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Lyapunov stability of non-isolated equilibria for strongly irreversible Allen–Cahn equations

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The present article is concerned with the Lyapunov stability of stationary solutions to the Allen–Cahn equation with *a strong irreversibility constraint*, which was first intensively studied in [2] and can be reduced to an evolutionary variational inequality of obstacle type. As a feature of the obstacle problem, the set of stationary solutions always includes accumulation points, and hence, it is rather delicate to determine the stability of such non-isolated equilibria. Furthermore, the strongly irreversible Allen–Cahn equation can also be regarded as a (generalized) gradient flow; however, standard techniques for gradient flows such as linearization and Łojasiewicz–Simon gradient inequalities are not available for determining the stability of stationary solutions to the strongly irreversible Allen–Cahn equation due to the non-smooth nature of the obstacle problem.

Keywords: Lyapunov stability of equilibria; non-isolated equilibria; obstacle problem; strongly irreversible Allen–Cahn equation; variational inequality

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1. Introduction

The present article concerns the following *strongly irreversible* Allen–Cahn equation:

$$u_t = \left(\Delta u - u^3 + \kappa u\right)_+ \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

where $(\cdot)_+$ stands for the positive-part function (i.e., $(s)_+ := \max\{s, 0\} \geq 0$ for $s \in \mathbb{R}$), κ is a positive constant, and Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and whose solutions are constrained to be non-decreasing in time (indeed, u_t is always non-negative). Equation (1.1) can be regarded as a variant of the classical Allen–Cahn equation, which has been well studied so far and is known as a phase separation model. Moreover, strongly irreversible evolution equations such as (1.1) also appear in some phase field models of damage and fracture. More precisely, the phase parameter represents the degree of damage, which is supposed to evolve monotonely due to the irreversible nature of damage and fracture (see [2, §2] and references therein). Furthermore, similar constrained models are also introduced to describe various irreversible phase transition phenomena (see, e.g., [14, 15] and also [24], where a mitochondrial swelling process is studied and it is a pure irreversible process). Equation (1.1) may be regarded as a simplified one, and moreover, it is worth studying (1.1) for figuring out various features of irreversible phase field models, although it is not in itself exactly from any physical model.

From mathematical points of view, Eq. (1.1) is classified as a fully nonlinear parabolic equation. On the other hand, it can be reduced to a (generalized) gradient flow of doubly nonlinear type. Indeed, applying the (multivalued) inverse mapping of the positive-part function $(\,\cdot\,)_+$ to both sides of (1.1), we see

$$u_t + \partial I_{[0,\infty)}(u_t) \ni \Delta u - u^3 + \kappa u \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

where $\partial I_{[0,\infty)}$ stands for the subdifferential of the indicator function $I_{[0,\infty)}$ over the half-line $[0,+\infty)$, that is,

$$\partial I_{[0,\infty)}(s) = \{\xi \in \mathbb{R} \colon \xi(\sigma - s) \le 0 \ \text{ for all } \ \sigma \ge 0\} = \begin{cases} \{0\} & \text{if } \ s > 0, \\ (-\infty, 0] & \text{if } \ s = 0 \end{cases} \tag{1.3}$$

for $s \in D(\partial I_{[0,\infty)}) = [0,\infty)$ (see [2] for more details). A similar problem was also studied in [5, 28, 29]. Equation (1.2) is classified as a *doubly-nonlinear evolution equation* of the form,

$$A(u_t) + B(u) = 0 \quad \text{in } X, \quad 0 < t < T$$

in a Banach space X with two nonlinear operators $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$. Doubly-nonlinear evolution equations were studied in [8, 12], and then, Colli–Visintin [21] and Colli [20] established a celebrated abstract theory, which has been applied to many nonlinear evolutionary problems arising from various phase-field models (see also, e.g., [9, 10, 22, 23, 36, 38–41, 43]). Furthermore, phase-field models in fracture mechanics (see [6, 7, 26, 27]) have also been vigorously studied in this direction (see, e.g., [13–15, 32, 34, 35, 37, 42] and references therein).

In this article, we are concerned with the Cauchy–Dirichlet problem (P) for (1.1), which is equivalently rewritten as

$$u_t + \mu - \Delta u + u^3 - \kappa u = 0, \quad \mu \in \partial I_{[0,\infty)}(u_t) \qquad \text{in } \ \Omega \times (0,\infty), \tag{1.4}$$

$$u = 0 \qquad \text{on } \ \partial\Omega \times (0,\infty), \tag{1.5}$$

$$u = u_0 \qquad \text{in } \ \Omega. \tag{1.6}$$

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Furthermore, comparing (1.4) with (1.1), one finds that

$$\mu = -\left(\Delta u - u^3 + \kappa u\right)_-,$$

where $(s)_- := \max\{-s, 0\} \geq 0$ for $s \in \mathbb{R}$. Then the energy functional $E : H_0^1(\Omega) \cap L^4(\Omega) \to \mathbb{R}$ defined by

$$E(w) := \frac{1}{2}\int_\Omega |\nabla w(x)|^2 \,\mathrm{d}x + \frac{1}{4}\int_\Omega |w(x)|^4 \,\mathrm{d}x - \frac{\kappa}{2}\int_\Omega |w(x)|^2 \,\mathrm{d}x \quad \text{for } w \in H_0^1(\Omega) \cap L^4(\Omega)$$

plays a role of Lyapunov functional, that is, $t \mapsto E(u(t))$ is non-increasing in time along the evolution of solutions $t \mapsto u(t)$ to (P) (see [2] for more details). The Cauchy–Dirichlet problem above was intensively studied in [2], where the well-posedness is proved in an L^2 formulation for initial data u_0 belonging to the closure $\overline{D}_r$ in $H_0^1(\Omega) \cap L^4(\Omega)$ of the set

$$D_r := \left\{u \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega) \colon \|(\Delta u - u^3 + \kappa u)_-\|_2^2 \leq r\right\}$$

for an arbitrary $r > 0$, and qualitative properties and asymptotic behaviours of strong solutions are studied. In particular, it is proved that (P) admits the unique strong solution $u = u(x, t)$ which also solves the Cauchy–Dirichlet problem for the evolutionary variational inequality of obstacle type,

$$u_t + \mu - \Delta u + u^3 - \kappa u = 0, \quad \mu \in \partial I_{[u_0(x),\infty)}(u) \quad \text{in } \Omega \times (0, \infty) \tag{1.7}$$

(see Definition 3.1 and Theorem 3.2 in §3). Therefore every equilibrium $\psi = \psi(x)$ of (1.4)–(1.6) turns out to solve

$$\partial I_{[u_0(x),\infty)}(\psi) - \Delta\psi + \psi^3 - \kappa\psi \ni 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega \tag{1.8}$$

(see [2, Theorem 10.1]). Throughout this article, we denote by $\mathcal{VI}(u_0)$ the set of all solutions $\psi \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$ to the variational inequality (1.8) with the obstacle function $u_0 = u_0(x)$, which coincides with the initial datum of (P). Then the set of stationary solutions to (P) for $u_0 \in \overline{D}_r$ is given by

$$\mathcal{VI} := \bigcup \left\{\mathcal{VI}(u_0) \colon u_0 \in \overline{D_r}\right\}.$$

In particular, all *super*solutions $\psi \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^6(\Omega)$ to the classical stationary Allen–Cahn equation,

$$-\Delta u + u^3 - \kappa u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.9}$$

(namely, ψ satisfies $\Delta\psi - \psi^3 + \kappa\psi \leq 0$ in Ω and $\psi = 0$ on $\partial\Omega$) belong to the set $\mathcal{VI}(\psi) \subset \mathcal{VI}$. Hence the set $\mathcal{VI}$ involves non-isolated equilibria (see Corollary 5.1 in §3). In contrast with isolated equilibria, it is more delicate to determine the Lyapunov stability (or instability) of such non-isolated equilibria (see Definition 3.3

in §3). Linearization and Łojasiewicz–Simon gradient inequalities are often used to overcome such difficulties (see [1] for applications of ŁS inequalities to stability analysis of non-isolated equilibria). However, both devices do not seem to be applicable to the present issue due to the severely nonlinear and non-smooth nature of the obstacle problem. The main purpose of this article is to investigate the Lyapunov stability of equilibria $\psi \in \mathcal{VI}$ under the Dynamical System (DS for short) generated by (P) = {(1.4)–(1.6)} (or {(1.7), (1.5), (1.6)} equivalently).

