Bull. Aust. Math. Soc. **90** (2014), 90–98
doi:10.1017/S000497271300110X

A NEW UNIQUE CONTINUATION PROPERTY FOR THE KORTEWEG–DE VRIES EQUATION

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(Received 26 October 2013; accepted 31 October 2013; first published online 10 January 2014)

Abstract

The aim of this paper is to obtain a new unique continuation property (UCP) for the Korteweg–de Vries equation posed on a finite interval. Compared with the previous UCP, we need fewer conditions on the solution. For this purpose, we have to establish a global Carleman estimate for the Korteweg–de Vries equation.

2010 *Mathematics subject classification*: primary 35Q53.

Keywords and phrases: Korteweg–de Vries equation, unique continuation property, Carleman estimate.

1. Introduction

This paper is concerned with the following Korteweg–de Vries (KdV) equation posed on a finite interval:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } I \times (0, T), \\ u(0, t) = u(l, t) = u_x(l, t) = 0 & \text{on } (0, T), \\ u(x, 0) = u_0(x) & \text{on } I, \end{cases} \tag{1.1}$$

where $T > 0$, $I = (0, l)$. In applications to physical problems, the independent variable x is often a coordinate representing position in the medium of propagation, t is proportional to elapsed time, and $u(x, t)$ is a velocity or an amplitude at point x at time t. The KdV equation was first derived by Korteweg and de Vries [3] in 1895 (or by Boussinesq [1] in 1876) as a model for propagation of some surface water waves along a channel.

The unique continuation property (UCP) is an important issue in the theory of partial differential equations. Its history may date back to the classical results of Holmgren and Carleman at the very beginning of the twentieth century. There are many articles concerned with the UCP for the KdV equation (see [6–8]).

Zhang [8] obtained that if $u \in L^\infty_{\text{loc}}(\mathbb{R}, H^3(\mathbb{R}))$ is a solution of the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

and vanishes on an open set of $\mathbb{R}_x \times \mathbb{R}_t$, then

$$u(x,t) \equiv 0 \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}.$$

Then, in [7], Saut and Scheurer proved that if $u \in L^2(0,T; H^3_{\text{loc}}(I))$ is the solution of (1.1) and

$$u \equiv 0 \quad \text{in } \omega \times (0,T),$$

then $u \equiv 0$ in $I \times (0,T)$. Here (and elsewhere) $\omega \subset I$ is a nonempty open set.

Later, Rosier and Zhang [6] studied the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0,l) \times (0,T), \\ u(0,t) = u(l,t) = 0 & \text{for a.e. } t \in (0,T). \end{cases} \tag{1.2}$$

They showed that if $u \in L^\infty(0,T; H^1(I))$ solves (1.2) and $u \equiv 0$ in $\omega \times (0,T)$, then $u \equiv 0$ in $I \times (0,T)$.

In this paper, we consider a new UCP for the KdV equation which is different from all the above results.

Theorem 1.1. *Let $u_0 \in L^2(I)$ and $u \in C([0,T]; L^2(I)) \cap L^2(0,T; H^1_0(I))$ be a solution of (1.1). If $u \equiv 0$ in $\omega \times (0,T)$, then $u \equiv 0$ in $I \times (0,T)$.*

Remark 1.2. Compared with the above results, Theorem 1.1 needs less regularity. Actually, the UCP by Rosier and Zhang [6] implies Theorem 1.1, when combined with a smoothing property, but we prove Theorem 1.1 through the Carleman estimate with internal observation, which is a new result for the KdV equation.

To prove Theorem 1.1, we introduce some functions. Let $\psi \in C^\infty(\bar{I})$ be such that $\psi > 0$ in I, $\psi(0) = \psi(l) = 0$, $\|\psi\|_{C(\bar{I})} = 1$, $|\psi_x| > 0$ in $\bar{I} \setminus \omega$, $\psi_x(0) > 0$ and $\psi_x(l) < 0$. For any given positive constants λ and μ, we set

$$a(x,t) = \frac{e^{\mu(\psi(x)+3)} - e^{5\mu}}{t(T-t)}, \quad \theta(x,t) = e^{\lambda a}, \quad \varphi(x,t) = \frac{e^{\mu(\psi(x)+3)}}{t(T-t)},$$

for all $(x,t) \in Q$. We write Q and Q^ω for $I \times (0,T)$ and $\omega \times (0,T)$, respectively.

