

A QUANTITATIVE VERSION OF JAMES'S COMPACTNESS THEOREM

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Abstract We introduce two measures of weak non-compactness Ja_E and Ja that quantify, via distances, the idea of boundary that lies behind James's Compactness Theorem. These measures tell us, for a bounded subset C of a Banach space E and for given $x^* \in E^*$, how far from E or C one needs to go to find $x^{**} \in \overline{C}^{w^*} \subset E^{**}$ with $x^{**}(x^*) = \sup x^*(C)$. A quantitative version of James's Compactness Theorem is proved using Ja_E and Ja , and in particular it yields the following result. *Let C be a closed convex bounded subset of a Banach space E and $r > 0$. If there is an element x_0^{**} in \overline{C}^{w^*} whose distance to C is greater than r , then there is $x^* \in E^*$ such that each $x^{**} \in \overline{C}^{w^*}$ at which $\sup x^*(C)$ is attained has distance to E greater than $\frac{1}{2}r$.* We indeed establish that Ja_E and Ja are equivalent to other measures of weak non-compactness studied in the literature. We also collect particular cases and examples showing when the inequalities between the different measures of weak non-compactness can be equalities and when the inequalities are sharp.

Keywords: Banach space; measure of weak non-compactness; James's Compactness Theorem

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1. Introduction

The celebrated James Compactness Theorem says that a closed convex subset C of a Banach space E is weakly compact whenever each $x^* \in E^*$ attains its supremum on C [11]. In particular, E is reflexive whenever each $x^* \in E^*$ attains its norm at some point of the closed unit ball B_E of E . In the present paper we prove a quantitative version of this theorem. Such a result not only fits into the recent research on quantitative versions of various famous theorems on compactness presented in, for example, [2, 5, 7–10], to which we relate our results here, but also yields a strengthening of James's Theorem itself. In particular, we get the following result.

Theorem 1.1. *Let E be a Banach space and let $C \subset E$ be a closed convex bounded set that is not weakly compact. Let $0 \leq c < \frac{1}{2}\hat{d}(\overline{C}^{w^*}, C)$ be arbitrary. Then there is*

some $x^* \in E^*$ such that for any $x^{**} \in \overline{C}^{w^*}$ satisfying $x^{**}(x^*) = \sup x^*(C)$ we have $\text{dist}(x^{**}, E) > c$.

We use the following notation: if A and B are non-empty subsets of a Banach space E , then $d(A, B)$ denotes the *usual inf distance* between A and B , and the *Hausdorff non-symmetrized distance* from A to B is defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

Notice that $\hat{d}(A, B)$ can be different from $\hat{d}(B, A)$ and that $\max\{\hat{d}(A, B), \hat{d}(B, A)\}$ is the Hausdorff distance between A and B . Notice further that $\hat{d}(A, B) = 0$ if and only if $A \subset \bar{B}$ and that

$$\hat{d}(A, B) = \inf\{\varepsilon > 0 : A \subset B + \varepsilon B_E\}.$$

Let us remark that we consider the space E canonically embedded into its bidual E^{**} and that by \overline{C}^{w^*} we mean the weak* closure of C in the bidual E^{**} .

When applying Theorem 1.1 for $c = 0$, we obtain the classical James Compactness Theorem. Our results in this paper go beyond Theorem 1.1. We should stress that what we really do in this paper is introduce several measures of weak non-compactness in Banach spaces related to distances to boundaries and then study their relationship with other well-known measures of weak non-compactness previously studied. Our main result is Theorem 3.1. Combination with known or easy results gives Corollary 3.5. Theorem 1.1 is then an immediate consequence.

The quantities that we introduce are the following.

Definition 1.2. Given a bounded subset H of a Banach space E we define

$$\text{Ja}_E(H) = \inf\{\varepsilon > 0 : \text{for every } x^* \in E^*, \text{ there is } x^{**} \in \overline{H}^{w^*} \\ \text{such that } x^{**}(x^*) = \sup x^*(H) \text{ and } d(x^{**}, E) \leq \varepsilon\}$$

and

$$\text{Ja}(H) = \inf\{\varepsilon > 0 : \text{for every } x^* \in E^*, \text{ there is } x^{**} \in \overline{H}^{w^*} \\ \text{such that } x^{**}(x^*) = \sup x^*(H) \text{ and } d(x^{**}, H) \leq \varepsilon\}.$$

Note that the definition of $\text{Ja}(H)$ is clearly inspired by the notion of a *boundary* that is hidden in James's Theorem. Recall that if Y is a Banach space and $K \subset Y^*$ is a convex weak*-compact set, then a subset $B \subset K$ is called a *boundary* of K if, for each $y \in Y$, there is $b^* \in B$ such that

$$b^*(y) = \sup_{k^* \in K} k^*(y).$$

James's Compactness Theorem can now be rephrased in the following way.