In §2, we shall give fundamental properties of equilibria. Section 3 is devoted to stating main results (see Theorem 3.7) of the present article, which is concerned with the Lyapunov stability of equilibria lying on a small neighbourhood of the positive least energy solution ϕ_{ac} for the classical elliptic Allen–Cahn equation. Here we emphasize that, even in small neighbourhoods of ϕ_{ac}, every equilibria ψ is non-isolated. So one cannot expect the asymptotic stability of ψ.

Notation. We denote by $\|\cdot\|_p$, $1 \le p \le \infty$ the $L^p(\Omega)$-norm, that is, $\|f\|_p := (\int_\Omega |f(x)|^p \,\mathrm{d}x)^{1/p}$ for $p \in [1, \infty)$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$. Moreover, we denote by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$. We often simply write $u(t)$ instead of $u(\cdot, t)$, which is regarded as a function from Ω to $\mathbb{R}$, for each fixed $t \ge 0$. Here and henceforth, we use the same notation $I_{[0,\infty)}$ for the indicator function over the half-line $[0, \infty)$ as well as for that over the closed convex set $K := \{u \in L^2(\Omega) \colon u \ge 0 \text{ a.e. in } \Omega\}$ in $L^2(\Omega)$, namely,

$$I_{[0,\infty)}(u) = \begin{cases} 0 & \text{if } \ u \in K, \\ \infty & \text{otherwise} \end{cases} \quad \text{for } \ u \in L^2(\Omega),$$

when no confusion can arise. Moreover, let $\partial I_{[0,\infty)}$ also denote the subdifferential operator in $\mathbb{R}$ (precisely, $\partial_{\mathbb{R}} I_{[0,\infty)}$) (see (1.3)) as well as that in $L^2(\Omega)$ (precisely, $\partial_{L^2} I_{[0,\infty)}$), that is,

$$\partial I_{[0,\infty)}(u) = \left\{\eta \in L^2(\Omega) : (\eta, u - v) \ge 0 \ \text{ for all } \ v \in K\right\}.$$

Here, we note that these two notions of subdifferentials are equivalent to each other in the following sense: for $u, \eta \in L^2(\Omega)$,

$$\eta \in \partial_{L^2} I_{[0,\infty)}(u) \quad \text{if and only if} \quad \eta(x) \in \partial_{\mathbb{R}} I_{[0,\infty)}(u(x)) \ \text{ a.e. in } \Omega$$

(see, e.g., [16]). Moreover, ϕ_{ac} stands for the *positive* least energy solution to (1.9). We denote by C a non-negative constant, which does not depend on the elements of the corresponding space or set and may vary from line to line.

2. Stationary problem

Let $u_0 \in H^1_0(\Omega)$. We are concerned with the stationary problem,

$$\psi \in H^1_0(\Omega), \quad \partial I_{[u_0(x),\infty)}(\psi) - \Delta\psi + \psi^3 - \kappa\psi \ni 0 \ \text{ in } L^2(\Omega). \tag{2.1}$$

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In particular, we address ourselves to *least energy solutions*, i.e., solutions to (2.1) achieving the minimum of the Lagrangian functional $J(\,\cdot\,; u_0)$ defined by

$$J(v; u_0) := I_{[u_0(x),\infty)}(v) + E(v) \quad \text{for} \quad v \in H^1_0(\Omega) \cap L^4(\Omega).$$

Let us start with proving existence of global minimizers of $J(\,\cdot\,; u_0)$.

LEMMA 2.1. *For each $u_0 \in H^1_0(\Omega)$, the functional $J(\,\cdot\,; u_0)$ has a global minimizer (i.e., minimizer over $H^1_0(\Omega) \cap L^4(\Omega)$).*

Proof. The existence of global minimizers of $J(\,\cdot\,; u_0)$ can be proved by the Direct Method. Indeed, the functional $v \mapsto J(v; u_0)$ is weakly lower semicontinuous in $H^1_0(\Omega) \cap L^4(\Omega)$ due to the compact embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$. Moreover, since $E(\cdot)$ is coercive in $H^1_0(\Omega) \cap L^4(\Omega)$, so is $J(\,\cdot\,; u_0)$. □

The next proposition ensures the H^2-regularity of global minimizers of $J(\,\cdot\,; u_0)$, provided that u_0 belongs to $H^2(\Omega) \cap L^6(\Omega)$ and Ω is smooth.

PROPOSITION 2.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let $\psi \in H^1_0(\Omega) \cap L^4(\Omega)$ be a global minimizer of $J(\,\cdot\,; u_0)$. If u_0 belongs to $H^2(\Omega) \cap H^1_0(\Omega) \cap L^6(\Omega)$, then ψ belongs to $H^2(\Omega) \cap L^6(\Omega)$ and the following pointwise expression holds true*:

$$(-\Delta\psi + \psi^3 - \kappa\psi)(\psi - u_0) = 0 \text{ a.e. in } \Omega, \tag{2.2}$$

$$-\Delta\psi + \psi^3 - \kappa\psi \geq 0, \quad \psi \geq u_0 \text{ a.e. in } \Omega. \tag{2.3}$$

In particular, ψ solves (2.1).

Proof. Let $\psi \in H^1_0(\Omega) \cap L^4(\Omega)$ be a global minimizer of $J(\,\cdot\,; u_0)$, that is, $\psi \in [\,\cdot \geq u_0]$ and

$$E(\psi) \leq E(z) \quad \text{for all } z \in [\,\cdot \geq u_0] := \{w \in H^1_0(\Omega) : w \geq u_0 \quad \text{a.e. in } \Omega\},$$

which implies

$$\begin{aligned}\int_\Omega \nabla z \cdot \nabla(\psi - z)\,\mathrm{d}x + \int_\Omega z^3(\psi - z)\,\mathrm{d}x &\leq \frac{1}{2}\|\nabla\psi\|_2^2 - \frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{4}\|\psi\|_4^4 - \frac{1}{4}\|z\|_4^4 \\ &\leq \frac{\kappa}{2}\left(\|\psi\|_2^2 - \|z\|_2^2\right) = \frac{\kappa}{2}(\psi + z, \psi - z)\end{aligned}$$

for all $z \in [\,\cdot \geq u_0]$. Here we also observe that

$$\begin{aligned}&\int_\Omega \nabla z \cdot \nabla(\psi - z)\,\mathrm{d}x + \int_\Omega z^3(\psi - z)\,\mathrm{d}x \\ &\quad\geq \int_\Omega \nabla\psi \cdot \nabla(\psi - z)\,\mathrm{d}x + \int_\Omega \psi^3(\psi - z)\,\mathrm{d}x \\ &\qquad - \|\nabla(\psi - z)\|_2^2 - \int_\Omega \left(\psi^3 - z^3\right)(\psi - z)\,\mathrm{d}z\end{aligned}$$

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$$\ge \int_\Omega \nabla\psi \cdot \nabla(\psi - z)\,\mathrm{d}x + \int_\Omega \psi^3(\psi - z)\,\mathrm{d}x$$
$$- \|\nabla(\psi - z)\|_2^2 - 3\int_\Omega (\psi^2 + z^2)|\psi - z|^2\,\mathrm{d}z.$$

Let $w \in [\,\cdot \ge u_0]$ and $\theta \in (0,1)$ be arbitrarily fixed and substitute $z = z_\theta := (1-\theta)\psi + \theta w \in [\,\cdot \ge u_0]$. Then dividing both sides by $\theta > 0$, we see that

$$\int_\Omega \nabla\psi \cdot \nabla(\psi - w)\,\mathrm{d}x + \int_\Omega \psi^3(\psi - w)\,\mathrm{d}x$$
$$\le \frac{\kappa}{2}(2\psi + \theta(w - \psi), \psi - w) + \theta\|\nabla(\psi - w)\|_2^2$$
$$+ 3\theta\int_\Omega (\psi^2 + z_\theta^2)|\psi - w|^2\,\mathrm{d}x.$$

Passing to the limit as $\theta \to 0_+$, we deduce that

$$\int_\Omega \nabla\psi \cdot \nabla(\psi - w)\,\mathrm{d}x + \int_\Omega \psi^3(\psi - w)\,\mathrm{d}x \le \kappa(\psi, \psi - w) \tag{2.4}$$

for any $w \in [\,\cdot \ge u_0]$. On the other hand, setting $f := \kappa\psi - \psi^3$ and recalling (2.4), one can rewrite the inequality above as

$$\int_\Omega \nabla\psi \cdot \nabla(\psi - w)\,\mathrm{d}x \le \int_\Omega f(\psi - w)\,\mathrm{d}x \tag{2.5}$$

for any $w \in [\,\cdot \ge u_0]$. Now, we are ready to apply the regularity theory for variational inequalities of obstacle type (see [33], [30] and [5, §3]). Then since $f = \kappa\psi - \psi^3 \in L^{4/3}(\Omega)$ and $u_0 \in H^2(\Omega)$, we deduce that ψ belongs to $W^{2,4/3}(\Omega)$ and the following pointwise expression holds true:

$$(-\Delta\psi - f)(\psi - u_0) = 0, \quad -\Delta\psi \ge f, \quad \psi \ge u_0 \ \text{ a.e. in } \Omega,$$

which implies (2.2) and (2.3) by the relation $f = \kappa\psi - \psi^3$. Hence we can derive

$$\partial I_{[u_0(x),\infty)}(\psi) - \Delta\psi + \psi^3 - \kappa\psi \ni 0 \ \text{ in } L^{4/3}(\Omega).$$