Let L denote the operator $Lu = u_t + u_x + u_{xxx}$ with its domain

$$\mathcal{D}(L) = \{u \in L^2(0,T; H^3(I)) \cap H^1(0,T; L^2(I)) : \\ u(0,t) = u(l,t) = u_x(l,t) = 0,\ \forall t \in (0,T)\}.$$

One of the main results in this paper is the following global Carleman estimate.

Theorem 1.3. *There exist constants $C_0 > 0$ and $C_1 > 0$ such that, for any $u \in \mathcal{D}(L)$ and all numbers $\lambda \geq C_0(T + T^2)$,*

$$\int_Q (\lambda^5 \theta^2 \varphi^5 u^2 + \lambda^3 \theta^2 \varphi^3 u_x^2 + \lambda \theta^2 \varphi u_{xx}^2)\, dx\, dt \\ \leq C_1 \left(\int_Q \theta^2 |Lu|^2\, dx\, dt + \int_{Q^\omega} (\lambda^5 \theta^2 \varphi^5 u^2 + \lambda^3 \theta^2 \varphi^3 u_x^2 + \lambda \theta^2 \varphi u_{xx}^2)\, dx\, dt \right). \tag{1.3}$$

REMARK 1.4. The Carleman estimate for the KdV equation was also considered in [2, 5]. However, to the best of our knowledge, there are few results about the Carleman estimate with internal observation.

The rest of this paper is organised as follows. Section 2 is devoted to the proof of Theorem 1.3. Then, combined with a smoothing property, we prove our main result.

2. Proof of Theorem 1.3

This section is motivated by [9].

As in [4], it is enough to derive (1.3) for $\widetilde{L}u = u_t + u_{xxx}$ with $u \in \mathcal{D}(L)$. In fact, assume that we have proved (1.3) for $\widetilde{L}u$. We have

$$\int_Q \theta^2 |\widetilde{L}u|^2 \, dx \, dt \le 2 \int_Q \theta^2 |Lu|^2 \, dx \, dt + 2 \int_Q \theta^2 u_x^2 \, dx \, dt.$$

By choosing $\lambda > 0$ large, it is possible to absorb $2 \int_Q \theta^2 u_x^2 \, dx \, dt$ with the left-hand side of (1.3), concluding that (1.3) also holds for Lu.

It is obvious that

$$\begin{aligned} \theta\widetilde{L}u = \theta(u_t + u_{xxx}) = v_t + v_{xxx} - 3\lambda a_x v_{xx} + (3\lambda^2 a_x^2 - 3\lambda a_{xx}) v_x \\ + (-\lambda a_t - \lambda^3 a_x^3 + 3\lambda^2 a_x a_{xx} - \lambda a_{xxx}) v. \end{aligned} \tag{2.1}$$

Define

$$\theta\widetilde{L}u = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= v_t + v_{xxx} + 3\lambda^2 a_x^2 v_x, \\ I_2 &= -3\lambda a_x v_{xx} - \lambda^3 a_x^3 v, \\ I_3 &= -3\lambda a_{xx} v_x + (-\lambda a_t + 3\lambda^2 a_x a_{xx} - \lambda a_{xxx}) v. \end{aligned}$$

It is easy to see that

$$2I_1 I_2 \le (I_1 + I_2)^2 = (\theta\widetilde{L}u - I_3)^2 \le 2\theta^2 |\widetilde{L}u|^2 + 2I_3.$$

Throughout this paper, $C(\psi)$ denotes a constant depending on ψ whose value can change from line to line.

