NON-UNITAL BANACH JORDAN ALGEBRAS AND C^*-TRIPLE SYSTEMS

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Introduction

The definition of a suitable Jordan analogue of C^*-algebras (which we call JB^*-algebras in this paper) was recently suggested by Kaplansky (see (**26**)). The theory of unital JB^*-algebras is now comparatively well understood due to the work of Alfsen, Shultz and Størmer (**1**) from which a Gelfand–Neumark theorem for unital JB^*-algebras can be obtained (**26**). Independently, from work on simply connected symmetric complex Banach manifolds with base point, Kaup introduced the definition of C^*-triple systems in (**14**) and subsequently in (**7**) it was shown that every unital JB^*-algebra is a C^*-triple system. In this paper, we wish to extend this result to show that every JB^*-algebra is a C^*-triple system.

We start by showing that every JB^*-algebra may be embedded into a JB^*-algebra with unit. This allows us to use the Gelfand Neumark theorem for unital JB^*-algebras to show that the double dual of a JB^*-algebra is a unital JB^*-algebra under the Arens product and an involution which is constructed using numerical range techniques. Our main result, that every JB^*-algebra is a C^*-triple system, then follows from this. As a corollary, we give a geometric characterisation of the existence of a unit in a JB^*-algebra using an algebraic characterisation of the extreme points of the closed unit ball of a JB^*-algebra.

1. Preliminaries

In this section we give the definitions and some of the known results on Banach Jordan algebras which we shall later require. A *Banach Jordan algebra* is a (real or complex) Jordan algebra W with a norm $\|\cdot\|$ under which W is a Banach space and

$$\|cd\| \leqq \|c\| \, \|d\|$$

for all c and d in W. If W has a unit 1 then W is called unital if $\|1\| = 1$.

Given a Banach algebra V with product $c.d$, we can make V into a Banach Jordan algebra by defining

$$cd = \tfrac{1}{2}(c.d + d.c). \tag{1}$$

However not every Banach Jordan algebra arises in this way, an example being M_3^8, the single exceptional formally real simple finite dimensional real Jordan algebra ((**12**),

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(**11**), (**1**)). Partly due to this example, no reasonable Jordan analogue of C^*-algebras was known until recently when Alfsen, Shultz and Størmer introduced and studied the class of JB-algebras in (**1**). A *JB-algebra* is a real Banach Jordan algebra E such that

$$\|x\|^2 = \|x^2\| \leqq \|x^2 + y^2\|$$

for all x and y in E.

Examples of JB-algebras include norm closed real Jordan subalgebras of self-adjoint elements of a C^*-algebra, called *JC-algebras* by Størmer (**25**), and M_3^8.

In this paper, we shall be more concerned with the complex Jordan analogues of C^*-algebras. Before we give their definition, we first recall that if J is a Jordan algebra and x, y and z are in J, then the Jordan triple product of x, y and z is defined by

$$\{x, y, z\} = x(yz) - y(xz) + z(xy).$$

A complex Banach Jordan algebra W with an involution * is a *JB^*-algebra* if, for all y in W,

$$\|\{y, y^*, y\}\| = \|y\|^3.$$

If $\mathcal{H}$ is a complex Hilbert space and $B(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators on $\mathcal{H}$ with the usual involution, then $B(\mathcal{H})$ with multiplication (1) is a JB^*-algebra. Another example of a JB^*-algebra is $\mathcal{M}_3^8$, the complexification of M_3^8 in a suitable norm ((**26**), Corollary 2.7), and more generally if S is a compact Hausdorff space and $C(S, \mathcal{M}_3^8)$ is the set of all continuous functions from S into $\mathcal{M}_3^8$, then $C(S, \mathcal{M}_3^8)$ with natural product and involution and the sup norm is a JB^*-algebra ((**7**) 1.10(iii)).