Now set $v = \psi - u_0 \ge 0$. Then v solves

$$\partial I_{[0,\infty)}(v) - \Delta v + v^3 \ni \Delta u_0 - 3v^2 u_0 - 3v u_0^2 - u_0^3 + \kappa(u_0 + v). \tag{2.6}$$

Let $\alpha_n : [0,\infty) \to [0,\infty)$ be a non-negative bounded increasing function of class C^2 satisfying

$$\alpha_n(s) = \begin{cases} (n+1)^3 & \text{if } \ s \ge n+2, \\ s^3 & \text{if } \ 0 \le s \le n, \end{cases} \qquad 0 \le \alpha_n(s) \le s^3 \ \text{ for } \ s \ge 0$$

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for each $n \in \mathbb{N}$ and test (2.6) by $\alpha_n(v) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then since $\eta\alpha_n(v) = 0$ if $\eta \in \partial I_{[0,\infty)}(v)$, using Young's inequality, we have

$$\begin{aligned}
&\int_\Omega \nabla v \cdot \nabla \alpha_n(v)\,\mathrm{d}x + \int_\Omega v^3 \alpha_n(v)\,\mathrm{d}x \\
&\quad = \int_\Omega (\Delta u_0)\alpha_n(v)\,\mathrm{d}x - 3\int_\Omega v^2 \alpha_n(v) u_0\,\mathrm{d}x - 3\int_\Omega v\alpha_n(v)u_0^2 \\
&\qquad - \int_\Omega \alpha_n(v)u_0^3 + \kappa \int_\Omega \alpha_n(v)(u_0+v)\,\mathrm{d}x \\
&\quad \le \int_\Omega (\Delta u_0)\alpha_n(v)\,\mathrm{d}x + \int_\Omega \left(\frac{1}{2}v^3 + C|u_0|^3\right)\alpha_n(v)\,\mathrm{d}x \\
&\qquad + \int_\Omega |u_0|^3\alpha_n(v) + \kappa\int_\Omega |v|^3(|u_0|+|v|)\,\mathrm{d}x \\
&\quad \le \|\Delta u_0\|_2 \|\alpha_n(v)\|_2 + \frac{1}{2}\int_\Omega v^3\alpha_n(v)\,\mathrm{d}x + C\int_\Omega |u_0|^3\alpha_n(v)\,\mathrm{d}x \\
&\qquad + \kappa\left(\|v\|_4^3\|u_0\|_4 + \|v\|_4^4\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_\Omega \nabla v \cdot \nabla\alpha_n(v)\,\mathrm{d}x + \frac{1}{2}\int_\Omega v^3\alpha_n(v)\,\mathrm{d}x \\
&\quad \le \|\Delta u_0\|_2\|\alpha_n(v)\|_2 + C\|u_0\|_6^3\|\alpha_n(v)\|_2 + \kappa\left(\|v\|_4^3\|u_0\|_4 + \|v\|_4^4\right).
\end{aligned}$$

Here we note that

$$\int_\Omega \nabla v\cdot\nabla\alpha_n(v)\,\mathrm{d}x \ge 0 \quad \text{and} \quad \int_\Omega v^3\alpha_n(v)\,\mathrm{d}x \ge \|\alpha_n(v)\|_2^2.$$

Therefore

$$\|\alpha_n(v)\|_2^2 \le C\left[\|\Delta u_0\|_2^2 + \|u_0\|_6^6 + \kappa\left(\|v\|_4^3\|u_0\|_4 + \|v\|_4^4\right)\right] < \infty,$$

which implies

$$\alpha_n(v) \to v^3 \quad \text{weakly in } L^2(\Omega).$$

In particular, we deduce that $v \in L^6(\Omega)$. Recalling that $v = \psi - u_0$ and $u_0 \in L^6(\Omega)$, we obtain $\psi \in L^6(\Omega)$. Now, going back to (2.5) and combining it with the improved regularity $f = \kappa\psi - \psi^3 \in L^2(\Omega)$, we deduce that $\psi \in H^2(\Omega)$ from the regularity result for obstacle problems. The proof is completed. □

REMARK 2.3. Let ψ be a global minimizer of $J(\,\cdot\,; u_0)$ again. In a similar fashion, one can also prove $\psi \in W^{2,r}(\Omega) \cap L^{3r}(\Omega)$ for $r \in (2,\infty)$, provided that $u_0 \in W^{2,r}(\Omega) \cap H_0^1(\Omega) \cap L^{3r}(\Omega)$. Hence, for $r > N$, the Hölder regularity $\psi \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - N/r \in (0,1)$ also follows. On the other hand, the $C^{1,1}$-regularity of ψ still seems open (cf. see, e.g., [19]).

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3. Lyapunov stability of equilibria

We are concerned with *strong solutions* for (P) defined in the following (see [2, Definition 3.1]):

DEFINITION 3.1. (Strong solution [2]) *A function $u \in C([0,\infty); L^2(\Omega))$ is said to be a solution (or a strong solution) of the Cauchy–Dirichlet problem* (P)*, if the following conditions are all satisfied*:

(i) *u belongs to $W^{1,2}(0,T;L^2(\Omega))$, $C([0,T];H_0^1(\Omega)\cap L^4(\Omega))$, $L^2(0,T;H^2(\Omega))$ and $L^6(0,T;L^6(\Omega))$ for any $0 < T < \infty$;*
(ii) *there exists $\eta \in L^\infty(0,\infty;L^2(\Omega))$ such that*

$$u_t + \eta - \Delta u + u^3 - \kappa u = 0, \quad \eta \in \partial I_{[0,\infty)}(u_t) \quad \text{for a.e. } (x,t) \in \Omega \times (0,\infty)$$

and $\eta = -\big(\Delta u - u^3 + \kappa u\big)_-$ for a.e. $(x,t) \in \Omega \times (0,\infty)$. Hence u also solves (1.1) *a.e. in $\Omega \times (0,\infty)$;*
(iii) *$u(\cdot,0) = u_0$ a.e. in Ω.*

We further recall the following (see [2, Theorems 3.2, 5.1, and 6.1]):

THEOREM 3.2 (Well-posedness [2]). *Let $r > 0$ be arbitrarily fixed and let u_0 belong to the closure $\overline{D_r}^{H_0^1\cap L^4}$ of D_r in $H_0^1(\Omega)\cap L^4(\Omega)$. Then the Cauchy–Dirichlet problem* (P) *admits the unique strong solution $u = u(x,t)$ which solves* (1.7) *a.e. in $\Omega \times (0,\infty)$ as well. Moreover, $u(t) = u(\cdot,t)$ lies on $\overline{D_r}^{H_0^1\cap L^4}$ for any $t \geq 0$.*

We set

$$X = H_0^1(\Omega) \cap L^4(\Omega)$$

equipped with the norm $\|\cdot\|_X := \|\nabla\cdot\|_2 + \|\cdot\|_4$. Fix $r > 0$ arbitrarily and let

$$D = \overline{D}_r,$$

which is the closure of D_r in the strong topology of X. Since D is an invariant set under the evolution of strong solutions to (1.4)–(1.6), one can define a DS generated by (1.4)–(1.6) on the phase set D.

Now, let us define notions of Lyapunov stability and instability of equilibria $\psi \in \mathcal{VI}$ in the following sense:

DEFINITION 3.3. (Lyapunov stability of $\psi \in \mathcal{VI}$). *Let $\psi \in \mathcal{VI}$.*

(i) *ψ is said to be stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any strong solution $u = u(x,t)$ of* (1.4) *and* (1.5)*, it holds that*

$$\sup_{t\geq 0} \|u(t) - \psi\|_X < \varepsilon,$$

whenever $u(0) \in D$ and $\|u(0) - \psi\|_X < \delta$;

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(ii) *ψ is said to be unstable, if ψ is not stable*;

(iii) *ψ is said to be asymptotically stable, if ψ is stable and there exists $\bar{\delta} > 0$ such that for any strong solution $u = u(x, t)$ of* (1.4) *and* (1.5), *it holds that*

$$\lim_{t\to\infty} \|u(t) - \psi\|_X = 0,$$

whenever $u(0) \in D$ and $\|u(0) - \psi\|_X < \bar{\delta}$.