Step 1. We have the inequality

$$\begin{aligned} \int_Q 2I_1 I_2 \, dx \, dt = \int_Q \Big((\cdot) v^2 + (\cdot) v_x^2 + (\cdot) v_{xx}^2 + 6\lambda a_{xx} v_x \theta L u \Big) \, dx \, dt \\ + \int_0^T (V(l,t) - V(0,t)) \, dt, \end{aligned} \tag{2.2}$$

where

$$
\begin{aligned}
(\cdot)v^2 &= (3\lambda^3 a_x^2 a_{xt} + \lambda^3 (a_x^3)_{xxx} + 15\lambda^5 a_x^4 a_{xx} + 3\lambda (Ba_{xx})_x) v^2, \\
(\cdot)v_x^2 &= (-3\lambda a_{xt} - 9\lambda^3 a_x^2 a_{xx} + 27\lambda^3 a_x^2 a_{xx} - 3\lambda a_{xxxx} \\
&\quad - 9\lambda^2 (a_x a_{xx})_x - 6\lambda A a_{xx}) v_x^2, \\
(\cdot)v_{xx}^2 &= (9\lambda a_{xx}) v_{xx}^2, \\
V(l,t) &= -3\lambda a_x v_{xx}^2 (l,t), \\
V(0,t) &= -3\lambda a_x v_{xx}^2 (0,t) - 6\lambda a_{xx} v_x(0,t) v_{xx}(0,t) \\
&\quad + (\lambda^3 a_x^3 - 9\lambda^3 a_x^3 + 3\lambda a_{xxx} + 9\lambda^2 a_x a_{xx}) v_x^2 (0,t).
\end{aligned}
$$

To prove (2.2), we shall calculate each term of $2I_1 I_2$.

Let I_{1i} $(i = 1, 2, 3)$ and I_{2j} $(j = 1, 2)$ denote the ith term of I_1 and the jth term of I_2, respectively.

It is not difficult to deduce that

$$
\begin{aligned}
2I_{11}I_{21} &= -6\lambda a_x v_t v_{xx} \\
&= (-6\lambda a_x v_t v_x)_x + 6\lambda a_{xx} v_t v_x - 3\lambda a_{xt} v_x^2 + (3\lambda a_x v_x^2)_t, && (2.3)\\
2I_{11}I_{22} &= -2\lambda^3 a_x^3 v v_t = (-\lambda^3 a_x^3 v^2)_t + 3\lambda^3 a_x^2 a_{xt} v^2, && (2.4)\\
2I_{12}I_{21} &= -6\lambda a_x v_{xx} v_{xxx} = 3\lambda a_{xx} v_{xx}^2 + (-3\lambda a_x v_{xx}^2)_x, && (2.5)\\
2I_{12}I_{22} &= -2\lambda^3 a_x^3 v v_{xxx} \\
&= -9\lambda^3 a_x^2 a_{xx} v_x^2 + \lambda^3 (a_x^3)_{xxx} v^2 + (\lambda^3 a_x^3 v_x^2)_x + (-2\lambda^3 a_x^3 v v_{xxx})_x \\
&\quad + (6\lambda^3 a_x^2 a_{xx} v v_x)_x - ((6\lambda^3 a_x a_{xx}^2 + 3\lambda^3 a_x^2 a_{xxx}) v^2)_x, && (2.6)\\
2I_{13}I_{21} &= -18\lambda^3 a_x^3 v_x v_{xx} = (-9\lambda^3 a_x^3 v_x^2)_x + 27\lambda^3 a_x^2 a_{xx} v_x^2, && (2.7)\\
2I_{13}I_{22} &= -6\lambda^5 a_x^5 v v_x = (-3\lambda^5 a_x^5 v^2)_x + 15\lambda^5 a_x^4 a_{xx} v^2. && (2.8)
\end{aligned}
$$

Rewrite (2.1) as

$$
\theta \widetilde{L} u = v_t + v_{xxx} - 3\lambda a_x v_{xx} + A v_x + B v, \tag{2.9}
$$

where

$$
\begin{aligned}
A &= 3\lambda^2 a_x^2 - 3\lambda a_{xx}, \\
B &= -\lambda a_t - \lambda^3 a_x^3 + 3\lambda^2 a_x a_{xx} - \lambda a_{xxx}.
\end{aligned}
$$

