit distribution of $\Lambda_{i,j}$. For each $i = 1,\dots,q$, let $I(\theta_i)$ (resp. $J(\theta_i)$) denote the set, with cardinality $\sum_{j=1}^{\alpha_i} \beta_{i,j}$, of indices in $\{1,\dots,2r\}$ corresponding to the first (resp. last) columns of the blocks $\mathbf{R}_{p_{i,j}}(\theta_i)$ $(1 \le j \le \alpha_i)$ in (2.10).

Remark 2.14 Note that the columns of $\mathbf{Q}$ (resp. of $(\mathbf{Q}^{-1})^*$) whose index belongs to $I(\theta_i)$ (resp. $J(\theta_i)$) are eigenvectors of $\mathbf{P}$ (resp. of $\mathbf{P}^*$) associated with θ_i (resp. $\overline{\theta_i}$). See [12, Remark 2.8].

Now let

$$ (m^{\theta_i}_{k,\ell})_{\substack{(k,\ell)\in J(\theta_i)\times I(\theta_i) \\ 1\le i\le q}} \tag{2.12} $$

be the random centered complex Gaussian vector with covariances

$$ \mathbb{E}\Big[m^{\theta_i}_{k,\ell} m^{\theta_{i'}}_{k',\ell'} \Big] = \Big(\frac{1}{\theta_i\theta_{i'} - \rho} - \frac{1}{\theta_i\theta_{i'}} \Big) \delta_{k,\ell'}\delta_{k',\ell}, $$
$$ \mathbb{E}\Big[m^{\theta_i}_{k,\ell} \overline{m^{\theta_{i'}}_{k',\ell'}} \Big] = \Phi_\rho(\widehat{\theta}_i, \widehat{\theta}_{i'}) \mathbf{e}_k \mathbf{Q}^{-1} (\mathbf{Q}^{-1})^* \mathbf{e}_{k'} \cdot \mathbf{e}_{\ell'} \mathbf{Q}^* \mathbf{Q} \mathbf{e}_\ell, $$

where $\mathbf{e}_1,\dots,\mathbf{e}_{2r}$ are the column vectors of the canonical basis of $\mathbb{C}^{2r}$ and

$$ \Phi_\rho(z,z') = \int \frac{1}{z-w} \frac{1}{\overline{z'} - \overline{w}} \mu_\rho(\mathrm{d}w) - \int \frac{1}{z-w}\mu_\rho(\mathrm{d}w) \int \frac{1}{\overline{z'}-\overline{w}} \mu_\rho(\mathrm{d}w). $$

Remark 2.15 In Section 3.3, the random vector of (2.12) will appear as the limit in the convergence:

$$ \Big(\sqrt{N}\mathbf{e}_k^* \mathbf{Q}^{-1} \Big(\big(\widehat{\theta}_i - \mathbf{X}_N\big)^{-1} - \frac{1}{\theta_i} \Big) \mathbf{Q}\mathbf{e}_\ell \Big)_{\substack{(k,\ell)\in J(\theta_i)\times I(\theta_i) \\ 1\le i\le q}} \xrightarrow{(\mathrm{d})} (m^{\theta_i}_{k,\ell})_{\substack{(k,\ell)\in J(\theta_i)\times I(\theta_i) \\ 1\le i\le q}}. $$

This convergence is a consequence of Theorem 2.3.

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Remark 2.16 When $\rho = 0$, one has

$$\Phi_0(z,z') = \frac{1}{\pi}\int_{|w|\le 1}\frac{1}{z-w}\frac{1}{\overline{z'}-\overline{w}}\mathrm{d}w - \frac{1}{\pi}\int_{|w|\le 1}\frac{1}{z-w}\mathrm{d}w\frac{1}{\pi}\int_{|w|\le 1}\frac{1}{\overline{z'}-\overline{w}}\mathrm{d}w$$
$$= \frac{1}{z\overline{z'}-1} - \frac{1}{z\overline{z'}} = \frac{1}{z\overline{z'}(z\overline{z'}-1)}.$$

We recover the expression of the covariance in the Ginibre case [12]. Also, the expression of Φ_1 corresponds to the covariance in the GUE case [44].

For each i, j, let $K(i,j)$ (resp. $K(i,j)^-$) be the set, with cardinality $\beta_{i,j}$ (resp. $\sum_{j'=1}^{j-1}\beta_{i,j'}$), of indices in $J(\theta_i)$ corresponding to a block of the type $\mathbf{R}_{p_{i,j}}(\theta_i)$ (resp. to a block of the type $\mathbf{R}_{p_{i,j'}}(\theta_i)$ for $j' < j$). In the same way, let $L(i,j)$ (resp. $L(i,j)^-$) be the set, with the same cardinality as $K(i,j)$ (resp. as $K(i,j)^-$), of indices in $I(\theta_i)$ corresponding to a block of the type $\mathbf{R}_{p_{i,j}}(\theta_i)$ (resp. to a block of the type $\mathbf{R}_{p_{i,j'}}(\theta_i)$ for $j' < j$). Note that $K(i,j)^-$ and $L(i,j)^-$ are empty if $j = 1$. Let us define the random matrices

$$\mathrm{M}_j^{\theta_i,\mathrm{I}} := [m_{k,\ell}^{\theta_i,n}]_{\substack{k\in K(i,j)^-\\ \ell\in L(i,j)^-}} \qquad \mathrm{M}_j^{\theta_i,\mathrm{II}} := [m_{k,\ell}^{\theta_i}]_{\substack{k\in K(i,j)^-\\ \ell\in L(i,j)}}$$
$$\mathrm{M}_{j,n}^{\theta_i,\mathrm{III}} := [m_{k,\ell}^{\theta_i}]_{\substack{k\in K(i,j)\\ \ell\in L(i,j)^-}} \qquad \mathrm{M}_j^{\theta_i,\mathrm{IV}} := [m_{k,\ell}^{\theta_i,n}]_{\substack{k\in K(i,j)\\ \ell\in L(i,j)}}$$

and then let us define the matrix $\mathbf{M}_j^{\theta_i}$ as

$$\mathbf{M}_j^{\theta_i} := \theta_i\big(\mathrm{M}_j^{\theta_i,\mathrm{IV}} - \mathrm{M}_j^{\theta_i,\mathrm{III}}(\mathrm{M}_j^{\theta_i,\mathrm{I}})\,\mathrm{M}_j^{\theta_i,\mathrm{II}}\big)$$

Remark 2.17 It follows from the fact that the matrix $\mathbf{Q}$ is invertible that $\mathrm{M}_j^{\theta_i,\mathrm{I}}$ is a.s. invertible and that so is $\mathbf{M}_j^{\theta_i}$.

Now we can state the result about the fluctuations of the outliers.

Theorem 2.18 (i) *As N goes to infinity, the random vector $(\Lambda_{i,j})_{1\le i\le q, 1\le j\le\alpha_i}$ defined at (2.11) converges in distribution to a random vector*

$$(\Lambda_{i,j}^\infty)_{\substack{1\le i\le q\\ 1\le j\le\alpha_i}}$$

with joint distribution defined by the fact that, for each $1 \le i \le q$ and $1 \le j \le \alpha_i$, $\Lambda_{i,j}^\infty$ is the collection of the $p_{i,j}$-th roots of the eigenvalues of the random matrix $\mathbf{M}_j^{\theta_i}$.