Before stating main results of this section, let us briefly recall some well-known facts for the classical (elliptic) Allen–Cahn equation (1.9): Let d be the infimum of $E(\cdot)$ over X. Since E is coercive and smooth in X, by the Direct Method, one can verify that E has a global minimizer $\phi_{ac} \in X$, which solves the Euler–Lagrange equation (1.9). Moreover, noting that $d < 0$, we observe that $\phi_{ac} \neq 0$. Since $|\phi_{ac}|$ also minimizes E over X, we find that ϕ_{ac} is sign-definite in Ω from the strong maximum principle. The uniqueness of positive solutions to (1.9) can be proved as in [18]. Thus the infimum d is achieved by the two sign-definite solutions $\pm\phi_{ac}$ to (1.9) only. Furthermore, $\pm\phi_{ac}$ is isolated in X from the other critical points of $E(\cdot)$. Indeed, as in [3, Lemma 4], one can verify that any sign-changing solutions to (1.9) are isolated in X from the sign-definite solutions due to the strong maximum principle. In what follows, for simplicity, we shall use the following notation:

$$\begin{aligned}[E \leq a] &:= \{w \in X : E(w) \leq a\},\\ B(u; r) &:= \{w \in X : \|w - u\|_X < r\}\end{aligned}$$

for $a \in \mathbb{R}$, $r > 0$ and $u \in X$.

Lemma 3.4. (Geometry of $E(\cdot)$)

(i) *For all $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $B(\phi_{ac}; r_\varepsilon) \cup B(-\phi_{ac}; r_\varepsilon) \subset [E \leq d + \varepsilon]$.*

(ii) *For all $r > 0$, there exists $\varepsilon_r > 0$ such that $[E \leq d+\varepsilon_r] \subset B(\phi_{ac}; r) \cup B(-\phi_{ac}; r)$.*

Proof. This lemma may be standard (cf. see [3, 4, 31]), but we give a proof for the completeness. Assertion (i) follows immediately from the continuity of $E(\cdot)$ in X. As for the assertion (ii), suppose to the contrary that there exist a number $r_0 > 0$ and a sequence (u_n) in X such that $E(u_n) \leq d + 1/n$ and $u_n \notin B(\phi_{ac}; r_0) \cup B(-\phi_{ac}; r_0)$ for $n \in \mathbb{N}$. Since E is coercive in X, (u_n) is bounded in X. Hence, up to a (not relabelled) subsequence, one can verify that $u_n \to u$ weakly in X and strongly in $L^2(\Omega)$ for some $u \in X$. Moreover, we find that

$$\begin{aligned}\frac{1}{2}\limsup_{n\to\infty}\|\nabla u_n\|_2^2 &\le \lim_{n\to\infty} E(u_n) - \frac{1}{4}\liminf_{n\to\infty}\|u_n\|_4^4 + \frac{\kappa}{2}\lim_{n\to\infty}\|u_n\|_2^2 \\ &\le d - \frac{1}{4}\|u\|_4^4 + \frac{\kappa}{2}\|u\|_2^2 \\ &\le E(u) - \frac{1}{4}\|u\|_4^4 + \frac{\kappa}{2}\|u\|_2^2 = \frac{1}{2}\|\nabla u\|_2^2,\end{aligned}$$

which along with the uniform convexity of $\|\nabla\cdot\|_2$ yields $u_n \to u$ strongly in $H_0^1(\Omega)$. We can also similarly prove the strong convergence of (u_n) in $L^4(\Omega)$. Thus, we obtain $u_n \to u$ strongly in X and $E(u) = d$. Since E has only two global minimizers $\pm\phi_{ac}$, the limit u must coincide with either of them; however, it is a contradiction to the assumption $u_n \notin B(\phi_{ac}; r_0) \cup B(-\phi_{ac}; r_0)$ for $n \in \mathbb{N}$. Thus (ii) follows. □

In what follows, we denote by ϕ_{ac} the *positive* least energy solution of (1.9). Here we note that ϕ_{ac} also belongs to $\mathcal{VI}$, e.g., $\phi_{ac} \in \mathcal{VI}(u_0)$ with $u_0 \equiv 0$. Let us start with the following simple observation:

Proposition 3.5. *The least energy solutions $\pm\phi_{ac}$ of* (1.9) *are stable in the sense of Definition* 3.3 *equations* (1.4). *However, $\pm\phi_{ac}$ is never asymptotically stable in the sense of Definition* 3.3.

Proof. This proposition can be proved as in [3, 4, 31], but we give a proof for the convenience of the reader. Let $r := \|(-\phi_{ac}) - \phi_{ac}\|_X > 0$. For any $\varepsilon \in (0, r)$, we claim that

$$c_\varepsilon := \inf\{E(w)\colon w \in X,\ \|w - \phi_{ac}\|_X = \varepsilon\} > d. \tag{3.1}$$

Indeed, suppose to the contrary that there exists a sequence (w_n) in X such that $\|w_n - \phi_{ac}\|_X = \varepsilon$ and $E(w_n) \to d$. Since E is coercive in X, one can take a (not relabelled) subsequence of (n) and $w_\infty \in X$ such that $w_n \to w_\infty$ strongly in X as in the proof of (ii) of Lemma 3.4. Hence, we have $E(w_\infty) = d$ (hence $w_\infty \in \{\pm\phi_{ac}\}$) and $\|w_\infty - \phi_{ac}\|_X = \varepsilon \in (0, r)$, which however contradicts $w_\infty \in \{\pm\phi_{ac}\}$. Thus, (3.1) follows. From (i) of Lemma 3.4, one can take $\delta \in (0, \varepsilon)$ small enough so that $E(w) < c_\varepsilon$ for $w \in B(\phi_{ac}; \delta)$. Let $u_0 \in B(\phi_{ac}; \delta)$ and let $u = u(x, t)$ be the strong solution to (1.4)–(1.6) with the initial datum u_0. Then, we observe that $E(u(t)) \le E(u_0) < c_\varepsilon$ for $t \ge 0$. We claim that $u(t) \in B(\phi_{ac}; \varepsilon)$ for $t \ge 0$. Indeed, if $u(t_0) \in \partial B(\phi_{ac}; \varepsilon)$ for some $t_0 > 0$, then $E(u(t_0)) \ge c_\varepsilon$, which is a contradiction. Therefore ϕ_{ac} turns out to be stable. As for the second half of the assertion, see the following remark. □

Remark 3.6. Comparison to the Allen–Cahn equation Concerning the usual Allen–Cahn equation, two global minimizers of E are asymptotically stable in Lyapunov's sense. However, as for the strongly irreversible Allen–Cahn equation, ϕ_{ac} is never asymptotically stable; indeed, if we take an initial data u_0 which is sufficiently close to ϕ_{ac} in X but slightly greater than ϕ_{ac} on a subset of Ω, then

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$u(t)$ never converges to ϕ_{ac} as $t \to \infty$ due to the presence of the obstacle function u_0 and the non-decrease of $t \mapsto u(x,t)$.

The main result of the present article is stated as follows:

THEOREM 3.7 (Lyapunov stability of equilibria close to $\pm\phi_{ac}$) *Assume that $N \le 4$. Let $\psi \in \mathcal{VI}$ be such that $\psi \in B(\phi_{ac};\delta_0) \cup B(-\phi_{ac};\delta_0)$ for some $\delta_0 > 0$ sufficiently small. Then ψ is stable in the sense of Definition 3.3.*

In order to prove Theorem 3.7, the assumption $N \le 4$ will be used only for deriving the continuous embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, which enables us to assure that the functional E is well defined on the Hilbert space $H_0^1(\Omega)$. Indeed, the Hilbert structure of the domain for E will essentially be used in the proofs of Lemmas 4.1 and 4.5.

4. Proof of Theorem 3.7

We shall prove the assertion for $\psi \in \mathcal{VI}$ lying in a small neighbourhood of ϕ_{ac} only. However, it can also be proved for $-\phi_{ac}$ in the same manner. Let us start with verifying the strict convexity of $E(\cdot)$ in a small neighbourhood of ϕ_{ac}.