According to (2.9),

$$
\begin{aligned}
6\lambda a_{xx} v_t v_x &= 6\lambda a_{xx} v_x (\theta \widetilde{L} u - v_{xxx} + 3\lambda a_x v_{xx} - A v_x - B v) \\
&= 6\lambda a_{xx} v_x \theta \widetilde{L} u + 6\lambda a_{xx} v_{xx}^2 + (3\lambda a_{xxx} v_x^2)_x - 3\lambda a_{xxxx} v_x^2 \\
&\quad + (-6\lambda a_{xx} v_x v_{xx})_x + (9\lambda^2 a_x a_{xx} v_x^2)_x - 9\lambda^2 (a_x a_{xx})_x v_x^2 \\
&\quad - 6\lambda A a_{xx} v_x + 3\lambda (B a_{xx})_x v^2 + (-3\lambda B a_{xx} v^2)_x.
\end{aligned} \tag{2.10}
$$

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Noting that $u(0,t) = u(l,t) = u_x(l,t) = 0$ and $\lim_{t\to 0^+} a(t,\cdot) = \lim_{t\to T^-} a(t,\cdot) = -\infty$,

$$v(0,t) = v(l,t) = v_x(l,t) = v(x,0) = v(x,T) \equiv 0, \tag{2.11}$$

for all $x, t \in (0,l)\times(0,T)$. Combining (2.3)–(2.11), we can obtain (2.2).

Step 2. We claim that there exists a positive constant C_2 such that, for all numbers $\lambda \geq C_2(T+T^2)$,

$$\begin{aligned}&\int_Q (\lambda^5\mu^6\varphi^5|\psi_x|^6 v^2 + \lambda\mu^2\varphi|\psi_x|^2 v_{xx}^2)\,dx\,dt\\ &\quad\leq C(\psi)\bigg(\int_Q \theta^2|\widetilde{L}u|^2\,dx\,dt + \int_Q(\lambda^5\mu^5\varphi^5 v^2 + \lambda^3\mu^3\varphi^3 v_x^2 + \lambda\mu\varphi v_{xx}^2)\,dx\,dt\bigg).\end{aligned} \tag{2.12}$$

We shall estimate each term in the right-hand side of (2.2).

Suppose that $\mu > 1$ is a constant which will be chosen later. By the definitions of a, φ and ψ, it is obvious that

$$\begin{aligned}&|a_x| \leq C(\psi)\mu\varphi, \quad |a_{xx}| \leq C(\psi)\mu^2\varphi, \quad |a_{xxx}| \leq C(\psi)\mu^3\varphi,\\ &|a_{xxxx}| \leq C(\psi)\mu^4\varphi, \quad |a_t| \leq CT\varphi^2, \quad |a_{xt}| \leq C(\psi)\mu T\varphi^2,\end{aligned}$$

and $\varphi \leq (T^2/4)\varphi^2$.

If we choose $\lambda \geq \mu C(\psi)(T+T^2)$ with $C(\psi)$ large enough, then

$$I_3^2 \leq C(\psi)\lambda^3\mu^3\varphi^3 v_x^2 + C(\psi)\lambda^5\mu^5\varphi^5 v^2, \tag{2.13}$$

$$(\cdot)v^2 = 15\lambda^5\mu^6\varphi^5|\psi_x|^6 v^2 + F_1, \tag{2.14}$$

$$(\cdot)v_x^2 = F_2, \tag{2.15}$$

$$(\cdot)v_{xx}^2 = 9\lambda\mu^2\varphi|\psi_x|^2 v_{xx}^2 + F_3, \tag{2.16}$$

where

$$\begin{aligned}|F_1| &\leq C(\psi)\lambda^5\mu^5\varphi^5 v^2,\\ |F_2| &\leq C(\psi)\lambda^3\mu^3\varphi^3 v_x^2,\\ |F_3| &\leq C(\psi)\lambda\mu\varphi v_{xx}^2.\end{aligned}$$