It is easy to see, as in (**28**), Lemma 4 and (**26**), that if W is a JB^*-algebra then $\|y^*\| = \|y\|$ for all y in W and every closed associative *-subalgebra of W is a C^*-algebra. This shows that the class of JB^*-algebras coincides with the class of Jordan C^*-algebras introduced by Kaplansky (see (**26**)). Moreover if W is a JB^*-algebra with a unit 1, then 1 is a self adjoint element of norm one. It is known that the set of self-adjoint elements of a unital JB^*-algebra form a unital JB-algebra, while conversely the main result of (**26**) states that the complexification of a unital JB-algebra in a suitable norm is a JB^*-algebra. In the next section we shall show that this also holds for non-unital algebras. To do this, we require the concept of the spectrum of an element of a complex unital Banach Jordan algebra which was introduced in (**8**). An element x of W is *invertible* with inverse $y \in W$ if

$$xy = 1 \quad \text{and} \quad x^2y = x.$$

Let W be a Jordan algebra with unit 1. The *spectrum* of x, denoted by $\sigma(x)$, is defined by

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ is not invertible in } W\},$$

and the *spectral radius* of x, denoted by $r(x)$, is defined by

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

2. Adjoining a unit to a JB^*-algebra

The main purpose of this section is to show that we can embed an arbitrary JB^*-algebra into a unital JB^*-algebra. This is proved using the characterisation of the

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complex unital Banach Jordan algebras which are the homeomorphic images of unital JB^*-algebras obtained in (**30**). We deduce from this some extensions of the theory of unital JB^*-algebras to non-unital JB^*-algebras which we shall require.

We first recall that if J is a Jordan algebra over a field F and J does not have a unit, we may formally adjoin a unit by defining an addition and multiplication on $J \oplus F$ which extends the Jordan algebra structure on J and under which $J \oplus F$ is a Jordan algebra with unit $u = (0, 1)$ (see (**11**), Theorem 1.6).

Theorem 1. *Let W be a JB^*-algebra which does not have a unit and let $V = W \oplus \mathbb{C}$ be the Jordan algebra obtained by adjoining a unit to W. Then there exists a norm and involution on V under which V is a JB^*-algebra.*

Proof. A routine argument shows that if we define a function $^*: V \to V$ and a function $p: V \to \mathbb{R}$ by

$$(x+\lambda)^* = x^* + \lambda^*$$

$$p(x+\lambda) = \|x\| + |\lambda|$$

for $x \in W$ and $\lambda \in \mathbb{C}$, then * is an involution on V and (V, p) is a complex unital Banach Jordan algebra such that $p(x) = \|x\|$ for all x in W. In general (V, p) is not a JB^*-algebra and so we have to show the existence of an equivalent norm on V under which V is a JB^*-algebra.

It was shown in (**28**), Lemma 4 that the involution on a unital JB^*-algebra is an isometry and a quick check reveals that the argument used also works for non-unital JB^*-algebras. It follows that

$$p((x+\lambda)^*) = \|x^*\| + |\lambda^*| = \|x\| + |\lambda| = p(x+\lambda) \tag{2}$$

for all $x \in W$ and $\lambda \in \mathbb{C}$, and so the involution on (V, p) is an isometry.

Let S be the set of self-adjoint elements of W and let T be the set of self-adjoint elements of V. Then S and T are real Banach Jordan algebras and T may be identified with the Jordan algebra obtained by adjoining a unit to S. Let $s, w \in S$, let $\mu, \lambda \in \mathbb{R}$ and let $t = s + \lambda$ and $x = w + \mu$. Let $P(s)$ denote the closed subalgebra of W generated by s. As every Jordan algebra is power associative ((**11**), Theorem 1.8) and multiplication is continuous, $P(s)$ is a commutative Banach algebra. Moreover as the involution on W is an isometry and s is self-adjoint, $P(s)$ is a self-adjoint subset. Hence $P(s)$ is a C^*-algebra. So

$$\begin{aligned} p(\exp it) &= p(\exp i(s+\lambda)) \\ &= p(\exp is) \\ &= 1 + \|\exp is - 1\| \\ &\leqq 3. \end{aligned}$$

Similarly $p(\exp -it) \leqq 3$ and so, as p is submultiplicative,

$$\tfrac{1}{3} \leqq p(\exp it) \leqq 3. \tag{3}$$

Moreover as s is a self-adjoint element of the C^*-algebra $P(s)$, it follows that there exists $y \in P(s)$ such that

$$s^2 + y + s^2 y = 0 = s^2 y + s^4 + s^4 y.$$

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Hence S is a Hermitian Banach Jordan algebra in the terminology of (**4**) and so, by (**4**) Corollary 1,

$$r(x+t) \leqq r(x)+r(t). \tag{4}$$

Finally by (2), (3), (4) and (**30**), Theorem 11, there is an equivalent norm on V under which V, with involution defined above, is a unital JB^*-algebra.