(ii) *The distributions of the random matrices $\mathbf{M}_j^{\theta_i}$ are absolutely continuous with respect to Lebesgue measure and the random vector $(\Lambda_{i,j}^\infty)_{1\le i\le q,1\le j\le\alpha_i}$ has no deterministic coordinate.*

Remark 2.19 Each non-zero complex number has exactly $p_{i,j}$ $p_{i,j}$-th roots, drawing a regular $p_{i,j}$-sided polygon. Moreover, by the second part of the theorem, the spectra

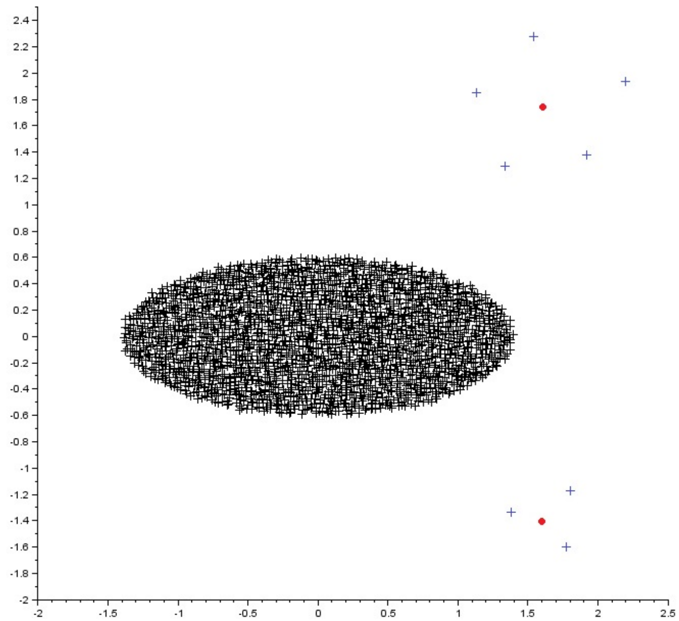

Figure 1: Spectrum of a Gaussian elliptic matrix of size $N = 2500$ with perturbation matrix $\mathbf{P} = \operatorname{diag}(\mathbf{R}_5(1.5 + 2.625\,\mathrm{i}), \mathbf{R}_3(1.5 - 1.5\,\mathrm{i}), 0, \ldots, 0)$. We see the blue crosses "+" (outliers) forming respectively a regular pentagon and an equilateral triangle around the red dots "•" (their limit). We also see a significant difference between the two rates of convergence, $N^{-1/10}$ and $N^{-1/6}$.

of $\mathbf{M}_j^{\theta_i}$ almost surely do not contain 0, so each $\Lambda_{i,j}^{\infty}$ is actually a complex random vector with $p_{i,j} \times \beta_{i,j}$ coordinates, which draw $\beta_{i,j}$ regular $p_{i,j}$-sided polygons.

Remark 2.20 We notice that in the particular case where the matrix $\mathbf{Q}$ is unitary, the covariance of the Gaussian variables $(m_{k,\ell}^{\theta_i})_{(k,\ell)\in J(\theta_i)\times I(\theta_i, 1\le i\le q}$ can be rewritten

$$\mathbb{E}\left[m_{k,\ell}^{\theta_i} m_{k',\ell'}^{\theta_{i'}}\right] = \left(\frac{1}{\theta_i\theta_{i'} - \rho} - \frac{1}{\theta_i\theta_{i'}}\right)\delta_{k,\ell'}\delta_{k',\ell},$$

$$\mathbb{E}\left[m_{k,\ell}^{\theta_i} \overline{m_{k',\ell'}^{\theta_{i'}}}\right] = \Phi(\widehat{\theta}_i, \widehat{\theta}_{i'})\delta_{k,k'}\delta_{\ell,\ell'},$$

which means that for any i, i' such that $i \neq i'$, the famliy $(m_{k,\ell}^{\theta_i})_{(k,\ell)\in J(\theta_i)\times I(\theta_i)}$ is independent of $(m_{k,\ell}^{\theta_{i'}})_{(k,\ell)\in J(\theta_{i'})\times I(\theta_{i'})}$. Indeed, since the Jordan blocks associated

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with θ_i are distinct from those associated with $\theta_{i'}$, the sets $I(\theta_i)$ and $J(\theta_i)$ do not share any common index with $I(\theta_{i'})$ and $J(\theta_{i'})$. We can deduce that in this particular case, all the fluctuations around θ_i are independent from those around $\theta_{i'}$ (see [12, §2.3.1.] for more details).

However, in the general case, there is no particular reason to have independence between the fluctuations around two spikes at the macroscopic distance. To illustrate this phenomenon, we can take the same particular example from [12, Example 2.17] since a Ginibre matrix is also a Gaussian elliptic matrix. In this example, the authors took a matrix $\mathbf{P}$ of the form $\mathbf{P} = \mathbf{Q}\left(\begin{smallmatrix} \theta & 0 \\ 0 & \theta' \end{smallmatrix}\right)\mathbf{Q}^{-1}, \mathbf{Q} = \left(\begin{smallmatrix} 1 & \kappa \\ \kappa & 1 \end{smallmatrix}\right)$, $\kappa \neq \pm 1$, and they empirically confirmed that, in the case $\kappa \neq 0$, the fluctuations of the outliers around θ are correlated with these around θ'.

3 Proofs of Theorem 2.1 and Theorem 2.3

3.1 Preliminary Result

Let $(\mathbf{B}_k)_{k\in K}$ be a collection of (implicitly depending on N) $N \times N$ random matrices such that

(1) for each $k \in K$, almost surely, $\operatorname{Tr} \mathbf{B}_k = 0$;
(2) for each $k \in K$, and each $p, q \geq 1$, $\frac{1}{N} \operatorname{Tr} |\mathbf{B}_k|^{2p}$ is bounded in L^q independently of N; and
(3) for each $k, k' \in K$, we have the following convergences to nonrandom variables in L^2
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathbf{B}_k \mathbf{B}_{k'} = \tau(k, k') \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathbf{B}_k \mathbf{B}_{k'}^* = \tau(k, \overline{k'}).$$
Let also $(\mathbf{M}_\ell)_{\ell\in L}$ be a collection of non-random matrices such that
(4) for each $\ell, \ell' \in L$, we have the following convergences
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathbf{M}_\ell \mathbf{M}_{\ell'} - \frac{1}{N} \operatorname{Tr} \mathbf{M}_\ell \cdot \frac{1}{N} \operatorname{Tr} \mathbf{M}_{\ell'} = \eta_{\ell\ell'},$$
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathbf{M}_\ell \mathbf{M}_{\ell'}^* - \frac{1}{N} \operatorname{Tr} \mathbf{M}_\ell \cdot \frac{1}{N} \operatorname{Tr} \mathbf{M}_{\ell'}^* = \beta_{\ell\ell'}.$$

Finally, let $\mathbf{U} = \mathbf{U}^{(N)}$ be an $N \times N$ Haar-distributed unitary random matrix independent of $(\mathbf{B}_k)_{k\in K}$.