Moreover,

$$6\lambda a_{xx}v_x\theta\widetilde{L}u \geq -9\lambda^2 a_{xx}^2 v_x^2 - \theta|\widetilde{L}u|^2. \tag{2.17}$$

We now estimate $V(l,t) - V(0,t)$. It is obvious that

$$V(l,t) = -3\lambda a_x v_{xx}^2(l,t) = -3\lambda\mu\varphi\psi_x(l)v_{xx}^2(l,t) \geq 0, \tag{2.18}$$

$$\begin{aligned}V(0,t) &= -6\lambda a_{xx}v_x(0,t)v_{xx}(0,t) - 3\lambda a_x v_{xx}^2(0,t)\\ &\quad + (\lambda^3 a_x^3 + 3\lambda a_{xxx} + 9\lambda^2 a_x a_{xx})v_x^2(0,t).\end{aligned}$$

For any $\varepsilon_0 > 0$, if we choose $\lambda \geq \mu C(\varepsilon_0, \psi)(T + T^2)$ with $C(\varepsilon_0, \psi)$ large enough, then, as in (2.13)–(2.16),

$$\begin{aligned}|-6\lambda a_{xx} v_x(0,t) v_{xx}(0,t)| &\leq C(\psi)\lambda\mu^2\varphi|v_x(0,t)||v_{xx}(0,t)| \\ &\leq \varepsilon_0\lambda^3\mu^3\varphi^3|v_x(0,t)|^2 + \varepsilon_0\lambda\mu\varphi|v_{xx}(0,t)|^2, \\ -3\lambda a_x v_{xx}^2(0,t) &= -3\lambda\mu\varphi\psi_x(0)v_{xx}^2(0,t), \\ (\lambda^3 a_x^3 + 3\lambda a_{xxx} + 9\lambda^2 a_x a_{xx})v_x^2(0,t) &= -8\lambda^3\mu^3\varphi^3\psi_x^3(0)v_x^2(0,t) + F_4,\end{aligned}$$

where

$$|F_4| \leq \varepsilon_0\lambda^3\mu^3\varphi^3 v_x^2(0,t).$$

If we take ε_0 small enough, there exist positive constants C_3 and C_4 such that

$$V(0,t) \leq -C_3\lambda^3\mu^3\varphi^3\psi_x^3(0)v_x^2(0,t) - C_4\lambda\mu\varphi\psi_x(0)v_{xx}^2(0,t) \leq 0. \tag{2.19}$$

Combining (2.13)–(2.19), we arrive at (2.12).

Step 3. We shall prove that

$$\int_Q \lambda^3\mu^4\varphi^3|\psi_x|^8 v_x^2\,dx\,dt \leq C(\psi)\int_Q (\lambda^5\mu^6\varphi^5|\psi_x|^6 v^2 + \lambda\mu^2\varphi|\psi_x|^2 v_{xx}^2)\,dx\,dt. \tag{2.20}$$

Using integration by parts and noting that $v(0,\cdot) = v(l,\cdot) = 0$,

$$\begin{aligned}&\int_Q \lambda^3\mu^4\varphi^3|\psi_x|^8 v_x^2\,dx\,dt \\ &\quad= \int_Q (-3\lambda^3\mu^4\varphi^3\psi_x^8 v v_{xx} - 3\lambda^3\mu^4\varphi^2\varphi_x|\psi_x|^8 v v_x - 8\lambda^3\mu^4\varphi^3\psi_x^7 v v_x)\,dx\,dt \\ &\quad\triangleq J_1 + J_2 + J_3.\end{aligned}$$

By the definition of φ and noting that $\mu \geq 1$, it is clear that

$$\begin{aligned}J_1 &\leq C(\psi)\int_Q \lambda^3\mu^4\varphi^3|\psi_x|^4|v v_{xx}|\,dx\,dt \\ &\leq C(\psi)\int_Q (\lambda^5\mu^6\varphi^5|\psi_x|^6 v^2 + \lambda\mu^2\varphi|\psi_x|^2 v_{xx}^2)\,dx\,dt, \\ J_2 &\leq C(\psi)\int_Q \lambda^3\mu^5\varphi^3|\psi_x|^7|v v_x|\,dx\,dt \triangleq J_4, \\ J_3 &\leq C(\psi)\int_Q \lambda^3\mu^4\varphi^3|\psi_x|^7|v v_x|\,dx\,dt \leq C(\psi)\int_Q \lambda^3\mu^5\varphi^3|\psi_x|^7|v v_x|\,dx\,dt = J_4.\end{aligned}$$

