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A CRITERION FOR THE HALL-CLOSURE OF FITTING CLASSES

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In a recent paper, Cusack has given a criterion, in terms of the Fitting class "join" operation, for a normal Fitting class to be closed under the taking of Hall π-subgroups. Here we show that Cusack's result can be slightly modified so as to give a criterion for *any* Fitting class of finite soluble groups to be closed under taking Hall π-subgroups.

1. Introduction

We will take our groups and classes of groups from the universe $\underline{\underline{S}}$ of all finite soluble groups. Let $\underline{\underline{F}}$ be a Fitting class and π be a set of primes, and let $\underline{\underline{S}}_\pi$ denote the class of all (finite, soluble) π-groups. Then $\underline{\underline{F}}$ is said to be *Hall* π-*closed* if whenever G belongs to $\underline{\underline{F}}$, then the Hall π-subgroups of G also belong to $\underline{\underline{F}}$. If we define $Y(\underline{\underline{S}}_\pi, \underline{\underline{F}})$ to be the class of all those groups whose Hall π-subgroups belong to $\underline{\underline{F}}$, then it is clear that $\underline{\underline{F}}$ is Hall π-closed if and only if $\underline{\underline{F}} \subseteq Y(\underline{\underline{S}}_\pi, \underline{\underline{F}})$. It is not hard to see that $Y(\underline{\underline{S}}_\pi, \underline{\underline{F}})$ is itself a Fitting class. If $\underline{\underline{G}}$ is a further Fitting class, then the *join*, $\underline{\underline{F}} \vee \underline{\underline{G}}$, is the smallest Fitting class to contain both $\underline{\underline{F}}$ and $\underline{\underline{G}}$. In [6], Lockett associates with each Fitting class $\underline{\underline{F}}$ the "new" Fitting classes $\underline{\underline{F}}^*$ and $\underline{\underline{F}}_*$, and shows that $\underline{\underline{S}}_*$ is the so-called smallest normal Fitting class introduced in [2]. Then the result of Cusack in which we are interested is the following.

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THEOREM [5, Theorem 5]. *Let* $\underline{\underline{F}}$ *be a normal Fitting class and* π *be a set of primes. Then* $\underline{\underline{F}}$ *is Hall* π*-closed if and only if*

$$\underline{\underline{F}} = \left(\underline{\underline{S}}_\pi \cap \underline{\underline{F}}\right) \vee \left(Y\left(\underline{\underline{S}}_\pi, \underline{\underline{S}}_*\right) \cap \underline{\underline{F}}\right) .$$

2. Preliminaries

If $\underline{\underline{F}}$ and $\underline{\underline{G}}$ are Fitting classes, then $\underline{\underline{FG}}$ denotes the class

$$\underline{\underline{FG}} = \left(X \in \underline{\underline{S}} : X/X_{\underline{\underline{F}}} \in \underline{\underline{G}}\right) ,$$

where $X_{\underline{\underline{F}}}$ denotes the $\underline{\underline{F}}$-radical of X . It is well-known that $\underline{\underline{FG}}$ is again a Fitting class.

We refer to [6] for the definitions of the classes $\underline{\underline{F}}^*$ and $\underline{\underline{F}}_*$; the following result, which is due to Lockett, collects the properties we need of these classes.

THEOREM 2.1 [6]. *Let* $\underline{\underline{F}}$ *be a Fitting class and let* $H \in \underline{\underline{F}}$. *Then*

(a) $\underline{\underline{F}}^*$ *and* $\underline{\underline{F}}_*$ *are Fitting classes with* $\underline{\underline{F}}_* \subseteq \underline{\underline{F}} \subseteq \underline{\underline{F}}^*$;

(b) $H' \leq H_{\underline{\underline{F}}_*}$;

(c) $(H \times H)_{\underline{\underline{F}}_*} = H_{\underline{\underline{F}}_*} \times H_{\underline{\underline{F}}_*} \langle \left(h^{-1}, h\right) : h \in H \rangle$; *and*

(d) if $\underline{\underline{G}}$ *is a further Fitting class then* $(\underline{\underline{F}} \cap \underline{\underline{G}})^* = \underline{\underline{F}}^* \cap \underline{\underline{G}}^*$.