In (**4**), Behncke showed that every JB-algebra can be embedded into a unital JB-algebra. If we could have proved that the set of self-adjoint elements of a JB^*-algebra formed a JB-algebra, then the proof of Theorem 1 would have been simplified. However we could not do this but it is now an immediate corollary.

Corollary 2. *If W is a JB^*-algebra then the set of self-adjoint elements of W forms a JB-algebra.*

Finally in this section we extend the main representation theorem for unital JB^*-algebras to non-unital ones. The proof is immediate from Theorem 1 and (**7**), Theorem 1.11(a).

Corollary 3. *Every JB^*-algebra is isometrically *-isomorphic to a closed self-adjoint Jordan subalgebra of $B(\mathscr{H}) \oplus C(S, \mathscr{M}_3^8)$ for some Hilbert space $\mathscr{H}$ and some compact Hausdorff space S, with the* max *norm on $B(\mathscr{H}) \oplus C(S, \mathscr{M}_3^8)$.*

3. The double dual of a JB^*-algebra

In this section we aim to show that the double dual of a JB^*-algebra is a unital JB^*-algebra. We follow the approach used in (**5**) for unital C^*-algebras but we have the additional problem of showing that the Arens product on the double dual of a JB^*-algebra satisfies the axioms of a Jordan algebra. As we shall be considering involutions on dual spaces, we shall use the following notation for dual spaces and adjoint maps.

Notation. If X and Y are Banach spaces and T is a bounded linear operator from X into Y, we denote the dual space of X by X' and the adjoint from Y' into X' by T'. Similarly we denote the dual space of X' by X'' and the double adjoint of T by T''. If M is a closed subset of Y, we denote by $M^\perp$ the annihilator of M in Y' defined by

$$M^\perp = \{f \in Y' : f(x) = 0 \text{ for all } x \text{ in } M\}.$$

We shall require the well known result (see (**21**) for example) that if X and Y are Banach spaces and T is an isometry of X into Y, then T'' is an isometry of X'' onto the weak*-closed subset $\{Tx : x \in X\}^{\perp\perp}$ of Y''.

If W is a Banach Jordan algebra we recall that the Arens product on W'' is the unique bilinear extension of the (Jordan) product on W such that

$\|cd\| \leqq \|c\|\,\|d\|$ for all c and d in W'';

$c \to cd$ is weak*-continuous for all c and d in W'';

$d \to cd$ is weak*-continuous for all c in W and d in W''.

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In general the Arens product is not symmetric in the two variables so it need not be commutative. However by (**23**), Theorem 1.2 and (**24**), Theorem 3.7, the double dual of a JB-algebra is again a JB-algebra. We shall show that the double dual of a JB^*-algebra W is a JB^*-algebra, but before we show that W'' under the Arens multiplication is a Jordan algebra, we first show the existence of a suitable involution on W''. The most convenient way to do this is by numerical range techniques. Let W be a complex unital Banach Jordan algebra. The set of states on W, denoted by $D(W, 1)$, is defined by

$$D(W, 1) = \{f \in W' : \|f\| = f(1) = 1\}.$$

If $p \in W$, the *numerical range* of p, denoted by $V(p)$, is defined by

$$V(p) = \{f(p) : f \in D(W, 1)\}.$$

If $p \in W$ and $V(p) \subseteq \mathbb{R}$ then p is called *Hermitian*. The set of Hermitian elements of W is denoted by Her W.

