A Problem in the Theory of Numbers.

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(*Read* 12*th June.* *MS. received* 19*th January* 1908.)

One of the well known properties of the number 7 is that when $\frac{1}{7}$ is reduced to a decimal, the periods of two digits are obtained to infinity by successive doubling. It is interesting to find for what other numbers this property is true.

Let n be any number, and r the base of notation, then

$$\frac{1}{n} = \frac{2n}{r^2} + \frac{4n}{r^4} + \frac{8n}{r^6} + \text{ to infinity}$$

$$= \frac{2n}{r^2 - 2}$$

Therefore $r^2 = 2n^2 + 2$ where r and n are integers.

To find a solution of this equation, write it in the form

$$r^2 - 2n^2 = 2$$

$$\therefore \quad (r + \sqrt{2}.n)(r - \sqrt{2}.n) = (2 - \sqrt{2})(2 + \sqrt{2})\{(1 + \sqrt{2})(1 - \sqrt{2})\}^{2p}$$

where p is any integer

$$\therefore \quad (r + \sqrt{2}.n)(r - \sqrt{2}.n)$$

$$= \left[\left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} + \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\} + \left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} - \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\}\right]$$

$$\times \left[\left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} + \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\} - \left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} - \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\}\right]$$

Now, from the symmetry of the expression, this equation is satisfied if we make

$$r = \frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} + \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p},$$

$$n = \frac{2-\sqrt{2}}{2\sqrt{2}}(1+\sqrt{2})^{2p} - \frac{2+\sqrt{2}}{2\sqrt{2}}(1-\sqrt{2})^{2p};$$

both expressions being integral. Now if

$$(2 - \sqrt{2})(1 + \sqrt{2})^{2p} = \mathrm{M}_p + \mathrm{N}_p\sqrt{2},$$

then $r_p = \mathrm{M}_p$ and $n_p = \mathrm{N}_p$, where p may have any positive integral value.

Expanding, and writing $\overset{p}{C}_k$ for the number of combinations of p things taken k together

$$r_p = (2p-1)2 + \overset{2p-1}{C_3} . 4 + \overset{2p-1}{C_5} . 8 + \overset{2p-1}{C_7} . 2^4 + \dots + (2p-1)\, 2^{p-1} + 2^p.$$

and $$n_p = 1 + \overset{2p-1}{C_2} . 2 + \overset{2p-1}{C_4} . 2^2 + \dots \qquad + \overset{2p-1}{C_2} . 2^{p-2} + 2^{p-1}.$$

On substituting for p in succession 1, 2, 3, etc., we get

$$\begin{aligned} r_1 &= 2, & n_1 &= 1. \\ r_2 &= 10, & n_2 &= 7. \\ r_3 &= 58, & n_3 &= 41. \\ r_4 &= 338, & n_4 &= 239. \\ r_5 &= 1970, & n_5 &= 1393, \text{ etc.} \end{aligned}$$

The first is an obvious illustration, as

$$\frac{2}{2^2} + \frac{4}{2^4} + \frac{8}{2^6} + \text{to infinity} = \frac{1}{1}.$$

That the third number has the same property, can be proved from the infinite geometrical progression, or may be tested by reducing $\frac{1}{41}$ to a radix fraction in scale 58.

If we adopt the following notation for numbers: 0 to 9 to be expressed as usual by arabic figures, ten to nineteen by t, t_1, t_2, .. t_9, twenty by T, thirty θ, forty f, fifty F, with subscript figures for the excess above multiples of ten; then f_1 represents forty-one, and

$$\frac{1}{f_1} = \dot{1}T_4 2 f_8 5 \theta_8 t_1 t_8 T_2 \theta_6 f_5 t_5 \theta_2 \theta_1 7 4 t_4 8 T_8 t_6 F_6 \theta_3 F_5 9 F_2 t_9 f_6 \theta_9 \theta_5 T_1 t_2 f_2 T_5 T_6 F F_3 f_3 f_4 T_9 \dot{f}_1$$

Now $2 \times f_1 = 1T_4$; $2 \times 1T_4 = 2f_8$, etc.

The number n and the base r are connected by various relations, such as

$$\begin{aligned} n_p &= 2r_{p-1} + 3n_{p-1}, \\ r_p - r_{p-1} &= n_{p-1} + n_p, \\ r_p - 6r_{p-1} + r_{p-2} &= 0, \end{aligned}$$

from which other forms of the series for n and r can be obtained and some properties of n proved. Thus, the number of recurring figures in $\frac{1}{n}$ is always $n-1$ and the last figure is always n.

https://doi.org/10.1017/S0013091500036622 Published online by Cambridge University Press