Theorem. *Let E be a Banach space and let $C \subset E$ be a bounded closed convex set. If C is a boundary of \overline{C}^{w^*} , then C is weakly compact.*

We will study the relationship of $\text{Ja}_E(C)$ and $\text{Ja}(C)$ to other quantities measuring weak non-compactness of C . The two most obvious quantities of this kind are $\hat{d}(\overline{C}^{w^*}, C)$ and $\hat{d}(\overline{C}^{w^*}, E)$. We stress that these two quantities can be different (see the examples in § 5). The first one can be called ‘measure of weak non-compactness’ of C ; the other one can be called ‘measure of relative weak non-compactness’ of C .

Using the notation introduced above, Theorem 1.1 says that $\text{Ja}_E(C) \geq \frac{1}{2} \hat{d}(\overline{C}^{w^*}, C)$ holds for any closed convex bounded subset C of a Banach space E .

In the following section we introduce several other quantities measuring weak non-compactness and summarize easy inequalities among them. In § 3 we formulate and prove our main result. As a corollary we obtain that all considered quantities measuring weak non-compactness are equivalent.

In § 4 we discuss the relationship to the quantitative version of Krein's Theorem. Section 5 contains examples showing that most of the inequalities are sharp. In the final section we study some particular cases in which some of the inequalities become equalities.

2. Measures of weak non-compactness

In this section we define and relate several quantities measuring weak non-compactness of a bounded set in a Banach space. Such quantities are called *measures of weak non-compactness*. Measures of non-compactness or weak non-compactness have been successfully applied to the study of compactness in operator theory, differential equations and integral equations (see, for example, [2, 3, 5–10, 12, 13]). An axiomatic approach to measures of weak non-compactness may be found in [4, 13]. But many of the natural quantities do not satisfy all the axioms, so we will not adopt this approach. Anyway, there is one property which should be pointed out: a measure of weak non-compactness should have value zero if and only if the respective set is relatively weakly compact.

Let (x_n) be a bounded sequence in a Banach space E . We define $\text{clust}_{E^{**}}((x_n))$ to be the set of all cluster points of this sequence in (E^{**}, w^*) , i.e.

$$\text{clust}_{E^{**}}((x_n)) = \bigcap_{n \in \mathbb{N}} \overline{\{x_m : m > n\}}^{w^*}.$$

Given a bounded subset H of a Banach space E we define

$$\gamma(H) = \sup \left\{ \left| \lim_n \lim_m x_m^*(x_n) - \lim_m \lim_n x_m^*(x_n) \right| : (x_m^*) \subset B_{E^*}, (x_n) \subset H \right\},$$

assuming the limits involved exist:

$$\text{ck}_E(H) = \sup_{(x_n) \subset H} d(\text{clust}_{E^{**}}((x_n)), E), \quad \text{ck}(H) = \sup_{(x_n) \subset H} d(\text{clust}_{E^{**}}((x_n)), H).$$

Properties of γ can be found in [2, 3, 5, 7, 13], whereas ck_E can be found in [2] (note that ck_E is denoted as ck in that paper; do not mistake it for ck above).

So for a bounded set $H \subset E$ we have the following quantities measuring weak non-compactness:

$$\hat{d}(\overline{H}^{w^*}, H), \quad \hat{d}(\overline{H}^{w^*}, E), \quad \text{ck}(H), \quad \text{ck}_E(H), \quad \gamma(H), \quad \text{Ja}(H), \quad \text{Ja}_E(H).$$

Let us stress the different nature of these quantities.

First, the quantities $\hat{d}(\overline{H}^{w^*}, H)$, $\text{ck}(H)$, $\gamma(H)$ and $\text{Ja}(H)$ do not depend directly on the space E . More exactly, if F is a Banach space and $H \subset E \subset F$, where E is a closed linear subspace of F and H a bounded subset of E , then these quantities are the same, no matter whether we consider H as a subset of E or as a subset of F . This is trivial for $\hat{d}(\overline{H}^{w^*}, H)$, and $\text{ck}(H)$ and follows from the Hahn–Banach Extension Theorem for $\gamma(H)$ and $\text{Ja}(H)$.

On the other hand, the quantities $\hat{d}(\overline{H}^{w^*}, E)$, $\text{ck}_E(H)$ and $\text{Ja}_E(H)$ may decrease if the space E is enlarged. More exactly, if $H \subset E \subset F$ are as above, then it may be that $\hat{d}(\overline{H}^{w^*}, F) < \hat{d}(\overline{H}^{w^*}, E)$ and similarly for the other quantities (see examples in § 5).

Since we are interested in James's Compactness Theorem, the most important case for us is the case of a closed convex bounded set H . Nonetheless, we define the quantities for an arbitrary bounded set and formulate results as generally as possible. Anyway, such generalizations do not yield new results in view of the following proposition.