As the numerical range is essentially a linear concept, many of the results on the numerical range of elements of a complex unital Banach algebra remain valid for complex unital Banach Jordan algebras. Several of these are given in (**28**), for instance that the set of Hermitian elements is a closed real linear subspace, but we shall also require the following lemma.

Lemma 4. *If W is a complex unital Banach Jordan algebra and $f \in W'$ then there exists $\alpha_k \in \mathbb{R}^+$ and $f_k \in D(W, 1)$ $(k = 1, 2, 3, 4)$ such that*

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leqq \sqrt{2}\, \mathrm{e}\, \|f\|;$$
$$f = \alpha_1 f_1 - \alpha_2 f_2 + i(\alpha_3 f_3 - \alpha_4 f_4).$$

Proof. As $D(W, 1)$ is a weak*-compact convex subset of W' such that

$$\|p\| \leqq \mathrm{e} \sup \{|f(p)| : f \in D(W, 1)\}$$

for all p in W by (**28**), Theorem 2, the required result follows from (**3**) Theorem 1.

The corresponding result for Banach algebras was obtained by Moore (see [**6**]) who used it to obtain a dual characterisation of unital C^*-algebras in (**18**). We now generalise some of his results, but it is first convenient to introduce the following notation. If W is a complex unital Banach Jordan algebra, we denote by $H'(W)$ the real linear span of $D(W, 1)$.

By Lemma 4 it follows that $W' = H'(W) + iH'(W)$.

Theorem 5. *Let Y be a unital JB^*-algebra such that Y'' with the Arens product is a Jordan algebra. Then*

(i) $Y = \mathrm{Her}\, Y \oplus i\, \mathrm{Her}\, Y$;
(ii) $y \in \mathrm{Her}\, Y$ *if and only if* y *is self-adjoint*;
(iii) $H'(Y) \cap iH'(Y) = \{0\}$;
(iv) $\mathrm{Her}\,(Y'') \oplus i\, \mathrm{Her}\,(Y'') = Y''$.

Proof. (i) and (ii) follow from (**28**), Theorem 7.

(iii). If $f \in H'(Y)$ then $f(y) \in \mathbb{R}$ for all $y \in \mathrm{Her}\, Y$. Hence if $f \in H'(Y) \cap iH'(Y)$, it follows that $f(y) \in \mathbb{R} \cap i\mathbb{R} = \{0\}$ for all $y \in \mathrm{Her}\, Y$. As $Y = \mathrm{Her}\, Y \oplus i\, \mathrm{Her}\, Y$, the required result follows by linearity.

(iv). By Lemma 4 and (iii), $Y' = H'(Y) \oplus iH'(Y)$ and each $f \in Y'$ has a unique expression of the form $f = \mathrm{re}\, f + i\, \mathrm{im}\, f$ where $\mathrm{re}\, f$ and $\mathrm{im}\, f$ are in $H'(Y)$ and

$$\|\mathrm{re}\, f\| \leqq 2\sqrt{2}\mathrm{e}\, \|f\|; \quad \|\mathrm{im}\, f\| \leqq 2\sqrt{2}\mathrm{e}\, \|f\|.$$

Hence we can define a continuous involution on Y' by

$$f^* = \mathrm{re}\, f - i\, \mathrm{im}\, f \tag{5}$$

for f in Y'. Next, we define a continuous involution on Y'' by

$$F^*(f) = (F(f^*))^* \tag{6}$$

for $f \in Y'$ and $F \in Y''$. Now, as $D(Y, 1)$ is a set of self-adjoint elements with respect to the involution on Y', it follows that

$$\tfrac{1}{2}(F + F^*)(f) \in \mathbb{R}; \quad \frac{1}{2i}(F - F^*)(f) \in \mathbb{R}$$

for all $f \in D(Y, 1)$ and all $F \in Y''$. By an analogue of the argument used in (**5**) Theorem 12.2, we can conclude that $\tfrac{1}{2}(F + F^*)$ and $1/(2i)(F - F^*)$ are Hermitian elements of Y''. Hence

$$Y'' = \mathrm{Her}\,(Y'') \oplus i\, \mathrm{Her}\,(Y'') \tag{7}$$

and the involution (6) is the natural involution associated with this direct sum decomposition.