LEMMA 4.1. *The functional $v \mapsto E(v)$ is strictly convex in a small neighbourhood $B(\phi_{ac};\delta_0)$ of ϕ_{ac}.*

Proof. The linearized operator $\mathcal{L}_{\phi_{ac}} : D(\mathcal{L}_{\phi_{ac}}) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ given by

$$\mathcal{L}_{\phi_{ac}}(u) := -\Delta u + 3\phi_{ac}^2 u - \kappa u \quad \text{for} \quad u \in H_0^1(\Omega)$$

is self-adjoint and has a compact resolvent. Hence $\mathcal{L}_{\phi_{ac}}$ possesses a sequence (λ_j, e_j) of eigenpairs such that $\lambda_j \nearrow \infty$ as $j \to \infty$ and (e_j) forms a complete orthonormal system (CONS for short) of $L^2(\Omega)$ (as well as a CONS of $H_0^1(\Omega)$ with different normalization). Moreover, the principal eigenvalue λ_1 of $\mathcal{L}_{\phi_{ac}}$ is positive; indeed, let (e_1, λ_1) be the principal eigenpair of $\mathcal{L}_{\phi_{ac}}$. Test $\mathcal{L}_{\phi_{ac}}(e_1) = \lambda_1 e_1$ by ϕ_{ac} and integrate by parts to observe that

$$(-\Delta\phi_{ac}, e_1) + 3\int_\Omega \phi_{ac}^3 e_1 \, \mathrm{d}x - \kappa(\phi_{ac}, e_1) = \lambda_1(\phi_{ac}, e_1),$$

which along with (1.9) gives

$$2\int_\Omega \phi_{ac}^3 e_1 \, \mathrm{d}x = \lambda_1 \int_\Omega \phi_{ac} e_1 \, \mathrm{d}x.$$

Since the principal eigenfunction e_1 and the least energy solution ϕ_{ac} of (1.9) are sign-definite, we obtain $\lambda_1 > 0$ (see, e.g., [25, §6.5.2], [17, §9.8] for properties of e_1). Therefore, we have

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$$\langle E''(\phi_{ac})(u), u\rangle \geq \lambda_1 \|u\|_2^2 \quad \text{for all} \;\; u \in H_0^1(\Omega),$$

where $E'' : H_0^1(\Omega) \to \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ denotes the second (Fréchet) derivative of E and which also implies

$$\begin{aligned}\langle [E''(\phi_{ac})](u), u\rangle &= \theta\left(\|\nabla u\|_2^2 + 3\int_\Omega \phi_{ac}^2 u^2 \,\mathrm{d}x - \kappa\|u\|_2^2\right) + (1-\theta)\langle E''(\phi_{ac})(u), u\rangle \\ &\geq \theta\|\nabla u\|_2^2 + \{(1-\theta)\lambda_1 - \kappa\theta\}\|u\|_2^2 \geq \theta\|\nabla u\|_2^2 \quad \text{for all} \;\; u \in H_0^1(\Omega)\end{aligned}$$

by choosing $\theta > 0$ small enough. Now, let $z \in B(\phi_{ac}; \delta_0)$, where $\delta_0 > 0$ will be chosen below. Since $E(\cdot)$ is obviously of class C^3 in $H_0^1(\Omega)$ from $N \leq 4$, one observes that

$$\begin{aligned}\langle E''(z)u, u\rangle &\geq \langle [E''(\phi_{ac})](u), u\rangle \\ &\quad - \sup_{w\in B(\phi_{ac};\delta_0)} \|E^{(3)}(w)\|_{\mathcal{L}(H_0^1(\Omega),\mathcal{L}(H_0^1(\Omega),H^{-1}(\Omega)))}\|\nabla z - \nabla\phi_{ac}\|_2\|\nabla u\|_2^2 \\ &\geq \left(\theta - \sup_{w\in B(\phi_{ac};\delta_0)} \|E^{(3)}(w)\|_{\mathcal{L}(H_0^1(\Omega),\mathcal{L}(H_0^1(\Omega),H^{-1}(\Omega)))}\delta_0\right)\|\nabla u\|_2^2 \\ &\geq \frac{\theta}{2}\|\nabla u\|_2^2 \quad \text{for} \;\; u \in H_0^1(\Omega),\end{aligned}$$

where $E^{(3)} : H_0^1(\Omega) \to \mathcal{L}(H_0^1(\Omega), \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega)))$ stands for the third-order (Fréchet) derivative of E, by choosing $\delta_0 > 0$ small enough. Thus, $E(\cdot)$ is strictly convex in the neighbourhood $B(\phi_{ac}; \delta_0)$. □

One can assume $\delta_0 > 0$ small enough so that

$$B(\phi_{ac}; \delta_0) \cap B(-\phi_{ac}; \delta_0) = \emptyset \tag{4.1}$$

without any loss of generality. Due to (ii) of Lemma 3.4, there exists $\varepsilon_0 > 0$ such that

$$[E \leq d + \varepsilon_0] \subset B(\phi_{ac}; \delta_0/2) \cup B(-\phi_{ac}; \delta_0/2). \tag{4.2}$$

Moreover, by (i) of Lemma 3.4, one can take $r_0 \in (0, \delta_0/2)$ such that

$$B(\phi_{ac}; r_0) \cup B(-\phi_{ac}; r_0) \subset [E \leq d + \varepsilon_0] \tag{4.3}$$

(cf. see [3, 4]).

Lemma 4.2. *Let $u = u(x,t)$ be a solution of (1.4)–(1.6). If u_0 lies on $B(\phi_{ac}; r_0)$, then $u(t)$ stays on $B(\phi_{ac}; \delta_0/2)$ for all $t \geq 0$.*

Proof. From the fact that $u_0 \in B(\phi_{ac}; r_0)$, we find by (4.3) that $u_0 \in [E \leq d + \varepsilon_0]$. Hence $u(t)$ lies on $B(\phi_{ac}; \delta_0/2) \cup B(-\phi_{ac}; \delta_0/2)$ by (4.2), since $E(u(t)) \leq E(u_0) \leq d + \varepsilon_0$ for all $t \geq 0$. We deduce by (4.1) that $u(t) \in B(\phi_{ac}; \delta_0/2)$ for all $t \geq 0$. □

The following lemma is concerned with existence and uniqueness of (local) minimizers of the functional $v \mapsto J(v; u_0)$.

LEMMA 4.3. *If $u_0 \in B(\phi_{ac};\delta_0)$, then $J(\,\cdot\,;u_0)$ has a unique minimizer over $\overline{B(\phi_{ac};\delta_0)}$.*

Proof. As in Lemma 2.1, one can prove the existence of a minimizer of $J(\,\cdot\,;u_0)$ over $\overline{B(\phi_{ac};\delta_0)}$ by employing the direct method. It remains to prove uniqueness. Suppose that there exist two different minimizers $u_1,\, u_2 \in \overline{B(\phi_{ac};\delta_0)}$ of $J(\,\cdot\,;u_0)$ over $\overline{B(\phi_{ac};\delta_0)}$. Then the convex combination $u_\theta := (1-\theta)u_1 + \theta u_2 \in \overline{B(\phi_{ac};\delta_0)}$ belongs to the closed convex set $[\,\cdot\, \geq u_0] := \{v \in H^1_0(\Omega) \colon v \geq u_0 \text{ a.e. in } \Omega\}$. Moreover, by the strict convexity of $E(\cdot)$ in $B(\phi_{ac};\delta_0)$ (recall Lemma 4.1) and by $u_1 \neq u_2$, it follows that

$$
\begin{aligned}
J(u_\theta;u_0) = E(u_\theta) &< (1-\theta)E(u_1) + \theta E(u_2) \\
&= (1-\theta)J(u_1;u_0) + \theta J(u_2;u_0) = \inf_{v\in B(\phi_{ac};\delta_0)} J(v;u_0),
\end{aligned}
$$

which contradicts the assumption that u_1, u_2 minimize $J(\,\cdot\,;u_0)$ over $\overline{B(\phi_{ac};\delta_0)}$. □

The next lemma provides a relation between solutions to (2.1) and (local) minimizers of $J(\,\cdot\,;u_0)$ for $u_0 \in H^1_0(\Omega)$.