Proposition 3.1 *Let us fix $p \geq 1$, $(k_1, \dots, k_p) \in K^p$ and $(\ell_1, \dots, \ell_p) \in L^p$. If* (1), (2), (3), and (4) *hold, then the centered vector*

$$(\operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i}))_{1\leq i\leq p} \tag{3.1}$$

converges in distribution as $N \to \infty$ to a complex centered Gaussian vector $(\mathcal{G}_i)_{1\leq i\leq p}$ such that, for all i, i', $\mathbb{E}\, \mathcal{G}_i \mathcal{G}_{i'} = \eta_{\ell_i \ell_{i'}} \tau(k_i, k_{i'})$ and $\mathbb{E}\, \mathcal{G}_i \overline{\mathcal{G}_{i'}} = \beta_{\ell_i \ell_{i'}} \tau(k_i, \overline{k_{i'}})$. Besides, for any sequence (Y_N) of bounded random variables such that Y_N is independent of $\mathbf{U}^{(N)}$, $\mathbb{E}\, Y_N$ has a limit L_Y, and any polynomial f in p complex variables and their conjugates, we have
$$\lim_{N\to\infty} \mathbb{E}[Y_N f(\operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i}), 1 \leq i \leq p)] = L_Y\, \mathbb{E}[f(\mathcal{G}_i, 1 \leq i \leq p)].$$

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Proof First, we can suppose that $\mathbf{B}_k$ and $\mathbf{M}_\ell$ are all Hermitian (which makes the entries of the vector of (3.1) real), up to changing

$$(\mathbf{B}_k)_{k\in K} \longrightarrow (\mathbf{B}_{(k,1)} := \frac{1}{2}(\mathbf{B}_k + \mathbf{B}_k^*), \mathbf{B}_{(k,2)} := \frac{1}{2\mathrm{i}}(\mathbf{B}_k - \mathbf{B}_k^*))_{(k,\varepsilon)\in K\times\{1,2\}},$$
$$(\mathbf{M}_\ell)_{\ell\in L} \longrightarrow (\mathbf{M}_{(\ell,1)} := \frac{1}{2}(\mathbf{M}_\ell + \mathbf{M}_\ell^*), bM_{(\ell,2)} := \frac{1}{2\mathrm{i}}(\mathbf{M}_\ell - \mathbf{M}_\ell^*))_{(\ell,\varepsilon)\in L\times\{1,2\}}.$$

Second, as all $\mathbf{B}_k$ have null trace up to changing $\mathbf{M}_\ell \to \mathbf{M}_\ell - \frac{1}{N}\operatorname{Tr}\mathbf{M}_\ell$, one can suppose that all $\mathbf{M}_\ell$ have null trace.

To prove the full proposition, it suffices to prove the convergence

$$\lim_{N\to\infty} \mathbb{E}\Big[\, Y_N \prod_{i=1}^{n} \operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i})\Big] = L_Y\, \mathbb{E}\Big[\prod_{i=1}^{n} \mathcal{H}_i\Big].$$

for any $n \geq 1$, $(k_1,\dots,k_n) \in K^n$, $(\ell_1,\dots,\ell_n) \in L^n$ and any sequence (Y_N) of bounded random variables independent from $\mathbf{U}^{(N)}$ such that $\lim_{N\to\infty} \mathbb{E}\, Y_N = L_Y$. Indeed, we can take each k as many times as we want in $(k_1,\dots,k_n)$ (and the same for ℓ), which implies the convergence of the expectation of any polynomials as wanted and consequently the convergence in distribution of finite-dimensional marginals.

Let $n \geq 1$, and $\mathfrak{S}_n$ be the n-th symmetric group, and let $\mathfrak{S}_{n,2}$ be the subset of permutations in $\mathfrak{S}_n$ with only cycles of length 2. We denote by $\#\sigma$ the number of cycles of $\sigma \in \mathfrak{S}_n$ and by $\operatorname{Fix}(\sigma)$ the number of fixed points of σ. The neutral element of $\mathfrak{S}_n$ is denoted by id_n. For any $\sigma \in \mathfrak{S}_n$, we set

$$\operatorname{Tr}_\sigma(\mathbf{N}_i)_{i=1}^n = \prod_{\substack{(t_1 t_2 \cdots t_m)\\ \text{cycle of } \sigma}} \operatorname{Tr}(\mathbf{N}_{t_1}\mathbf{N}_{t_2}\cdots\mathbf{N}_{t_m})$$

For example, for $\sigma \in \mathfrak{S}_6$, $\sigma := (1,2,3,4,5,6) \mapsto (3,2,4,1,6,5)$

$$\operatorname{Tr}_\sigma(\mathbf{N}_i)_{i=1}^6 = \operatorname{Tr}(\mathbf{N}_1\mathbf{N}_3\mathbf{N}_4)\operatorname{Tr}(\mathbf{N}_2)\operatorname{Tr}(\mathbf{N}_5\mathbf{N}_6).$$

Lemma 3.2 *Let $n \geq 1$, $(k_1,\dots,k_n) \in K^n$, $(\ell_1,\dots,\ell_n) \in L^n$, and let (Y_N) be any sequence of bounded random variables such that $\lim_{N\to\infty} \mathbb{E}\, Y_N = L_Y$. With the above assumptions on $(\mathbf{M}_\ell)_{\ell\in L}$ and $(\mathbf{B}_k)_{k\in K}$, we have, for all γ and σ in $\mathfrak{S}_n$,*

$$\operatorname{Tr}_\gamma(\mathbf{M}_{\ell_i})_{i=1}^n = \mathbb{1}_{\operatorname{Fix}(\gamma)=0} \times O(N^{n/2})$$

and

$$\mathbb{E}[Y_N \operatorname{Tr}_\sigma(\mathbf{B}_{k_i})_{i=1}^n] = \mathbb{1}_{\sigma\in\mathfrak{S}_{n,2}} N^{n/2} L_Y \prod_{\substack{(i,j)\\ \text{cycle of } \sigma}} \tau(k_i,k_j) + o(N^{n/2}).$$

Proof Because $\mathbf{B}_k$ and $\mathbf{M}_\ell$ have null traces, the formulas are true in the presence of fixed points. Thus, we can assume that σ and γ have no fixed point.

The first result comes from Lemma A.1 and from the fact that, for each ℓ, $\operatorname{Tr}\mathbf{M}_\ell^2 = O(N)$.

The second result can be proved in two steps. First, if $\sigma \notin \mathfrak{S}_{n,2}$, the non-commutative Hölder inequality [1, Appendix A.3] and Hypothesis (ii) say that

$$|\mathbb{E}[Y_N \operatorname{Tr}_\sigma\big(\mathbf{B}_{k_{i_j}}\big)_{j=1}^n]| = O(N^{\#\sigma}) = o(N^{n/2}).$$

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If $\sigma \in \mathfrak{S}_{n,2}$ (and $n > 0$ is even), we decompose σ in 2-cycles $\sigma = (i_1\ j_1)\cdots(i_{n/2}\ j_{n/2})$. By the classical Hölder's inequality, the absolute difference between

$$N^{-n/2}\,\mathbb{E}[Y_N \operatorname{Tr}_\sigma(\mathbf{B}_{k_j})_{j=1}^n] = \mathbb{E}\Big[\, Y_N \prod_{t=1}^{n/2} \frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_t}} \mathbf{B}_{k_{j_t}} \Big]$$

and

$$\mathbb{E}\Big[\, Y_N \prod_{t=1}^{n/2-1} \frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_t}} \mathbf{B}_{k_{j_t}} \Big]\, \mathbb{E}\Big[\, \frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_1}} \mathbf{B}_{k_{j_1}} \Big]$$

is less than

$$\mathbb{E}[Y_N^{2n}]^{1/2n} \prod_{t=1}^{n/2-1} \mathbb{E}\Big[\Big(\frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_t}} \mathbf{B}_{k_{j_t}}\Big)^{2n}\Big]^{1/2n} \mathbb{V}\mathrm{ar}\Big(\frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_1}} \mathbf{B}_{k_{j_1}}\Big)$$

and consequently converges to 0, using again the non-commutative Hölder's inequality (see [1, Appendix A.3]) and Hypothesis (ii) to control $\mathbb{E}(\frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_t}} \mathbf{B}_{k_{j_t}})^{2n}$. By a direct induction on $n/2$, it means that the expectation of product

$$N^{-n/2}\,\mathbb{E}\big[\, Y_N \operatorname{Tr}_\sigma(\mathbf{B}_{k_j})_{j=1}^n\big] = \mathbb{E}\Big[\, Y_N \prod_{t=1}^{n/2} \frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_t}} \mathbf{B}_{k_{j_t}} \Big]$$

has the same limit as the product of expectation $\mathbb{E}[Y_N] \prod_{t=1}^{n/2} \mathbb{E}[\frac{1}{N} \operatorname{Tr} \mathbf{B}_{k_{i_t}} \mathbf{B}_{k_{j_t}}]$, and the result follows. ∎