Now it remains only to estimate J_4, for $\lambda \geq \mu C(\psi)(T + T^2)$. With $C(\psi)$ large enough, we can deduce that

$$\begin{aligned}J_4 &\leq C(\psi)\int_Q \lambda^5\mu^6\varphi^5|\psi_x|^6 v^2\,dx\,dt + C(\psi)\int_Q \lambda\mu^4\varphi|\psi_x|^8 v_x^2\,dx\,dt \\ &\leq C(\psi)\int_Q \lambda^5\mu^6\varphi^5|\psi_x|^6 v^2\,dxdt + \frac{1}{4}\int_Q \lambda^3\mu^4\varphi^3|\psi_x|^8 v_x^2\,dx\,dt.\end{aligned}$$

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We conclude that (2.20) follows immediately from the above estimates.

Step 4. According to (2.12) and (2.20), there exists a positive constant C_5 such that for all numbers $\lambda \geq C_5(T + T^2)$,

$$\int_Q (\lambda^5\mu^6\varphi^5|\psi_x|^6 v^2 + \lambda^3\mu^4\varphi^3|\psi_x|^8 v_x^2 + \lambda\mu^2\varphi|\psi_x|^2 v_{xx}^2)\,dx\,dt \\ \leq C(\psi)\left(\int_Q \theta^2|\widetilde{L}u|^2\,dx\,dt + \int_Q (\lambda^5\mu^5\varphi^5 v^2 + \lambda^3\mu^3\varphi^3 v_x^2 + \lambda\mu\varphi v_{xx}^2)\,dx\,dt\right).$$

Recall that $|\psi_x| > 0$ in $\bar{I} \setminus \omega$. Then there exists a constant $\widetilde{C}(\psi)$ such that

$$\int_{Q\setminus Q^\omega} (\lambda^5\mu^6\varphi^5 v^2 + \lambda^3\mu^4\varphi^3 v_x^2 + \lambda\mu^2\varphi v_{xx}^2)\,dx\,dt \\ \leq \widetilde{C}(\psi)\left(\int_Q \theta^2|\widetilde{L}u|^2\,dx\,dt + \int_Q (\lambda^5\mu^5\varphi^5 v^2 + \lambda^3\mu^3\varphi^3 v_x^2 + \lambda\mu\varphi v_{xx}^2)\,dx\,dt\right).$$

If we choose $\mu = \widetilde{C}(\psi) + 1$, then

$$\int_{Q\setminus Q^\omega} (\lambda^5\varphi^5 v^2 + \lambda^3\varphi^3 v_x^2 + \lambda\varphi v_{xx}^2)\,dx\,dt \\ \leq C(\psi)\left(\int_Q \theta^2|\widetilde{L}u|^2\,dx\,dt + \int_{Q^\omega} (\lambda^5\varphi^5 v^2 + \lambda^3\varphi^3 v_x^2 + \lambda\varphi v_{xx}^2)\,dx\,dt\right). \tag{2.21}$$

Adding $\int_{Q^\omega}(\lambda^5\varphi^5 v^2 + \lambda^3\varphi^3 v_x^2 + \lambda\varphi v_{xx}^2)\,dx\,dt$ to both sides of (2.21),

$$\int_Q (\lambda^5\varphi^5 v^2 + \lambda^3\varphi^3 v_x^2 + \lambda\varphi v_{xx}^2)\,dx\,dt \\ \leq C(\psi)\left(\int_Q \theta^2|\widetilde{L}u|^2\,dx\,dt + \int_{Q^\omega} \lambda^5\varphi^5 v^2 + \lambda^3\varphi^3 v_x^2 + \lambda\varphi v_{xx}^2\,dx\,dt\right).$$

Taking $v = e^{\lambda a}u$, we can obtain (1.3), completing the proof of Theorem 1.3. □

REMARK 2.1. Actually, in dimension 1, only one parameter λ is needed in the Carleman estimate (see [2, 5]). It should be said that μ is considered here just to make the construction of the weights easier.