LEMMA 4.4. *Let z be a solution of (2.1) with the obstacle $u_0 \in H^1_0(\Omega)$ such that $z \in B(\phi_{ac};\delta_0)$. Then z minimizes $J(\,\cdot\,;u_0)$ over $B(\phi_{ac};\delta_0)$.*

Proof. Define a convex functional $\varphi : H^1_0(\Omega) \to \mathbb{R}$ by

$$
\varphi(w) := \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{4}\|w\|_4^4 \quad \text{for} \;\; w \in H^1_0(\Omega).
$$

By (2.1) and the definition of subdifferential, it follows that

$$
\begin{aligned}
&I_{[u_0(x),\infty)}(v) + \varphi(v) - I_{[u_0(x),\infty)}(z) - \varphi(z) \\
&\quad \geq \kappa(z, v-z) = \frac{\kappa}{2}\|v\|_2^2 - \frac{\kappa}{2}\|z\|_2^2 - \frac{\kappa}{2}\|v-z\|_2^2
\end{aligned}
$$

for all $v \in L^2(\Omega)$, that is,

$$
J(v;u_0) - J(z;u_0) \geq -\frac{\kappa}{2}\|v-z\|_2^2 \quad \text{for all} \;\; v \in L^2(\Omega). \tag{4.4}
$$

By Lemma 4.1, the functional $J(\,\cdot\,;u_0)$ is (strictly) convex in $B(\phi_{ac};\delta_0)$, and hence, let $w \in B(\phi_{ac};\delta_0)$, $\theta \in (0,1)$ and substitute $v = (1-\theta)z + \theta w \in B(\phi_{ac};\delta_0)$ to (4.4). Then we see that

$$
J(w;u_0) - J(z;u_0) \geq -\frac{\kappa\theta}{2}\|w-z\|_2^2 \quad \text{for all} \;\; w \in B(\phi_{ac};\delta_0).
$$

Letting $\theta \to 0$, we deduce that z minimizes $J(\,\cdot\,;u_0)$ over $B(\phi_{ac};\delta_0)$. □

Moreover, we have

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Lemma 4.5. *If $u_{0,n} \to u_0$ strongly in $H^1_0(\Omega)$ and $u_0 \in B(\phi_{ac};\delta_0)$, then each sequence (ψ_n) of minimizers of $J(\,\cdot\,;u_{0,n})$ over $\overline{B(\phi_{ac};\delta_0)}$ converges, up to a subsequence, to the minimizer ψ of $J(\,\cdot\,;u_0)$ over $\overline{B(\phi_{ac};\delta_0)}$ strongly in $H^1_0(\Omega)$.*

Proof. Assume that $u_{0,n} \to u_0$ strongly in $H^1_0(\Omega)$ and denote $K_n := [\,\cdot \geq u_{0,n}] \cap \overline{B(\phi_{ac};\delta_0)}$ and $K := [\,\cdot \geq u_0] \cap \overline{B(\phi_{ac};\delta_0)}$. Then we claim that

$$K_n \to K \text{ on } H^1_0(\Omega) \text{ in the sense of Mosco as } \; n \to +\infty$$

(in other words, I_{K_n} converges to I_K on $H^1_0(\Omega)$ in the sense of Mosco, see, e.g., [11]), more precisely, it holds that

(i) (*Existence of strong-recovery sequences*) For any $u \in K$, there exists a sequence (u_n) such that $u_n \in K_n$ for all $n \in \mathbb{N}$ and $u_n \to u$ strongly in $H^1_0(\Omega)$.
(ii) (*Weak-liminf condition*) Let $u \in H^1_0(\Omega)$ and let (u_n) be a sequence such that $u_n \in K_n$ for all $n \in \mathbb{N}$ and $u_n \to u$ weakly in $H^1_0(\Omega)$. Then u belongs to K.

We first prove (i). Let us $u \in K$. If $u = u_0$, one can take $u_n := u_{0,n} \in K_n$ to satisfy the desired property of (i). Hence we assume $u \neq u_0$. We set $w_n := u - u_0 + u_{0,n}$. Then $w_n \in [\,\cdot \geq u_{0,n}]$ and $w_n \to u$ strongly in $H^1_0(\Omega)$. In case $\|u - \phi_{ac}\|_{H^1_0(\Omega)} < \delta_0$, for $n \in \mathbb{N}$ large enough, $u_n := w_n$ belongs to K_n. In case $\|u - \phi_{ac}\|_{H^1_0(\Omega)} = \delta_0$, we set $u_n := \theta_n w_n + (1-\theta_n)u_{0,n} \in K_n$, where $\theta_n \in [0,1]$ is determined as follows: If $\|w_n - \phi_{ac}\|_{H^1_0(\Omega)} \leq \delta_0$, then we set $\theta_n = 1$ (and hence, $u_n = w_n$). Otherwise, we take $\theta_n \in [0,1)$ such that $\|u_n - \phi_{ac}\|_{H^1_0(\Omega)} = \delta_0$ (indeed, it is possible since $u_{0,n} \in B(\phi_{ac};\delta_0)$ for n large enough), namely, we have

$$\delta_0^2 = \|u_n - \phi_{ac}\|^2_{H^1_0(\Omega)} = \|u_{0,n} - u + \theta_n(u - u_0) + u - \phi_{ac}\|^2_{H^1_0(\Omega)}. \tag{4.5}$$

By $\theta_n \in [0,1]$, up to a (not relabelled) subsequence, it holds that $\theta_n \to \theta \in [0,1]$. In case $\theta = 1$, we obtain

$$u_n \to u \quad \text{strongly in } H^1_0(\Omega).$$

In case $\theta < 1$, we see that

$$0 = (1-\theta)\|u_0 - u\|^2_{H^1_0(\Omega)} - 2(u_0 - u, \phi_{ac} - u)_{H^1_0(\Omega)} \tag{4.6}$$

from the fact that $\|u - \phi\|_{H^1_0(\Omega)} = \delta_0$ as well as (4.5). We also note that[1.]

$$2\left(\frac{u_0 - u}{\|u_0 - u\|_{H^1_0(\Omega)}}, \phi_{ac} - u\right)_{H^1_0(\Omega)} \geq \|u - u_0\|_{H^1_0(\Omega)}, \tag{4.7}$$

[1.] Indeed, since $u_0 \in B(\phi_{ac};\delta_0)$ and $u \in \partial B(\phi_{ac};\delta_0)$, one can take $k > 1$ such that $\tilde{u} := u + k(u_0 - u) \in \partial B(\phi_{ac};\delta_0)$. Moreover, we note that $\|\tilde{u} - u\|_{H^1_0(\Omega)} = 2((\tilde{u} - u)/\|\tilde{u} - u\|_{H^1_0(\Omega)}, \phi_{ac} - u)_{H^1_0(\Omega)}$ from the parallelogram law in $H^1_0(\Omega)$. Thus, (4.7) follows from $\tilde{u} - u = k(u_0 - u)$ and $k > 1$.

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which along with (4.6) yields $\theta=0$. Therefore, $\lim_{n\to\infty}(u_n - u_{0,n}) = \lim_{n\to\infty}\theta_n(w_n - u_{0,n}) = 0$, and hence, $\delta_0 = \|u_0 - \phi_{ac}\|_{H_0^1(\Omega)}$. However, it is a contradiction to the fact $u_0 \in B(\phi_{ac};\delta_0)$. Hence we conclude that $\theta=1$, and hence, $u_n \to u_0$ strongly in $H_0^1(\Omega)$. As for (ii), if $u_n \in K_n$ and $u_n \to u$ weakly in $H_0^1(\Omega)$, then $\|u - \phi_{ac}\|_{H_0^1(\Omega)} \le \delta_0$, and moreover, by the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that $u_n \to u$ strongly in $L^2(\Omega)$, and therefore, $u \ge u_0$ a.e. in Ω.

By using the Mosco convergence, we shall prove the convergence of minimizers ψ_n and identify the limit. One first finds that (ψ_n) is bounded in $H_0^1(\Omega)$. Therefore $\psi_n \to \psi$ weakly in $H_0^1(\Omega)$ up to a subsequence. Due to the weak-liminf condition (see (ii) above), it follows that $\psi \in K$. Noting that

$$J(\psi_n;u_{0,n}) \le J(v;u_{0,n}) \quad \text{for all} \quad v \in H_0^1(\Omega), \tag{4.8}$$

we can deduce that

$$J(\psi;u_0) \le J(v;u_0) \quad \text{for all} \quad v \in H_0^1(\Omega).$$

Indeed, for any $v \in K$, one can take a strong-recovery sequence (v_n) in K_n such that $v_n \to v$ strongly in $H_0^1(\Omega)$ (see (i) above). Substitute $v = u_{0,n}$ to (4.8). Then by letting $n \to \infty$, the right-hand side converges as follows:

$$\lim_{n\to\infty} J(v_n;u_{0,n}) = \lim_{n\to\infty} E(v_n) = E(v) = J(v;u_0).$$