Let $n \geq 1$, $(k_1, \ldots, k_n) \in K^n$, $(\ell_1, \ldots, \ell_n) \in L^n$, and (Y_N) be any sequence of bounded random variables such that $\lim_{N\to\infty} \mathbb{E}\, Y_N = L$. Using [36, Proposition 3.4] (and, first, an integration with respect to the randomness of $\mathbf{U}$, and then a "full expectation"), we have

(3.2)
$$\mathbb{E}\Big[\, Y_N \prod_{i=1}^n \operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i})\Big] = \sum_{\sigma,\gamma\in\mathfrak{S}_n} \operatorname{Wg}(\sigma\gamma^{-1})\, \mathbb{E}[Y_N \operatorname{Tr}_\sigma(\mathbf{B}_{k_i})_{i=1}^n] \operatorname{Tr}_\gamma(\mathbf{M}_{\ell_i})_{i=1}^n,$$

where Wg is the Weingarten function. We know from [24, Corollary 2.7] and [40, Proposition 23.11] that, for any $\tau \in \mathfrak{S}_n$,

$$\operatorname{Wg}(\tau) = O(N^{\#\tau-2n}) \quad \text{and} \quad \operatorname{Wg}(1_n) = N^{-n} + O(N^{-n-2}).$$

By Lemma 3.2 it implies that for $\sigma, \gamma \in \mathfrak{S}_n$,

$$\begin{aligned}\operatorname{Wg}(\sigma\gamma^{-1})\, &\mathbb{E}[Y_N \operatorname{Tr}_\sigma(\mathbf{B}_{k_i})_{i=1}^n] \operatorname{Tr}_\gamma(\mathbf{M}_{\ell_i})_{i=1}^n \\ &= \mathbb{1}_{\sigma\in\mathfrak{S}_{n,2}} \mathbb{1}_{\operatorname{Fix}(\gamma)=0}\, O(N^{(\#(\sigma\gamma^{-1})-n)}) = \mathbb{1}_{\gamma=\sigma\in\mathfrak{S}_{n,2}} O(1),\end{aligned}$$

and more precisely, using the exact asymptotic for $\gamma = \sigma \in \mathfrak{S}_{n,2}$, that

$$\begin{aligned}\operatorname{Wg}(\sigma\gamma^{-1})\, &\mathbb{E}[Y_N \operatorname{Tr}_\sigma(\mathbf{B}_{k_i})_{i=1}^n] \operatorname{Tr}_\gamma(\mathbf{M}_{\ell_i})_{i=1}^n \\ &= \mathbb{1}_{\gamma=\sigma\in\mathfrak{S}_{n,2}} L_Y \prod_{\substack{(i,j)\\ \text{cycle of } \sigma}} \tau(k_i, k_j)\eta_{\ell_i\ell_j} + o(1).\end{aligned}$$

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As a consequence, we can rewrite (3.2) as

$$\mathbb{E}\Big[\, Y_N \prod_{i=1}^{n} \operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i})\Big] = L_Y \sum_{\sigma\in\mathfrak{S}_{n,2}} \prod_{\substack{(i,j)\\ \text{cycle of } \sigma}} \tau(k_i,k_j)\eta_{\ell_i\ell_j} + o(1),$$

which is the convergence needed to prove the proposition, since

$$\mathbb{E}\Big[\prod_{i=1}^{n} \mathcal{G}_i\Big] = \sum_{\sigma\in\mathfrak{S}_{n,2}} \prod_{\substack{(i,j)\\ \text{cycle of } \sigma}} \tau(k_i,k_j)\eta_{\ell_i\ell_j}. \qquad \blacksquare$$

Proof of Theorem 2.3 First, note that for each $k\in K$, $\mathbb{E}\,\mathbf{A}_k = \frac{1}{N}\mathbb{E}[\operatorname{Tr}(\mathbf{A}_k)]I$. Hence for $\mathbf{B}_k := \mathbf{A}_k - \frac{1}{N}\operatorname{Tr}(\mathbf{A}_k)I$ and $T_k := \frac{1}{N}\operatorname{Tr}(\mathbf{A}_k) - \mathbb{E}\,\frac{1}{N}\operatorname{Tr}(\mathbf{A}_k)$, one can write

$$(\mathbf{A}_k - \mathbb{E}\,\mathbf{A}_k)_{b\in K} = (\mathbf{B}_k + T_k I)_{b\in K}.$$

Let us now introduce a Haar-distributed unitary matrix $\mathbf{U}$ (implicitly depending on N) independent of the collection $\mathbf{A}$. By unitary invariance, we get

$$(\mathbf{A}_k - \mathbb{E}\,\mathbf{A}_k)_{b\in K} = (\mathbf{B}_k + T_k I)_{b\in K} \overset{\text{law}}{=} (\mathbf{U}\mathbf{B}_k\mathbf{U}^* + T_k I)_{b\in K}.$$

Then by Proposition 3.1, we know that, for any $n\geq 1$, any $k_1,\dots,k_n\in K$, and any $\ell_1,\dots,\ell_n\in L$, the random vector $(\operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i}))_{1\leq i\leq n}$ converges in distribution to a complex centered Gaussian vector $(\mathcal{H}_i)_{1\leq i\leq n}$ such that, for all i,i',

$$\begin{aligned}
\mathbb{E}\,\mathcal{H}_i\mathcal{H}_{i'} &= \Big(\tau(k_i,k_{i'}) - \tau(k_i)\tau(k_{i'})\Big)\,(\eta_{\ell_i\ell_{i'}} - \alpha_{\ell_i}\alpha_{\ell_{i'}}),\\
\mathbb{E}\,\mathcal{H}_i\overline{\mathcal{H}_{i'}} &= \Big(\tau(k_i,\overline{k_{i'}}) - \tau(k_i)\overline{\tau(k_{i'})}\Big)\,(\eta_{\ell_i\ell_{i'}} - \alpha_{\ell_i}\overline{\alpha_{\ell_{i'}}}).
\end{aligned}$$