3. Proof of Theorem 1.1

For any $\varepsilon > 0$, let $Q^\varepsilon = (\varepsilon, T) \times I$, $Q_0^\varepsilon = (\varepsilon, T) \times \omega$ and let L_ε denote the operator $L_\varepsilon u = u_t + u_x + u_{xxx}$ with its domain

$$\mathcal{D}(L_\varepsilon) = \{u \in L^2(\varepsilon, T; H^3(I)) \cap H^1(\varepsilon, T; L^2(I)) : \\ u(0, t) = u(l, t) = u_x(l, t),\ t \in (\varepsilon, T)\}.$$

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Set

$$a_\varepsilon(x,t) = \frac{e^{\mu(\psi(x)+3)} - e^{5\mu}}{(t-\varepsilon)(T-t)}, \quad \theta_\varepsilon(x,t) = e^{\lambda a_\varepsilon}, \quad \varphi_\varepsilon(x,t) = \frac{e^{\mu(\psi(x)+3)}}{(t-\varepsilon)(T-t)},$$

for all $(x,t) \in Q$.

By the same method as in Section 2, we have the following result.

Proposition 3.1. *For any $u \in \mathcal{D}(L_\varepsilon)$ and all numbers $\lambda \geq C_0(T+T^2)$,*

$$\begin{aligned} &\int_{Q^\varepsilon} (\lambda^5 \theta_\varepsilon^2 \varphi_\varepsilon^5 u^2 + \lambda^3 \theta_\varepsilon^2 \varphi_\varepsilon^3 u_x^2 + \lambda \theta_\varepsilon^2 \varphi_\varepsilon u_{xx}^2)\, dx\, dt \\ &\quad \leq C_1 \bigg(\int_{Q^\varepsilon} \theta_\varepsilon^2 |L_\varepsilon u|^2\, dx\, dt + \int_{Q_0^\varepsilon} (\lambda^5 \theta_\varepsilon^2 \varphi_\varepsilon^5 u^2 + \lambda^3 \theta_\varepsilon^2 \varphi^3 u_x^2 + \lambda \theta_\varepsilon^2 \varphi_\varepsilon u_{xx}^2)\, dx\, dt \bigg), \end{aligned} \tag{3.1}$$

where C_0 and C_1 are the same as in Theorem 1.3.

The following proposition, which reveals a strong smoothing property, follows directly from [6, Corollary 2.10].

Proposition 3.2. *For any $u_0 \in L^2(I)$, (1.1) has a unique solution $u \in C([0,T]; L^2(I)) \cap L^2(0,T; H_0^1(I))$. Moreover, for any $\varepsilon > 0$, u belongs to the space $C([\varepsilon, T]; H^3(I)) \cap L^2(\varepsilon, T; H^4(I))$.*

We can now prove Theorem 1.1.

Proof of Theorem 1.1. According to Theorem 3.2, it is easy to see that the solution u of (1.1) belongs to $\mathcal{D}(L_\varepsilon)$. By (3.1),

$$\begin{aligned} &\int_{Q^\varepsilon} (\lambda^5 \theta_\varepsilon^2 \varphi_\varepsilon^5 u^2 + \lambda^3 \theta_\varepsilon^2 \varphi_\varepsilon^3 u_x^2 + \lambda \theta_\varepsilon^2 \varphi_\varepsilon u_{xx}^2)\, dx\, dt \\ &\quad \leq C_1 \bigg(\int_{Q^\varepsilon} \theta_\varepsilon^2 |u u_x|^2\, dx\, dt + \int_{Q_0^\varepsilon} (\lambda^5 \theta_\varepsilon^2 \varphi_\varepsilon^5 u^2 + \lambda^3 \theta_\varepsilon^2 \varphi^3 u_x^2 + \lambda \theta_\varepsilon^2 \varphi_\varepsilon u_{xx}^2)\, dx\, dt \bigg). \end{aligned} \tag{3.2}$$