Recall that if G and H are groups and $N \leq G \times H$, then N is said to be *subdirect* in $G \times H$ if $N(1 \times H) = G \times H = (G \times 1)N$. It is clear that any subgroup of $H \times H$ which contains $\langle \left(h^{-1}, h\right) : h \in H \rangle$ is subdirect in $H \times H$. We need the following result of Cusack.

THEOREM 2.2 [4, Corollary 2.6]. *Let* $\underline{\underline{U}}$ *and* $\underline{\underline{V}}$ *be Fitting classes such that* $\underline{\underline{U}} \subseteq \underline{\underline{V}}^*$. *Then a group* G *lies in* $\underline{\underline{U}} \vee \underline{\underline{V}}$ *if and only if there exists a group* $H \in \underline{\underline{U}}$ *such that* $(G \times H)_{\underline{\underline{V}}}$ *is subdirect in* $G \times H$.

The following facts about $Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}\right)$ can be found in [3] (where $Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}\right)$ is called $K_\pi(\underline{\underline{F}})$). Note that part *(b)* has also appeared in [1].

THEOREM 2.3. *Let* $\underline{\underline{F}}$ *be a Fitting class,* π *be a set of primes and* G *be a group. Then*

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(a) $G_{Y(\underline{\underline{S}}_\pi, \underline{\underline{F}})} \cap H = H_{\underline{\underline{F}}}$ *for any Hall* π*-subgroup* H *of* G ;

(b) $Y(\underline{\underline{S}}_\pi, \underline{\underline{F}}^*) = (Y(\underline{\underline{S}}_\pi, \underline{\underline{F}}))^*$;

(c) $Y(\underline{\underline{S}}_\pi, \underline{\underline{F}}) = Y(\underline{\underline{S}}_\pi, \underline{\underline{F}})\underline{\underline{S}}_{\pi'}$; *and*

(d) if $\underline{\underline{F}}$ *is Hall* π*-closed, then so also are* $\underline{\underline{F}}^*$ *and* $\underline{\underline{F}}_*$.

3. The theorem

We model our proof on Cusack's; in particular, the three results below correspond, in order, to Lemma 3, Theorem 4 and Theorem 5 of [5]. The main difference is that here we use Theorem 2.3.

LEMMA 3.1. *Let* π *be a set of primes and* $\underline{\underline{W}}$ *be a Hall* π*-closed Fitting class. Suppose that* $G \in \underline{\underline{W}}\underline{\underline{S}}_\pi$ *and that* H *is a Hall* π*-subgroup of* G . *Then* $G_{\underline{\underline{W}}}H_{\underline{\underline{W}}}$ *is the* $Y(\underline{\underline{S}}_\pi, \underline{\underline{W}})$*-radical of* G .

Proof. Let $\underline{\underline{Y}}$ denote $Y(\underline{\underline{S}}_\pi, \underline{\underline{W}})$; then $G_{\underline{\underline{W}}} \le G_{\underline{\underline{Y}}}$ since $\underline{\underline{W}}$ is Hall π-closed. Now $G = G_{\underline{\underline{W}}}H$ by hypothesis, while $G_{\underline{\underline{Y}}} \cap H = H_{\underline{\underline{W}}}$ by Theorem 2.3 *(a)*. Applying Dedekind's law, we find that $G_{\underline{\underline{Y}}} = G_{\underline{\underline{W}}}(H \cap G_{\underline{\underline{Y}}}) = G_{\underline{\underline{W}}}H_{\underline{\underline{W}}}$, as claimed.

PROPOSITION 3.2. *Let* π *be a set of primes and* $\underline{\underline{F}}$ *be a Hall* π*-closed Fitting class. Then* $\underline{\underline{F}} = (\underline{\underline{S}}_\pi \cap \underline{\underline{F}}) \vee (Y(\underline{\underline{S}}_\pi, \underline{\underline{F}}_*) \cap \underline{\underline{F}})$.