Proposition 3.1 also says that $(\operatorname{Tr}(\mathbf{U}\mathbf{B}_{k_i}\mathbf{U}^*\mathbf{M}_{\ell_i}))_{1\leq i\leq n}$ is asymptotically independent of $(T_{k_i}\operatorname{Tr}(\mathbf{M}_{\ell_i}))_{1\leq i\leq n}$, which converges in distribution to $(\alpha_{\ell_i}\mathcal{T}_{k_i})_{1\leq i\leq n}$, by Hypothesis (iv). As it is clear from the covariance of $(\mathcal{G}_i)_{1\leq i\leq n}$ that for $(\mathcal{H}_i)_{1\leq i\leq n}$ independent from $(\alpha_{\ell_i}\mathcal{T}_{k_i}\alpha_{\ell_i})_{1\leq i\leq n}$, we have $(\mathcal{G}_i)_{1\leq i\leq n} \overset{\text{law}}{=} (\mathcal{H}_i)_{1\leq i\leq n} + (\alpha_{\ell_i}\mathcal{T}_{k_i})_{1\leq i\leq n}$; the theorem is proved. $\blacksquare$

Proof of Theorem 2.1 It is a direct application of Proposition 3.1 that if $\operatorname{Tr}\mathbf{M} = 0$, then $\operatorname{Tr}(\mathbf{A}\mathbf{M}) = \operatorname{Tr}\Big[\Big(\mathbf{A} - \frac{1}{N}(\operatorname{Tr}\mathbf{A})\mathbf{I}\Big)\mathbf{M}\Big]$, so that one can assume that $\operatorname{Tr}\mathbf{A} = 0$. $\blacksquare$

3.2 Proof of Corollary 2.8

We just need to show that the hypotheses of the corollary imply Hypotheses 1 and 2. The proof of Hypothesis 1 comes down to the following computations, where we introduce a Haar-distributed unitary matrix $\mathbf{U}$ independent of $(\mathbf{A}_k)_{k\in K}$ and use [13, (33)]. We have

$$\begin{aligned}
\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}\mathbf{A}_k\Big|^2 &= \frac{1}{N^2}\,\mathbb{E}\left[\mathbb{E}_{\mathbf{U}}[\operatorname{Tr}(\mathbf{U}\mathbf{A}_k)\operatorname{Tr}(\mathbf{A}_k^*\mathbf{U}^*)]\right]\\
&= \frac{1}{N^3}(1+o(1))\,\mathbb{E}\left[\operatorname{Tr}(\mathbf{A}_k\mathbf{A}_k^*)\right] = O\Big(\frac{1}{N^2}\Big),
\end{aligned}$$

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and

$$\begin{aligned}
&\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(\mathbf{A}_k\mathbf{A}_{k'})\Big|^2\\
&\quad=\frac{1}{N^2}\,\mathbb{E}\left[\mathbb{E}_{\mathbf{U}}[\operatorname{Tr}(\mathbf{A}_k\mathbf{U}\mathbf{A}_{k'}\mathbf{U})\operatorname{Tr}(\mathbf{A}_{k'}\mathbf{U}^*\mathbf{A}_k\mathbf{U}^*)]\right]\\
&\quad=\frac{1}{N^4}(1+o(1))\,\mathbb{E}\Big(\operatorname{Tr}(\mathbf{A}_k\mathbf{A}_k^*)\operatorname{Tr}(\mathbf{A}_{k'}\mathbf{A}_{k'}^*)+\operatorname{Tr}(\mathbf{A}_k\mathbf{A}_{k'}^*)\operatorname{Tr}(\mathbf{A}_{k'}\mathbf{A}_k^*)\Big)\\
&\quad=O\Big(\frac{1}{N^2}\Big).
\end{aligned}$$

Now in order to show Hypothesis 2, we want to prove that, for any fixed r,

$$\Big(\operatorname{Tr}(\mathbf{A}_{k_1}),\dots,\operatorname{Tr}(\mathbf{A}_{k_r})\Big)_{i=1}^{r}$$

is asymptotically Gaussian. Let $n\geq 1$ and $i_1,j_1,\dots,i_n,j_n\in\{1,\dots,r\}$. Using [36, Proposition 3.4], we have

$$\begin{aligned}
\mathbb{E}\Big[\prod_{\ell=1}^{n}\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}})\operatorname{Tr}(\mathbf{A}^*_{k_{j_\ell}})\Big]&=\mathbb{E}\Big[\mathbb{E}_{\mathbf{U}}\prod_{\ell=1}^{n}\operatorname{Tr}(\mathbf{U}\mathbf{A}_{k_{i_\ell}})\operatorname{Tr}(\mathbf{A}^*_{k_{j_\ell}}\mathbf{U}^*)\Big]\\
&=\frac{1}{N^n}\sum_{\sigma\in\mathfrak{S}_n}\mathbb{E}\Big[\prod_{\ell=1}^{n}\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}}\mathbf{A}^*_{k_{j_{\sigma(\ell)}}})\Big]+o(1).
\end{aligned}$$

Then one can prove that

$$\frac{1}{N^n}\sum_{\sigma\in\mathfrak{S}_n}\mathbb{E}\Big[\prod_{\ell=1}^{n}\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}}\mathbf{A}^*_{k_{j_{\sigma(\ell)}}})\Big]=\frac{1}{N^n}\sum_{\sigma\in\mathfrak{S}_n}\prod_{\ell=1}^{n}\mathbb{E}\Big[\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}}\mathbf{A}^*_{k_{j_{\sigma(\ell)}}})\Big]+o(1). \tag{3.3}$$

Indeed, similarly to the above, we use the classical Hölder inequality to state that the difference between

$$N^{-n}\,\mathbb{E}\Big[\prod_{\ell=1}^{n}\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}}\mathbf{A}^*_{k_{j_{\sigma(\ell)}}})\Big]$$

and

$$N^{-(n-1)}\,\mathbb{E}\Big[\prod_{i=1}^{n-1}\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}}\mathbf{A}^*_{k_{j_{\sigma(\ell)}}})\Big]\,\mathbb{E}\Big[\frac{1}{N}\operatorname{Tr}(\mathbf{A}_{k_{i_n}}\mathbf{A}^*_{k_{j_{\sigma(n)}}})\Big]$$

is lower than

$$\prod_{i=1}^{n-1}\mathbb{E}\Big[\Big(\frac{1}{N}\operatorname{Tr}(\mathbf{A}_{k_{i_\ell}}\mathbf{A}^*_{k_{j_{\sigma(\ell)}}})\Big)^{2(n-1)}\Big]^{\frac{1}{2(n-1)}}\,\mathbb{V}\mathrm{ar}\Big(\frac{1}{N}\operatorname{Tr}(\mathbf{A}_{k_{i_n}}\mathbf{A}^*_{k_{j_{\sigma(n)}}})\Big),$$

which tends to 0 thanks to the non-commutative Hölder inequality and the fact that $\frac{1}{N}\operatorname{Tr}(\mathbf{A}_n\mathbf{A}^*_{\sigma(n)})$ converges in probability to a constant. We conclude the proof of (3.3) with a simple induction. Once we have (3.3), we can conclude using the Wick Formula. ■

3.3 Proofs of Theorem 2.12 and Theorem 2.18.

In this section, we will directly apply [44, Theorem 2.3 and Theorem 2.10] in order to prove both Theorems 2.12 and 2.18. To do so, we only need to prove that the Gaussian elliptic ensemble satisfies the assumptions of [44, Theorem 2.3 and Theorem 2.10]. This is the purpose of the following proposition.

https://doi.org/10.4153/CJM-2017-024-8 Published online by Cambridge University Press

Proposition 3.3 *Let $\mathbf{X}_N := \frac{1}{\sqrt{N}}\mathbf{Y}_N$ where $\mathbf{Y}_N$ is an $N \times N$ Gaussian elliptic matrix of parameter ρ. Then as $N \to \infty$, we have the following.*

(i) $\|\mathbf{X}_N\|_{\mathrm{op}}$ *converges in probability to* $1 + |\rho|$.