On the other hand,

$$\liminf_{n\to\infty} J(\psi_n;u_{0,n}) = \liminf_{n\to\infty} E(\psi_n) \ge E(\psi) = J(\psi;u_0)$$

due to the weak lower-semicontinuity of $E(\cdot)$ in $H_0^1(\Omega)$. Thus $J(\psi;u_0) \le J(v;u_0)$ for all $v \in K$. We finally prove the strong convergence of (ψ_n) in $H_0^1(\Omega)$ as $n \to \infty$. Let $(\hat{\psi}_n)$ be a recovery sequence in $H_0^1(\Omega)$ of ψ (hence, $\widehat{\psi}_n \in K_n$ and $\hat{\psi}_n \to \psi$ strongly in $H_0^1(\Omega)$). Then

$$\begin{aligned}\frac{1}{2}\limsup_{n\to\infty}\|\nabla\psi_n\|_2^2 &\le \limsup_{n\to\infty} J(\psi_n;u_{0,n}) - \frac{1}{4}\liminf_{n\to\infty}\|\psi_n\|_4^4 + \frac{\kappa}{2}\lim_{n\to\infty}\|\psi_n\|_2^2 \\ &\le \limsup_{n\to\infty} J(\hat{\psi}_n;u_{0,n}) - \frac{1}{4}\|\psi\|_4^4 + \frac{\kappa}{2}\|\psi\|_2^2 \\ &\le J(\psi;u_0) - \frac{1}{4}\|\psi\|_4^4 + \frac{\kappa}{2}\|\psi\|_2^2 = \frac{1}{2}\|\nabla\psi\|_2^2,\end{aligned}$$

which together with the uniform convexity of $H_0^1(\Omega)$ ensures that $\psi_n \to \psi$ strongly in $H_0^1(\Omega)$ as $n \to \infty$. □

Furthermore, we observe that

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LEMMA 4.6. *Let $u_0 \in H^1_0(\Omega)$ and let ψ be a minimizer of $J(\,\cdot\,; u_0)$ over $\overline{B(\phi_{ac}; \delta_0)}$. Let $[u_0, \psi] := \{v \in H^1_0(\Omega) \colon u_0 \le v \le \psi$ a.e. in $\Omega\}$. Then ψ also minimizes $J(\,\cdot\,; v)$ over $\overline{B(\phi_{ac}; \delta_0)}$ for any $v \in [u_0, \psi]$. In particular, it holds that*

$$\psi = \arg\min\{J(v; \psi) \colon v \in \overline{B(\phi_{ac}; \delta_0)}\}.$$

Proof. We observe that

$$J(\psi; v) = E(\psi) = J(\psi; u_0) \le J(u; u_0) = E(u)$$

for all $u \in [\,\cdot \ge u_0] \cap \overline{B(\phi_{ac}; \delta_0)}$. In particular, for any $v \in [u_0, \psi]$ and $u \in [\cdot \ge v] \cap \overline{B(\phi_{ac}; \delta_0)}$, it holds that

$$J(\psi; v) \le E(u) = J(u; v).$$

Therefore, ψ also minimizes $J(\,\cdot\,; v)$ over $\overline{B(\phi_{ac}; \delta_0)}$. □

LEMMA 4.7. *Let $\psi \in \mathcal{VI}$ be such that $\psi \in B(\phi_{ac}; r_0)$. For any $\varepsilon > 0$, there exists a constant $\delta \in (0, r_0)$ such that for each $u_0 \in B(\psi; \delta)$, the minimizer $\hat\psi$ of $J(\,\cdot\,; u_0)$ over $\overline{B(\phi_{ac}; \delta_0)}$ belongs to $B(\psi; \varepsilon)$.*

Proof. By assumption, ψ is a solution of (2.1) with some $u_0 \in D$. Then, ψ also solves (2.1) with u_0 replaced by ψ; indeed, $\partial I_{[u_0(x),\infty)}(\psi(x)) \subset (-\infty, 0] = \partial I_{[\psi(x),\infty)}(\psi(x))$. Hence by $\psi \in B(\phi_{ac}; r_0)$ and Lemma 4.4, ψ turns out to be the (unique) minimizer of $J(\,\cdot\,; \psi)$ over $\overline{B(\phi_{ac}; \delta_0)}$, and it lies on $B(\phi_{ac}; \delta_0/2)$. We prove the assertion by contradiction. Suppose to the contrary that there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$ one can take $u_{0,n} \in B(\psi; 1/n)$ so that minimizers $\hat\psi_n$ of $J(\,\cdot\,; u_{0,n})$ over $\overline{B(\phi_{ac}; \delta_0)}$ do not belong to $B(\psi; \varepsilon_0)$. Then $u_{0,n} \to \psi$ strongly in $H^1_0(\Omega)$. By Lemma 4.5, up to a subsequence, $\hat\psi_n$ converges strongly in $H^1_0(\Omega)$ to a minimizer of $J(\,\cdot\,; \psi)$ over $\overline{B(\phi_{ac}; \delta_0)}$, which is uniquely determined (due to the fact $\psi \in B(\phi_{ac}; r_0)$ and Lemma 4.3) and nothing but ψ. However, it contradicts the fact that $\hat\psi_n \notin B(\psi; \varepsilon_0)$. □

We finally set up the following lemma:

LEMMA 4.8. *Let $\psi \in \mathcal{VI}$ be such that $\psi \in B(\phi_{ac}; r_0/2)$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any solution $u(x, t)$ of (P), it holds that*

$$\sup_{t \ge 0} \|u(t) - \psi\|_4 < \varepsilon,$$

whenever $\|\nabla u(0) - \nabla\psi\|_2 < \delta$ and $u(0) \in D$.

Proof. First, note that ψ is the minimizer of $J(\,\cdot\,; \psi)$ over $\overline{B(\phi_{ac}; \delta_0)}$ (see Lemmas 4.4 and 4.6). Let $\varepsilon > 0$ and let $u_0 \in B(\psi; \delta) \cap D$ be fixed for some $\delta \in (0, (r_0/2) \wedge \varepsilon)$. Then $u_0 \in B(\phi_{ac}; r_0)$. By Lemma 4.2, $u(t)$ lies on $B(\phi_{ac}; \delta_0/2)$ for all $t \ge 0$. On the other hand, $u(t)$ converges to a solution $\hat\psi \in B(\phi_{ac}; \delta_0)$ of (2.1) (with u_0) strongly in $H^1_0(\Omega)$ as $t \to \infty$ (see [2, Theorem 10.1]). Then Lemma 4.4 ensures that $\hat\psi$

minimizes $J(\,\cdot\,;u_0)$ over $\overline{B(\phi_{ac};\delta_0)}$. Hence, by Lemma 4.7, if $\delta > 0$ is small enough, then the minimizer $\hat{\psi}$ also lies on the ε-neighbourhood $B(\psi;\varepsilon)$ of ψ.

Now, recalling the non-decrease of $u(x,t)$ in t, we deduce that

$$u_0 - \psi \le u(t) - \psi \le \hat{\psi} - \psi \text{ a.e. in } \Omega$$

for all $t \ge 0$, and therefore, since $H^1_0(\Omega) \hookrightarrow L^4(\Omega)$, we obtain

$$\sup_{t\ge 0} \|u(t) - \psi\|_4 \le \max\{\|u_0 - \psi\|_4, \|\psi - \hat{\psi}\|_4\} < C\varepsilon.$$

This completes the proof. □

Now, we are in a position to prove the assertion of Theorem 3.7.

Proof of Theorem 3.7. Subtract (2.1) from (1.4) and test it by $w_t = (u-\psi)_t = u_t$, where $w := u - \psi$. Then we see that

$$\|w_t\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}\|u\|_4^4 - (\psi^3, w)\right) \le \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2.$$

Here we used the facts that

$$(\eta, w_t) = (\eta, u_t) = 0, \quad (-\zeta, u_t) \ge 0$$

for $\eta \in \partial I_{[0,\infty)}(u_t)$ and $\zeta \in \partial I_{[u_0(x),\infty)}(\psi)$. Hence

$$\begin{aligned}\frac{1}{2}\|\nabla w(t)\|_2^2 &+ \frac{1}{4}\|u(t)\|_4^4 - (\psi^3, w(t)) - \frac{\kappa}{2}\|w(t)\|_2^2 \\ &\le \frac{1}{2}\|\nabla w(0)\|_2^2 + \frac{1}{4}\|u(0)\|_4^4 - (\psi^3, w(0)) - \frac{\kappa}{2}\|w(0)\|_2^2.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\|\nabla w(t)\|_2^2 + \frac{1}{4}\|u(t)\|_4^4 \\ &\quad\le \frac{1}{2}\|\nabla w(0)\|_2^2 + \frac{1}{4}\|u(0)\|_4^4 - (\psi^3, w(0)) - \frac{\kappa}{2}\|w(0)\|_2^2 + (\psi^3, w(t)) + \frac{\kappa}{2}\|w(t)\|_2^2 \\ &\quad\le \frac{1}{2}\|\nabla w(0)\|_2^2 + \frac{1}{4}\|u(0)\|_4^4 + \|\psi\|_4^3\|w(0)\|_4 + \|\psi\|_4^3\|w(t)\|_4 + \frac{\kappa}{2}\|w(t)\|_2^2,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{2}\|\nabla w(t)\|_2^2 \\ &\quad\le \frac{1}{2}\|\nabla w(0)\|_2^2 + \|\psi\|_4^3\|w(0)\|_4 + \|\psi\|_4^3\|w(t)\|_4 + \frac{\kappa}{2}\|w(t)\|_2^2 \\ &\qquad + \frac{1}{4}\int_\Omega \left(|u(x,0)|^2 + |u(x,t)|^2\right)\left(|u(x,0)| + |u(x,t)|\right)|u(x,t) - u(x,0)|\,\mathrm{d}x \\ &\quad\le \frac{1}{2}\|\nabla w(0)\|_2^2 + \|\psi\|_4^3\|w(0)\|_4 + \|\psi\|_4^3\|w(t)\|_4 + \frac{\kappa}{2}\|w(t)\|_2^2 \\ &\qquad + C\left(\|u(t)\|_4^2 + \|u(0)\|_4^2\right)\left(\|u(t)\|_4 + \|u(0)\|_4\right)\left(\|w(t)\|_4 + \|w(0)\|_4\right).\end{aligned}$$