In fact, as the next result will show, if Y is a unital JB^*-algebra then Y'' with the Arens product is a Jordan algebra. This is the main result of this section but as the first part of the proof is similar to (**23**), Theorem 1.2, we omit many of the details.

Theorem 6. *Let W be a JB^*-algebra. Then W'', with the Arens product, is a unital JB^*-algebra.*

Proof. By Corollary 3, there is a complex Hilbert space $\mathscr{H}$, a compact Hausdorff space S and an isometrical *-isomorphism $j: W \to Y$ where $Y = B(\mathscr{H}) \oplus C(S, \mathscr{M}_3^8)$. Now $B(\mathscr{H})''$ is also a C^*-algebra and hence the Arens extension of the Jordan product on $B(\mathscr{H})$ is again a JB^*-algebra. Moreover as $\mathscr{M}_3^8$ is finite dimensional, $(C(S, \mathscr{M}_3^8))''$ is isometrically *-isomorphic to $C(T, \mathscr{M}_3^8)$ where T is the maximal ideal space of the commutative C^*-algebra $(C(S))''$. Hence $(C(S, \mathscr{M}_3^8))''$ with the Arens product is also a JB^*-algebra. As Y'' is isometrically isomorphic to $(B(\mathscr{H}))'' \oplus (C(S, \mathscr{M}_3^8))''$ with the max norm, it follows that Y'' is a unital JB^*-algebra under the Arens product and an involution which extends the given one on Y.

By (**2**) Theorem 2.5, $j'': W'' \to Y''$ is an isometrical isomorphism which preserves the Arens product. Hence W'' is a Jordan algebra. Moreover as $M = \{j(w) : w \in W\}$ is a self-adjoint subset of Y, it successively follows that $M^\perp$ and $M^{\perp\perp}$ are self-adjoint subsets of Y' and Y'' under the involutions (5) and (6) respectively. So by Theorem 5(i) and (7), $M^{\perp\perp}$ is a subset of Y'' which is self-adjoint with respect to the involution under which Y'' is a JB^*-algebra. On identifying $M^{\perp\perp}$ with W'' it follows that W'' is a JB^*-algebra.

It remains to show that W'' has a unit. Let $\{a_\lambda\}$ be a net of self-adjoint elements of $M^{\perp\perp}$ which converges in the weak* topology to a in $M^{\perp\perp}$. Then, for all f in $D(Y, 1)$, $a(f) = \lim a_\lambda(f) \in \mathbb{R}$ and so a is self-adjoint by the analogue of (**5**), Theorem 12.2. Hence the JB-algebra of self-adjoint elements of $M^{\perp\perp}$ is weak*-closed and so has a unit by (**9**), Lemma 1. Thus W'' has a unit.

Parts of the above proof are similar to the corresponding result for JB-algebras given in (**23**), Theorem 1.2 and (**24**), Theorem 3.7, but we had also to ensure the existence of a suitable involution. If the JB^*-algebra W had a unit the proof of (3.16) could be simplified using the Vidav-Palmer theorem for Banach Jordan algebras ((**29**), Theorem 8 and (**19**), Theorem 2.2). It can also be shown that (**6**), Theorem 31.10 can be generalized to complex unital Banach Jordan algebras to complete the duality characterisation of unital JB^*-algebras which was started in Theorem 5. After completing this article, we found that this had independently been noted in (**20**). However the methods used in (**20**) are different from the above.

Remark. An alternative proof of the result that the double dual of a unital JB^*-algebra W is a Jordan algebra under the Arens product can be obtained by noting that (Her W)$'$ is homeomorphically isomorphic to $H'(W)$ and $(H'(W))'$ is homeomorphically isomorphic to Her(W''). Moreover the induced isomorphism from (Her W)$''$ onto Her(W'') preserves the Arens product, so we may apply (**23**), Theorem 1.2 to conclude that Her(W''), and hence W'', is a Jordan algebra. To put in all the details we would have to consider the numerical range in general non-associative algebras. While this is not hard, we shall not require it in the sequel and so we do not pursue this further.