Proof. Let $\underline{\underline{Y}}$ denote $Y(\underline{\underline{S}}_\pi, \underline{\underline{F}}_*)$. It follows from Theorem 2.3 *(c)*, *(d)* that $\underline{\underline{F}}_*\underline{\underline{S}}_{\pi'} \subseteq \underline{\underline{Y}}\underline{\underline{S}}_{\pi'} = \underline{\underline{Y}}$, and so

$$\underline{\underline{F}}_*\underline{\underline{S}}_{\pi'} \cap \underline{\underline{F}} \subseteq \underline{\underline{Y}} \cap \underline{\underline{F}} . \tag{3.3}$$

Now let $G \in \underline{\underline{F}}_*\underline{\underline{S}}_\pi \cap \underline{\underline{F}}$, and let H be a Hall π-subgroup of G ; then $H \in \underline{\underline{F}}$. Form $G \times H \in \underline{\underline{F}}$. Applying Lemma 3.1 with $\underline{\underline{W}} = \underline{\underline{F}}_*$, and Theorem 2.1 *(c)* with $H \in \underline{\underline{F}}$, we find that

$$\begin{aligned}(G \times H)_{\underline{\underline{Y}}} &= (G \times H)_{\underline{\underline{F}}_*}(H \times H)_{\underline{\underline{F}}_*} \\ &\ge (G_{\underline{\underline{F}}_*}H_{\underline{\underline{F}}_*} \times H_{\underline{\underline{F}}_*})\langle (h^{-1}, h) : h \in H\rangle .\end{aligned}$$

But clearly $G = G_{\underline{\underline{F}}_*}H$, and it follows that $(G \times H)_{\underline{\underline{Y}}}$ is subdirect in

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$G \times H$. Since $G \times H \in \underline{\underline{F}}$, this says that $(G \times H)_{\underline{\underline{Y}} \cap \underline{\underline{F}}}$ is subdirect in $G \times H$.

We now wish to apply Theorem 2.2 with $\underline{\underline{U}} = \underline{\underline{S}}_\pi \cap \underline{\underline{F}}$ and $\underline{\underline{V}} = \underline{\underline{Y}} \cap \underline{\underline{F}}$. Note that by Theorem 2.1 *(d)* and Theorem 2.3 *(b)*, *(d)*, we have

$$\underline{\underline{V}}^* = \underline{\underline{Y}}^* \cap \underline{\underline{F}}^* = Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}^*\right) \cap \underline{\underline{F}}^* = \underline{\underline{F}}^* .$$

Thus $\underline{\underline{U}} \subseteq \underline{\underline{V}}^*$, and Theorem 2.2 implies that

$$\underline{\underline{F}}_*\underline{\underline{S}}_\pi \cap \underline{\underline{F}} \subseteq \left(\underline{\underline{S}}_\pi \cap \underline{\underline{F}}\right) \vee \left(\underline{\underline{Y}} \cap \underline{\underline{F}}\right) . \tag{3.4}$$

But it follows from Theorem 2.1 *(b)* that

$$\underline{\underline{F}} = \left(\underline{\underline{F}}_*\underline{\underline{S}}_\pi \cap \underline{\underline{F}}\right) \vee \left(\underline{\underline{F}}_*\underline{\underline{S}}_{\pi'} \cap \underline{\underline{F}}\right) ,$$

and so, combining (3.3) and (3.4), we conclude that

$$\underline{\underline{F}} = \left(\underline{\underline{S}}_\pi \cap \underline{\underline{F}}\right) \vee \left(\underline{\underline{Y}} \cap \underline{\underline{F}}\right) ,$$

as required.

THEOREM 3.5. *Let* π *be a set of primes and* $\underline{\underline{F}}$ *be a Fitting class. Then* $\underline{\underline{F}}$ *is Hall* π*-closed if and only if*

$$\underline{\underline{F}} = \left(\underline{\underline{S}}_\pi \cap \underline{\underline{F}}\right) \vee \left(Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}_*\right) \cap \underline{\underline{F}}\right) .$$

Proof. The "only if" assertion has been proved above. Thus suppose that $\underline{\underline{F}} = \left(\underline{\underline{S}}_\pi \cap \underline{\underline{F}}\right) \vee \left(Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}_*\right) \cap \underline{\underline{F}}\right)$. Since $\underline{\underline{S}}_\pi \cap \underline{\underline{F}} \subseteq Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}\right)$, and since the operator $Y\left(\underline{\underline{S}}_\pi, \ \right)$ clearly respects inclusions, then Theorem 2.1 *(a)* implies that

$$\underline{\underline{F}} \subseteq Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}\right) \vee \left(Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}\right) \cap \underline{\underline{F}}\right) = Y\left(\underline{\underline{S}}_\pi, \underline{\underline{F}}\right) .$$

Thus $\underline{\underline{F}}$ is Hall π-closed, and the proof is complete.

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