(ii) *For any $\delta > 0$, as N goes to infinity, we have the convergence in probability*

$$\sup_{|z|>1+|\rho|+\delta} \max_{1\le i,j\le 2r} \left|\mathbf{e}_i^* \left(z\mathbf{I} - \mathbf{X}_N\right)^{-1} \mathbf{e}_j - \delta_{ij} m(z)\right| \longrightarrow 0,$$

where $m(z) := \int \frac{1}{z-w}\mu_\rho(\mathrm{d}w)$.

(iii) *For any z such that $|z| > 1 + |\rho| + \varepsilon$, we have the convergence in probability*

$$\sqrt{N}\Big(\frac{1}{N}\operatorname{Tr}\left(z - \mathbf{X}_N\right)^{-1} - m(z)\Big) \longrightarrow 0.$$

(iv) *the finite marginals of random process*

$$\Big(\sqrt{N}\Big(\mathbf{e}_i^* \left(z - \mathbf{X}_N\right)^{-1} \mathbf{e}_j - \delta_{ij}\frac{1}{N}\operatorname{Tr}\left(z - \mathbf{X}_N\right)^{-1}\Big)\Big)_{\substack{|z|>1+|\rho|+\varepsilon \\ 1\le i,j\le 2r}}$$

converge to those of the complex centered Gaussian process

$$\left(\mathcal{G}_{i,j,z}\right)_{\substack{|z|>1+|\rho|+\varepsilon \\ 1\le i,j\le 2r}}$$

satisfying

$$\begin{aligned}
&\mathbb{E}[\mathcal{G}_{i,j,z}\mathcal{G}_{i',j',z'}] \\
&\quad = \delta_{ij'}\delta_{i'j}\Big(\int \frac{1}{(z-w)(z'-w)}\mu_\rho(\mathrm{d}w) - \int \frac{1}{z-w}\mu_\rho(\mathrm{d}w)\int \frac{1}{z'-w}\mu_\rho(\mathrm{d}w)\Big), \\
&\mathbb{E}[\mathcal{G}_{i,j,z}\overline{\mathcal{G}_{i',j',z'}}] \\
&\quad = \delta_{ii'}\delta_{jj'}\Big(\int \frac{1}{(z-w)(\overline{z'}-\overline{w})}\mu_\rho(\mathrm{d}w) - \int \frac{1}{z-w}\mu_\rho(\mathrm{d}w)\overline{\int \frac{1}{z'-w}\mu_\rho(\mathrm{d}w)}\Big).
\end{aligned}$$

(v) *for any $p \ge 1$, any $1 \le i, j \le 2r$, and any $|z| > 1 + |\rho| + \varepsilon$, the sequence*

$$\sqrt{N}\Big(\mathbf{e}_i^*(z - \mathbf{X}_N)^{-p}\mathbf{e}_j - \delta_{ij}\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-p}\Big)$$

is tight.

Remark 3.4 One should be careful about the fact that our $m(z)$ is not the same as [46, Lemma 4.3], but the opposite. Moreover, for any $|\theta| > 1$, we still have (see [46, (5.2) and (5.3)]) $m(z) = \frac{1}{\theta} \leftrightarrow z = \theta + \frac{\rho}{\theta}$, so that it is easy to compute $\mathbb{E}[\mathcal{G}_{i,j,z}\mathcal{G}_{i',j',z'}]$ in (iv) for $z = \theta + \frac{\rho}{\theta}$ and $z' = \theta' + \frac{\rho}{\theta'}$. Indeed

$$\begin{aligned}
&\int \frac{1}{(z-w)(z'-w)}\mu_\rho(\mathrm{d}w) - \int \frac{1}{z-w}\mu_\rho(\mathrm{d}w)\int \frac{1}{z'-w}\mu_\rho(\mathrm{d}w) \\
&\qquad\qquad = -\frac{m(z)-m(z')}{z-z'} - m(z)m(z') = \frac{1}{\theta\theta'-\rho} - \frac{1}{\theta\theta'}.
\end{aligned}$$

Also, for any $|z| > 2\sqrt{|\rho|}$, it might be useful to write $m(z) = \sum_{k\ge 0}\rho^k \operatorname{Cat}(k)z^{-2k-1}$, where $\operatorname{Cat}(k) = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

Proof of Proposition 3.3 First, (i) is an adaptation of [46, Theorem 2.2] to the complex case whose proof goes along exactly the same lines (as we work with Gaussian entries, the proof is even easier). It implies that with a probability tending to one for any fixed $|z| > 1 + |\rho|$, one can write $(z - \mathbf{X}_N)^{-1} = \sum_{k\geq 0} z^{-k-1}\mathbf{X}_N^k$. Moreover, if we apply Theorem 2.3 with

$$(\mathbf{M}_\ell) = (\sqrt{N}\mathbf{E}_{ji})_{1\leq i,j\leq 2r}$$

and

$$(\mathbf{A}_k) = \Big((z - \mathbf{X}_N)^{-1} - \frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1}\mathbf{I}\Big)_{|z|>1+|\rho|+\varepsilon},$$

(since $\mathbf{X}_N$ is invariant in distribution by unitary conjugation, so is $(z - \mathbf{X}_N)^{-p}$ for any $p \geq 1$), we easily obtain (iv). The same for (v) by changing the exponent -1 into $-p$. At last, we just need to prove (ii) and (iii).

(iii) First let us write, for any $\eta > 0$,

$$\mathbb{P}\Big(\sqrt{N}\Big|\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1} - m(z)\Big| > \eta\Big) \leq \frac{4N}{\eta^2}\Big(\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1} - \mathbb{E}\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1}\Big|^2 + \Big|\mathbb{E}\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1} - m(z)\Big|^2\Big),$$

which means that we only need to prove that

$$\Big|\mathbb{E}\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1} - m(z)\Big|^2 = o\Big(\frac{1}{N}\Big), \tag{3.4}$$

and

$$\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1} - \mathbb{E}\frac{1}{N}\operatorname{Tr}(z - \mathbf{X}_N)^{-1}\Big|^2 = o\Big(\frac{1}{N}\Big). \tag{3.5}$$

Proof of (3.4). We know from [5, Theorem 1.1] that (3.4) would be true had $\mathbf{X}_N$ been a Gaussian Wigner matrix instead of an elliptic one. Here the idea of the proof is to use the fact that the Stieltjes transform of the semicircular law of variance $\sigma^2 = \rho$ is equal to $m(z)$ outside the ellipse $\mathcal{E}_\rho$ when $\rho > 0$. First we shall suppose that $\rho \geq 0$ up to changing $\mathbf{X}_N$ into $\mathrm{i}\,\mathbf{X}_N$. For any $i \neq j$, we have

$$\mathbb{E}\,x_{ii}^2 = \mathbb{E}\,x_{ij}x_{ji} = \frac{\rho}{N}, \quad \text{and} \quad \mathbb{E}\,x_{ij}^2 = 0. \tag{3.6}$$

One can notice that if $\mathbf{W}_N$ is a real symmetric Gaussian matrix of variance ρ with i.i.d. entries such that for any $i \neq j$, $\mathbb{E}\,w_{ii}^2 = \mathbb{E}\,w_{ij}w_{ji} = \mathbb{E}\,w_{ij}^2 = \rho/N$, then we have by the Wick formula applied to the expansion of the traces,

- $\mathbb{E}\big[\operatorname{Tr}\mathbf{W}_N^k\big] \geq 0$ and $\mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^k\big] \geq 0$,
- $\mathbb{E}\big[\operatorname{Tr}\mathbf{W}_N^k\big] \geq \mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^k\big]$ since there are more non-zero terms for $\mathbf{W}_N$ than for $\mathbf{X}_N$.