4. JB^*-algebras and bounded symmetric homogeneous domains

In (**14**), a Jordan theoretic characterisation of the category of all symmetric complex Banach manifolds with base point was given. This category was equivalent to the category of Hermitian Jordan triple systems. At present, a full analysis of all Hermitian Jordan triple systems seems unattainable but the more promising subcategory of C^*-triple systems was also introduced in (**14**). Subsequently it was shown in (**7**) that all unital JB^*-algebras are C^*-triple systems under the natural Jordan triple product, and a partial converse was also obtained. We shall use this to show that all JB^*-algebras are C^*-triple systems. This implies in particular that the open unit ball of a JB^*-algebra is a bounded symmetric homogeneous domain. We conclude with a characterisation of the JB^*-algebras of which the open unit ball is holomorphically equivalent to a tube domain or a Siegel domain.

We first recall that a Hermitian Jordan triple system is a complex Banach space W together with a continuous map $\Gamma: W \times W \times W \to W$ such that

(i) Γ is symmetric complex linear in the outer variables and conjugate linear in the inner variable;

(ii) $\Gamma(\Gamma(v, \alpha, w), \beta, z) + \Gamma(\Gamma(v, \alpha, z), \beta, z)$
$- \Gamma(v, \alpha, \Gamma(w, \beta, z)) = \Gamma(w, \Gamma(\alpha, v, \beta), z)$ for all $\alpha, \beta, v, w, z \in W$;

(iii) if $\alpha \in W$ and $\alpha \square \alpha^* \in B(W)$ is defined by $(\alpha \square \alpha^*)(z) = \Gamma(\alpha, \alpha, z)$ for z in W, then $\alpha \square \alpha^* \in \text{Her}\, B(W)$ for all $\alpha \in W$.

If in addition

(iv) $\sigma(\alpha \square \alpha^*) \subseteq \mathbb{R}^+$ for all $\alpha \in W$;

(v) $\|\alpha \square \alpha^*\| = \|\alpha\|^2$ for all $\alpha \in W$,

then W is called a C^*-triple system.

By (**7**), Theorem 3.3, every unital JB^*-algebra forms a C^*-triple system. We now extend this result to non-unital algebras.

Theorem 7. *Let W be a JB^*-algebra. If $\Gamma: W \times W \times W \to W$ is defined by $\Gamma(x, y, z) = \{x, y^*, z\}$ for x, y and z in W, then (W, Γ) is a C^*-triple system.*

Proof. As W is a Jordan *-algebra, properties (i) and (ii) are easily verified. By Theorem 6, W'' is a unital JB^*-algebra. Let $j: W \to W''$ be the natural inclusion. By the construction of the Arens product,

$$(\alpha \square \alpha^*)'' = j(\alpha) \square (j(\alpha))^* \tag{8}$$

for all $\alpha \in W$. By (**7**), Theorem 3.3, $j(\alpha) \square (j(\alpha))^* \in \text{Her}\, B(W'')$ and $\sigma(j(\alpha) \square (j(\alpha))^*) \subseteq \mathbb{R}^+$. Hence by (**5**), Theorem 5.14, $V(j(\alpha) \square (j(\alpha))^*) \subseteq \mathbb{R}^+$. By (8) and (**5**), Corollary 9.6, it follows that $V(\alpha \square \alpha^*) \subseteq \mathbb{R}^+$. So by (**5**), Theorem 2.6, $\sigma(\alpha \square \alpha^*) \subseteq \mathbb{R}^+$ and $\alpha \square \alpha^* \in \text{Her}\, B(W)$. Finally as

$$\|\{x, y^*, z\}\| \leqq \|x\|\,\|y\|\,\|z\|$$

for all x, y and z in W by (**27**), Corollary 2.5, (**28**), Lemma 4 and Theorem 6, it follows that

$$\|\alpha\|^3 = \|(\alpha \square \alpha^*)(\alpha)\| \leqq \|\alpha \square \alpha^*\|\,\|\alpha\| \leqq \|\alpha\|^3$$

for all $\alpha \in W$. Hence (W, Γ) is a C^*-triple system.