Note that $u \equiv 0$ in $\omega \times (0,T)$ and $u \in L^\infty(Q^\varepsilon)$, so (3.2) is reduced to

$$\int_{Q^\varepsilon} (\lambda^5 \theta_\varepsilon^2 \varphi_\varepsilon^5 u^2 + \lambda^3 \theta_\varepsilon^2 \varphi_\varepsilon^3 u_x^2 + \lambda \theta_\varepsilon^2 \varphi_\varepsilon u_{xx}^2)\, dx\, dt \leq C(\|u\|_{L^\infty(Q^\varepsilon)}) \int_{Q^\varepsilon} \theta_\varepsilon^2 |u_x|^2\, dx\, dt.$$

For $\lambda \geq C_6(T+T^2)$ with C_6 sufficiently large,

$$\begin{aligned} \int_{Q^\varepsilon} \theta_\varepsilon^2 |u_x|^2\, dx\, dt &\leq \frac{1}{64} \int_{Q^\varepsilon} T^6 \theta_\varepsilon^2 \varphi_\varepsilon^3 |u_x|^2\, dx\, dt \\ &\leq \frac{1}{2C(\|u\|_{L^\infty(Q^\varepsilon)})} \int_{Q^\varepsilon} \lambda^3 \theta_\varepsilon^2 \varphi_\varepsilon^3 |u_x|^2\, dx\, dt. \end{aligned}$$

This implies that

$$\int_{Q^\varepsilon} (\lambda^5 \theta_\varepsilon^2 \varphi_\varepsilon^5 u^2 + \lambda^3 \theta_\varepsilon^2 \varphi_\varepsilon^3 u_x^2 + \lambda \theta_\varepsilon^2 \varphi_\varepsilon u_{xx}^2)\, dx\, dt = 0.$$

It follows that $u \equiv 0$ in $I \times (\varepsilon, T)$. Since $\varepsilon > 0$ is arbitrary, we have $u \equiv 0$ in $I \times (0,T)$.

https://doi.org/10.1017/S000497271300110X Published online by Cambridge University Press

The proof of Theorem 1.1 is complete. □

Acknowledgements

We sincerely thank Professor Yong Li for many useful suggestions and the anonymous referee for helpful comments.

References

[1] J. Boussinesq, 'Essai sur la théorie des eaux courantes', *Mémoires présentés par divers savants à l'Acad. des Sci. Inst. Nat. France* **23** (1877), 1–680.
[2] O. Glass and S. Guerrero, 'Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit', *Asymptot. Anal.* **60** (2008), 61–100.
[3] D. J. Korteweg and G. de Vries, 'On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves', *Philos. Mag.* **39** (1895), 422–443.
[4] P. G. Meléndez, 'Lipschitz stability in an inverse problem for the mian coefficient of a Kuramoto-Sivashinsky type equation', *J. Math. Anal. Appl.* **408** (2013), 275–290.
[5] L. Rosier, 'Control of the surface of a fluid by a wavemaker', *ESAIM Control Optim. Cal. Var.* **10** (2004), 346–380.
[6] L. Rosier and B. Y. Zhang, 'Global stabilization of the generalized Korteweg–de Vries equation posed on a finite domain', *SIAM J. Control Optim.* **45** (2006), 927–956.
[7] J. C. Saut and B. Scheurer, 'Unique continuation for some evolution equations', *J. Differential Equations* **66** (1987), 118–139.
[8] B. Y. Zhang, 'Unique continuation for the Korteweg–de Vries equation', *SIAM J. Math. Anal.* **23** (1992), 55–71.
[9] Z. Zhou, 'Observability estimate and null controllability for one-dimensional fourth order parabolic equation', *Taiwanese J. Math.* **16** (2012), 1991–2017.

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https://doi.org/10.1017/S000497271300110X Published online by Cambridge University Press