Also, we know that, for any z such that $|z| > 1 + \rho + \varepsilon$,

$$\mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{W}_N)^{-1} = \sum_{k\geq 0} z^{-k-1}\,\mathbb{E}\Big[\frac{1}{N}\operatorname{Tr}\mathbf{W}_N^k\Big],$$

$$\mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1} = \sum_{k\geq 0} z^{-k-1}\,\mathbb{E}\Big[\frac{1}{N}\operatorname{Tr}\mathbf{X}_N^k\Big]$$

converge to the same limit $m(z) = \sum_{k\geq 0}\operatorname{Cat}(k)\rho^k z^{-2k-1}$, where $\operatorname{Cat}(k)$ is the k-th Catalan number. Moreover, by the Wick formula again, if $\mathcal{P}_2(2k)$ (resp. $NC_2(2k)$) is the set of pairings (resp. non-crossing pairings) of $\{1,\dots,2k\}$, then

$$\begin{aligned}\mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^{2k}\big] &= \sum_{1\leq i_1,\dots,i_{2k}\leq N}\ \sum_{\pi\in\mathcal{P}_2(2k)}\ \prod_{\{s,t\}\in\pi}\mathbb{E}\big[x_{i_s i_{s+1}}x_{i_t i_{t+1}}\big]\\ &= \sum_{\pi\in\mathcal{P}_2(2k)}\ \sum_{1\leq i_1,\dots,i_{2k}\leq N}\ \prod_{\{s,t\}\in\pi}\mathbb{E}\big[x_{i_s i_{s+1}}x_{i_t i_{t+1}}\big].\end{aligned}$$

Note that using the Dyck path interpretation of $NC_2(2k)$ (see [40]), one can easily see that in the previous sum, the term associated with each $\pi\in NC_2(2k)$ is precisely ρ^k. Hence as the cardinality of $NC_2(2k)$ is $\operatorname{Cat}(k)$ (see [40] again) and each $\mathbb{E}\big[x_{i_s i_{s+1}}x_{i_t i_{t+1}}\big]$ is non negative, we have $\mathbb{E}\big[\frac{1}{N}\operatorname{Tr}\mathbf{X}_N^{2k}\big]\geq\operatorname{Cat}(k)\rho^k$. At last, we know from [5, Theorem 1.1] that, for any z such that $|z|>1+\rho+\varepsilon$, we have

$$\mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{W}_N)^{-1} - m(z) = o\Big(\frac{1}{\sqrt{N}}\Big),$$

so that, to conclude, it suffices to write

$$\begin{aligned}\Big|\mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1} - m(z)\Big| &\leq \sum_{k\geq 0}\Big(\mathbb{E}\,\frac{1}{N}\operatorname{Tr}\mathbf{X}_N^{2k} - \operatorname{Cat}(k)\rho^k\Big)|z|^{-2k-1}\\ &\leq \sum_{k\geq 0}\Big(\mathbb{E}\,\frac{1}{N}\operatorname{Tr}\mathbf{W}_N^{2k} - \operatorname{Cat}(k)\rho^k\Big)|z|^{-2k-1}\\ &= \mathbb{E}\,\frac{1}{N}\operatorname{Tr}(|z|-\mathbf{W}_N)^{-1} - m(|z|) = o\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Proof of (3.5). We apply the same idea, but this time $\mathbf{W}_N$ is a real symmetric Gaussian matrix of variance ρ with i.i.d. entries such that for any $i\neq j$,

$$\mathbb{E}\,w_{ii}^2 = \frac{1}{N} \quad\text{and}\quad \mathbb{E}\,w_{ij}w_{ji} = \mathbb{E}\,w_{ij}^2 = \frac{\rho}{N}. \tag{3.7}$$

From [5, Theorem 1.1], we know that for all $|z|>1+|\rho|+\varepsilon$,

$$\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(z-\mathbf{W}_N)^{-1} - \mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{W}_N)^{-1}\Big|^2 = o\Big(\frac{1}{N}\Big).$$

Moreover, we can write

$$\begin{aligned}&\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1} - \mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1}\Big|^2\\ &\qquad= \mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1}\Big|^2 - \Big|\mathbb{E}\,\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1}\Big|^2\\ &\qquad= \frac{1}{N^2}\sum_{k,\ell\geq 0} z^{-k-1}\bar{z}^{-\ell-1}\Big(\mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^k\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell}\big] - \mathbb{E}\operatorname{Tr}\mathbf{X}_N^k\,\mathbb{E}\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell}\Big).\end{aligned}$$

By the Wick formula, we see that, for all k,ℓ,

$$0 \leq \mathbb{E}[\operatorname{Tr}\mathbf{X}_N^k \overline{\operatorname{Tr}\mathbf{X}_N^\ell}] - \mathbb{E}\operatorname{Tr}\mathbf{X}_N^k\, \mathbb{E}\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell} \leq \mathbb{E}\big[\operatorname{Tr}\mathbf{W}_N^k \overline{\operatorname{Tr}\mathbf{W}_N^\ell}\big] - \mathbb{E}\operatorname{Tr}\mathbf{W}_N^k\, \mathbb{E}\,\overline{\operatorname{Tr}\mathbf{W}_N^\ell}.$$

Indeed,

$$\begin{aligned}\mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^k \overline{\operatorname{Tr}\mathbf{X}_N^\ell}\big] &= \sum_{\substack{1\leq i_1,\dots,i_k\leq N\\ 1\leq i'_{k+1},\dots,i'_{k+\ell}\leq N}} \mathbb{E}\big[x_{i_1 i_2}\cdots x_{i_k i_1}\overline{x_{i'_{k+1}i'_{k+2}}}\cdots\overline{x_{i'_{k+\ell}i'_{k+1}}}\big] \\ &= \sum_{\substack{1\leq i_1,\dots,i_k\leq N\\ 1\leq i'_{k+1},\dots,i'_{k+\ell}\leq N}} \sum_{\pi\in\mathcal{P}_2(k+\ell)} \prod_{\{a,b\}\in\pi} \mathbb{E}[x_a x_b]\end{aligned} \tag{3.8}$$

where

$$x_a = \begin{cases} x_{i_a i_{a+1}} & \text{if } 1\leq a\leq k-1,\\ x_{i_k i_1} & \text{if } a=k,\\ \overline{x_{i'_a i'_{a+1}}} & \text{if } k+1\leq a\leq k+\ell-1,\\ \overline{x_{i'_{k+\ell} i'_{k+1}}} & \text{if } a=k+\ell.\end{cases}$$