The following result is an immediate corollary to Theorem 7 and (**14**), Proposition 5.2.

Corollary 8. *If W is a JB^*-algebra, then the open unit ball of W is a bounded symmetric homogeneous domain.*

This allows us to characterise the extreme points of the closed unit ball of a JB^*-algebra.

Theorem 9. *Let W be a JB^*-algebra and let $w \in W$. Then the following are equivalent*:

(i) *$w \square w^*$ is invertible in $B(W)$ and $\{w, w^*, w\} = w$;*

(ii) *$y - 2\{w, w^*, y\} + \{w, \{w^*, y, w^*\}, w\} = 0$ for all $y \in W$;*

(iii) *w is a real extreme point of the closed unit ball of W;*

(iv) *w is a complex extreme point of the closed unit ball of W.*

Proof. The equivalence of (i) and (iv), and (i) and (ii) is given in (**16**), Theorem 3.5 and (**16**), Lemma 3.2. That (iii) implies (iv) is clear and so it only remains to show that

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(ii) implies (iii). As W is a JB^*-algebra, W'' is a unital JB^*-algebra and by the construction of the Arens product and the weak*-density of W in W'' it follows that

$$y-2\{w, w^*, y\}+\{w,\{w^*, y, w^*\}, w\}=0 \tag{9}$$

for all $y \in W''$, where we have identified w with its canonical image in W''. Hence by (**7**), (4.1), w is a real extreme point of the closed unit ball of W''. In particular, w is a real extreme point of the closed unit ball of W.

Corollary 10. *Let W be a JB^*-algebra. Then W has a unit if and only if the closed unit ball of W has a real extreme point.*

Proof. If W has a unit then by Theorem 9, u is a real extreme point of the closed unit ball of W. Conversely if w is a real extreme point of the closed unit ball of W then by (9), for all y in W'',

$$y=2\{w, w^*, y\}-\{w,\{w^*, y, w^*\}, w\}.$$

In particular as W'' has a unit u, it follows that

$$u=2ww^*-\{w,(w^*)^2, w\}\in W.$$

Hence W has a unit.

The characterisation of the extreme points of the unit ball of a B^*-algebra was obtained by Kadison (**13**), Miles (**17**) and Sakai (**22**). The proof of Corollary 10 is an adaptation of the argument used in (**17**). An alternative proof, directly related to the homogeneity of the open unit ball can be obtained by modifying a proof of Harris (**10**), Theorem 11.

The characterisation of the extreme points of the unit ball of a JB-algebra was obtained by Edwards (**9**) who gave an analogue of Corollary 10 for JB-algebras.

Our final corollary gives a geometric characterisation of the existence of a unit in a JB^*-algebra. For the definition of a Siegel domain or a tube domain, we refer to (**16**), (2.2).

Corollary 11. *Let W be a JB^*-algebra and let V be the open unit ball of W. Then the following are equivalent:*

(i) *V is biholomorphically equivalent to a Siegel domain;*
(ii) *V is biholomorphically equivalent to a tube domain;*
(iii) *W has a unit*

Proof. This is immediate from (**16**), Theorem 3.7 and Corollary 10.

If (W,Γ) is a C^*-triple system and E is a J^*-subsystem of W, that is, E is a closed subspace such that $\Gamma(x, y, z)\in E$ whenever x, y and z are in E, then E is also a C^*-triple system. Conversely in (**15**) it is essentially shown that every finite dimensional C^*-triple system is a J^*-subsystem of some finite dimensional JB^*-algebra. However a J^*-subsystem need not be a self-adjoint subspace even if it is a quadratic ideal of the JB^*-algebra.

Example. Let W be the JB^*-algebra of all two by two matrices over the complex field with Jordan product (1) and natural involution. Let J be the one dimensional subspace of W consisting of all matrices with zeros on the diagonal and the bottom left hand entry. Then J is a quadratic ideal and hence a J^*-subsystem of W, but J is clearly not a self-adjoint set.

In view of this and the above remarks we conclude with the following problem.

Problem. Is every C^*-triple system a J^*-subsystem of a unital JB^*-algebra?

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