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Combining this fact with Lemma 4.8 and $u \in L^\infty(0,T;L^4(\Omega))$, for any $\varepsilon > 0$, one can take $\delta > 0$ small enough so that

$$\|\nabla w(t)\|_2^2 \le C\left(\|\nabla w(0)\|_2^2 + \|w(0)\|_4 + \|w(t)\|_4 + \|w(t)\|_2^2\right) < \varepsilon^2,$$

provided that $\|\nabla w(0)\|_2 < \delta$ and $u(0) \in D$ (here we also used continuous embeddings $H_0^1(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^2(\Omega)$ since Ω is bounded and $N \le 4$). Therefore, ψ is stable in the sense of Definition 3.3. This completes the proof. □

5. Corollaries

As a by-product of the proof for Theorem 3.7 in the last section, one can also prove

COROLLARY 5.1. *Assume $N \le 4$. Any $\psi \in \mathcal{VI} \cap B(\phi_{ac}; r_0)$ is an accumulation point of equilibria in the strong topology of $H_0^1(\Omega)$.*

Proof. Indeed, by Lemmas 4.3 and 4.6, ψ is the unique minimizer of $J(\,\cdot\,;\psi)$ over $\overline{B(\phi_{ac};\delta_0)}$. Now, let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ be a positive function, e.g., a principal eigenfunction of the Dirichlet Laplacian, and let $u_{0,n} := \psi + \frac{1}{n}w$. Then one can check that $u_{0,n}$ lies on $B(\phi_{ac}; r_0)$ for n large enough and $u_{0,n} \to \psi$ strongly in $H^2(\Omega) \cap H_0^1(\Omega)$ as $n \to \infty$, since ψ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$. Hence thanks to Lemma 4.5, we assure that the sequence (ψ_n) of minimizers of $J(\,\cdot\,; u_{0,n})$ over $\overline{B(\phi_{ac};\delta_0)}$ also converges to ψ strongly in $H_0^1(\Omega)$, up to a subsequence, as $n \to \infty$. Moreover, noting that $\psi_n \ge u_{0,n} > \psi$ in Ω, we conclude that ψ is an accumulation point of equilibria. □

We shall further characterize equilibria $\psi \in \mathcal{VI}$ near the global minimizer ϕ_{ac} of the energy E. The following lemma ensures that by choosing $\delta_0 > 0$ sufficiently small, one can leave the δ_0-neighbourhood of $-\phi_{ac}$ out of consideration in order to minimize $J(\,\cdot\,; u_0)$.

LEMMA 5.2. *There exists $\delta_0 > 0$ small enough so that*

$$B(-\phi_{ac};\delta_0) \cap [\,\cdot \ge u_0] = \emptyset \quad \textit{for all} \ \ u_0 \in B(\phi_{ac};\delta_0).$$

Proof. We first observe that $-\phi_{ac} \notin [\,\cdot \ge \phi_{ac}]$ by $\phi_{ac} > 0$. Hence by virtue of the Hahn–Banach separation theorem, there exist $\xi \in H^{-1}(\Omega) \setminus \{0\}$ and $\alpha, \beta \in \mathbb{R}$ such that

$$f(-\phi_{ac}) =: \alpha < \beta \le f(u) \quad \text{for all} \ \ u \in [\,\cdot \ge \phi_{ac}]$$

where $f(w) := \langle \xi, w\rangle_{H_0^1(\Omega)}$ for $w \in H_0^1(\Omega)$. Now, choose $\delta_0 > 0$ small enough so that

$$0 < \delta_0 \le \frac{\beta - \alpha}{4\|\xi\|_{H^{-1}(\Omega)}}.$$

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Then for any $w \in B(-\phi_{ac}; \delta_0)$, one finds that

$$f(w) = f(-\phi_{ac}) + \langle \xi, \phi_{ac} + w \rangle_{H_0^1(\Omega)} \leq f(-\phi_{ac}) + \|\xi\|_{H^{-1}(\Omega)} \|\phi_{ac} + w\|_{H_0^1(\Omega)} \\ \leq \alpha + \frac{\beta - \alpha}{4} =: \alpha'.$$

Similarly, for any $v \in [\,\cdot \geq u_0]$, we obtain

$$f(v) = f(v + \phi_{ac} - u_0) - \langle \xi, \phi_{ac} - u_0 \rangle_{H_0^1(\Omega)} \geq \beta - \|\xi\|_{H^{-1}(\Omega)} \|\phi_{ac} - u_0\|_{H_0^1(\Omega)} \\ \geq \beta - \frac{\beta - \alpha}{4} =: \beta'.$$

From the fact that $\alpha' < \beta'$, one deduces that

$$f(w) \leq \alpha' < \beta' \leq f(v)$$

for any $w \in B(-\phi_{ac}; \delta_0)$ and $v \in [\,\cdot \geq u_0]$. Consequently, we conclude that $B(-\phi_{ac}; \delta_0) \cap [\,\cdot \geq u_0] = \emptyset$. □

Moreover, we have

LEMMA 5.3 *Assume $N \leq 4$. If $u_0 \in B(\phi_{ac}; r_0)$, then the set $B(\phi_{ac}; \delta_0)$ includes a unique global minimizer of $J(\,\cdot\,; u_0)$, and moreover, it lies on $B(\phi_{ac}; \delta_0/2)$.*

Proof. This lemma can be obtained as a corollary of Lemma 4.3. Indeed, by Lemma 4.3, for any $u_0 \in B(\phi_{ac}; r_0)$, the functional $J(\,\cdot\,; u_0)$ admits a unique minimizer ψ over $\overline{B(\phi_{ac}; \delta_0)}$. Then, we note that

$$J(\psi; u_0) = \inf_{v \in B(\phi_{ac}; \delta_0)} J(v; u_0) \leq J(u_0; u_0) = E(u_0) \overset{(4.3)}{\leq} d + \varepsilon_0,$$

since u_0 belongs to $B(\phi_{ac}; r_0)$. Hence $\psi \in B(\phi_{ac}; \delta_0/2)$ by (4.2). Furthermore, (4.2) yields

$$B(\phi_{ac}; \delta_0/2)^c \cap B(-\phi_{ac}; \delta_0/2)^c \subset [E > d + \varepsilon_0]. \tag{5.1}$$

Therefore thanks to Lemma 5.2, ψ minimizes $J(\,\cdot\,; u_0)$ over the whole of $H_0^1(\Omega)$. The uniqueness of global minimizers of $J(\,\cdot\,; u_0)$ follows from that of (local) minimizers over the set $\overline{B(\phi_{ac}; \delta_0)}$ (see Lemma 4.3) as well as (5.1) and Lemma 5.2. □

Thus, we obtain

COROLLARY 5.4. *Assume $N \leq 4$. Let δ_0 and r_0 be small positive constants given above. Let $u_0 \in B(\phi_{ac}; r_0)$ and let $\psi \in \mathcal{VI}$ be such that $\psi \in B(\phi_{ac}; \delta_0)$. Then ψ (globally) minimizes $J(\,\cdot\,; u_0)$ over $H_0^1(\Omega)$, and moreover, it is the unique global minimizer.*

REMARK 5.5. The statement above may not hold true for the negative ground state $-\phi_{ac}$. Indeed, if $u_0 \in B(-\phi_{ac}; r_0)$, then the cone $[\,\cdot \geq u_0]$ may include the positive ground state ϕ_{ac}, and therefore, $J(\,\cdot\,; u_0)$ achieves the minimum at ϕ_{ac}.

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