We also have

$$\mathbb{E}\operatorname{Tr}\mathbf{X}_N^k\,\mathbb{E}\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell} = \sum_{\substack{1\leq i_1,\dots,i_k\leq N\\ 1\leq i'_{k+1},\dots,i'_{k+\ell}\leq N}} \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \mu\in\mathcal{P}_2(\ell)}} \prod_{\{a,b\}\in\pi} \mathbb{E}[x_a x_b] \prod_{\{c,d\}\in\mu} \mathbb{E}[x_c x_d],$$

which is a subsum of (3.8). Hence, as all $\mathbb{E}[x_a x_b]$ are non-negative (see (3.6)), we conclude that $\mathbb{E}[\operatorname{Tr}\mathbf{X}_N^k \overline{\operatorname{Tr}\mathbf{X}_N^\ell}] \geq \mathbb{E}\operatorname{Tr}\mathbf{X}_N^k\,\mathbb{E}\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell}$. Since, for all a and b, $\mathbb{E}[x_a x_b] \leq \mathbb{E}[w_a w_b]$ (see (3.7)), we deduce that

$$\mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^k \overline{\operatorname{Tr}\mathbf{X}_N^\ell}\big] - \mathbb{E}\operatorname{Tr}\mathbf{X}_N^k\, \mathbb{E}\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell} \leq \mathbb{E}\big[\operatorname{Tr}\mathbf{W}_N^k \overline{\operatorname{Tr}\mathbf{W}_N^\ell}\big] - \mathbb{E}\operatorname{Tr}\mathbf{W}_N^k\, \mathbb{E}\,\overline{\operatorname{Tr}\mathbf{W}_N^\ell}.$$

At last, we can write

$$\begin{aligned}&\mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1} - \mathbb{E}\frac{1}{N}\operatorname{Tr}(z-\mathbf{X}_N)^{-1}\Big|^2 \\ &\quad\leq \frac{1}{N^2}\sum_{k,\ell\geq 0} |z|^{-k-1}|z|^{-\ell-1}\Big(\mathbb{E}\big[\operatorname{Tr}\mathbf{X}_N^k \overline{\operatorname{Tr}\mathbf{X}_N^\ell}\big] - \mathbb{E}\operatorname{Tr}\mathbf{X}_N^k\,\mathbb{E}\,\overline{\operatorname{Tr}\mathbf{X}_N^\ell}\Big) \\ &\quad\leq \frac{1}{N^2}\sum_{k,\ell\geq 0} |z|^{-k-1}|z|^{-\ell-1}\Big(\mathbb{E}\big[\operatorname{Tr}\mathbf{W}_N^k \overline{\operatorname{Tr}\mathbf{W}_N^\ell}\big] - \mathbb{E}\operatorname{Tr}\mathbf{W}_N^k\,\mathbb{E}\,\overline{\operatorname{Tr}\mathbf{W}_N^\ell}\Big) \\ &\quad= \mathbb{E}\Big|\frac{1}{N}\operatorname{Tr}(|z|-\mathbf{W}_N)^{-1} - \mathbb{E}\frac{1}{N}\operatorname{Tr}(|z|-\mathbf{W}_N)^{-1}\Big|^2 = o\Big(\frac{1}{N}\Big).\end{aligned}$$ ∎

Proof of (ii). Let $\eta > 0$ and let i,j be two integers less than $2r$. Since $\|\mathbf{X}_N\|_{\mathrm{op}}$ is bounded, we know that $\|(z-\mathbf{X}_N)^{-1}\|_{\mathrm{op}}$ goes to 0 when $|z|\to\infty$, as the function $m(z)$, so that we know there is a positive constant M such that

$$\begin{aligned}&\mathbb{P}\Big(\sup_{|z|>1+\rho+\varepsilon} |\mathbf{e}_i^*(z-\mathbf{X}_N)^{-1}\mathbf{e}_j - \delta_{ij}m(z)| > \eta\Big) \\ &\qquad\qquad = \mathbb{P}\Big(\sup_{1+\rho+\varepsilon<|z|<M} |\mathbf{e}_i^*(z-\mathbf{X}_N)^{-1}\mathbf{e}_j - \delta_{ij}m(z)| > \eta\Big) + o(1).\end{aligned}$$

Then for any $\eta' > 0$, the compact set $K = \{1 + \rho + \varepsilon \le |z| \le M\}$ admits an η'-net that we denote by $S_{\eta'}$, which is a finite set of K such that

$$\forall z \in K,\ \exists z' \in S_{\eta'}, \quad |z - z'| < \eta',$$

so that, using the uniform boundedness of the derivative of $m(z)$ and $\mathbf{e}_i^* (z - \mathbf{X}_N)^{-1} \mathbf{e}_j$ on K, we have for a small enough η'

$$\mathbb{P}\Big(\sup_{z \in K} |\mathbf{e}_i^* (z - \mathbf{X})^{-1} \mathbf{e}_j - \delta_{ij} m(z)| > \eta \Big) = \mathbb{P}\Big(\bigcup_{z \in S_{\eta'}} \big\{ |\mathbf{e}_i^* (z - \mathbf{X})^{-1} \mathbf{e}_j - \delta_{ij} m(z)| > \eta/2 \big\} \Big)$$

At last, we write, for any $z \in S_{\eta'}$,

$$\begin{aligned} &\mathbb{P}(|\mathbf{e}_i^* (z - \mathbf{X})^{-1} \mathbf{e}_j - \delta_{ij} m(z)| > \eta/2) \\ &\qquad \le \mathbb{P}(|\mathbf{e}_i^* (z - \mathbf{X})^{-1} \mathbf{e}_j - \delta_{ij} \frac{1}{N} \operatorname{Tr}(z - \mathbf{X})^{-1}| > \eta/4) \\ &\qquad\qquad + \mathbb{P}(\delta_{ij} |\frac{1}{N} \operatorname{Tr}(z - \mathbf{X})^{-1} - m(z)| > \eta/4). \end{aligned}$$

The first term vanishes thanks to Theorem 2.3 with $\mathbf{M} = \sqrt{N} \mathbf{E}_{ji}$ and the second one vanishes by (iii).

A A Matrix Inequality

Lemma A.1 *For any $k \ge 2$ and any Hermitian matrix $\mathbf{H}$, $|\operatorname{Tr} \mathbf{H}^k| \le (\operatorname{Tr} \mathbf{H}^2)^{k/2}$. More generally, for any family of Hermitian matrices $\mathbf{H}_1, \dots, \mathbf{H}_k$,*

$$|\operatorname{Tr}(\mathbf{H}_1 \cdots \mathbf{H}_k)| \le \prod_{i=1}^k \sqrt{\operatorname{Tr} \mathbf{H}_i^2}.$$

Proof We know that for any non negative Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, one has $\operatorname{Tr} \mathbf{AB} \le \operatorname{Tr} \mathbf{A} \operatorname{Tr} \mathbf{B}$ so that for any $p \ge 1$, $\operatorname{Tr} \mathbf{H}^{2p} \le (\operatorname{Tr} \mathbf{H}^2)^p$, also

$$\operatorname{Tr} \mathbf{H}^{2p+1} \le \sqrt{\operatorname{Tr} \mathbf{H}^2} \sqrt{\operatorname{Tr} \mathbf{H}^{4p}} \le \sqrt{\operatorname{Tr} \mathbf{H}^2} \sqrt{(\operatorname{Tr} \mathbf{H}^2)^{2p}} = (\operatorname{Tr} \mathbf{H}^2)^{(2p+1)/2}.$$

Then using the non-commutative Hölder inequality (see [1, A.3]), we deduce that

$$|\operatorname{Tr}(\mathbf{H}_1 \cdots \mathbf{H}_k)| \le \prod_{i=1}^k (\operatorname{Tr} |\mathbf{H}_i|^k)^{1/k} \le \prod_{i=1}^k \Big((\operatorname{Tr} \mathbf{H}_i^2)^{k/2} \Big)^{1/k}. \qquad \blacksquare$$

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