Proposition 2.1. *Let E be a Banach space and let $H \subset E$ be a bounded subset.*

- (i) *All the above defined quantities have the same value for H and for \overline{H} .*
- (ii) *The quantities $\hat{d}(\overline{H}^{w^*}, E)$, $\text{Ja}_E(H)$ and $\gamma(H)$ have the same value for H and for the weak closure of H .*
- (iii) *$\text{Ja}_E(\text{co } H) \leq \text{Ja}_E(H)$ and $\gamma(\text{co } H) = \gamma(H)$.*

Proof. Assertion (i) is obvious.

Let us proceed with assertion (iii). The first inequality is trivial. The second equality is not easy at all; it is proved in [7, Theorem 13] (see [5, Theorem 3.3] for a different proof).

Finally, let us show (ii). The case of $\gamma(H)$ follows from (i) and (iii). The other cases are trivial. \square

As for the quantities not covered by this proposition, it does not seem to be clear whether $\text{ck}_E(H)$ has the same value for H and for the weak closure of H . The quantities $\text{ck}_E(H)$ and $\hat{d}(\overline{H}^{w^*}, E)$ may increase when passing to $\text{co } H$: this follows from the results of [8, 10] (see Example 5.6).

We do not know whether the quantity $\text{Ja}_E(H)$ may really decrease when passing to $\text{co } H$. This question seems not to be easy. Indeed, in view of the obvious inequalities $\text{Ja}_E(\text{co } H) \leq \text{Ja}_E(H)$, $\text{ck}_E(H) \leq \text{ck}_E(\text{co } H)$ and taking into account $\text{Ja}_E(H) \leq \text{ck}_E(H)$ (see Proposition 2.2), if we had $\text{Ja}_E(\text{co } H) < \text{Ja}_E(H)$, then we would conclude that $\text{Ja}_E(\text{co } H) < \text{ck}_E(\text{co } H)$. The only example of a convex set C satisfying $\text{Ja}_E(C) < \text{ck}_E(C)$ known to us is given in Example 5.6 and it seems that it cannot easily be improved.

As for the quantities $\hat{d}(\overline{H}^{w^*}, H)$, $\text{ck}(H)$ and $\text{Ja}(H)$, they are natural in the case of a convex set H . If H is not convex, they are not measures of weak non-compactness in the above sense, since they may be strictly positive even if H is relatively weakly compact. This is demonstrated by Example 2.3.

The following proposition summarizes the easy inequalities.

Proposition 2.2. *Let E be a Banach space.*

(i) *Let $H \subset E$ be a bounded set. Then the following inequalities hold true:*

$$\text{Ja}_E(H) \leq \text{ck}_E(H) \leq \hat{d}(\overline{H}^{w^*}, E) \leq \gamma(H).$$

(ii) *Let $C \subset E$ be a convex bounded set. Then the following inequalities hold true:*

$$\begin{array}{ccccccc} \text{ck}_E(C) & \leq & \hat{d}(\overline{C}^{w^*}, E) & & & & \\ \leq & & \leq & & \leq & & \\ \text{Ja}_E(C) & \leq & \text{Ja}(C) & \leq & \text{ck}(C) & \leq & \hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C) \end{array}$$

Proof. Let us start with (i). The inequality $\text{ck}_E(H) \leq \hat{d}(\overline{H}^{w^*}, E)$ is trivial. The inequality $\hat{d}(\overline{H}^{w^*}, E) \leq \gamma(H)$ is proved in [7, Proposition 8(ii)] (see also [5, Corollary 4.3]). Let us show that $\text{Ja}_E(H) \leq \text{ck}_E(H)$.

Note first that if $\text{Ja}_E(H) = 0$, then inequality $0 \leq \text{ck}_E(H)$ trivially holds. Assume that $0 < \text{Ja}_E(H)$ and take an arbitrary $0 < \varepsilon < \text{Ja}_E(H)$. By definition there exists $x^* \in E^*$ such that for any $x^{**} \in \overline{H}^{w^*}$ with $x^{**}(x^*) = \sup x^*(H)$ we have that $\varepsilon < d(x^{**}, E)$. Fix a sequence (x_n) in H satisfying $\sup x^*(H) = \lim_n x^*(x_n)$. Then each weak* cluster point x^{**} of (x_n) satisfies $x^{**}(x^*) = \sup x^*(H)$; hence $\varepsilon \leq d(\text{clust}_{E^{**}}((x_n)), E)$ and therefore $\varepsilon \leq \text{ck}_E(H)$. This finishes the proof for $\text{Ja}_E(H) \leq \text{ck}_E(H)$.

Now let us proceed with part (ii). All inequalities are obvious except $\text{Ja}_E(C) \leq \text{ck}_E(C)$, $\text{Ja}(C) \leq \text{ck}(C)$ and $\hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C)$. The former follows from (i). The latter can be proved in the same way.

Now we prove that $\hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C)$. Suppose that $r > 0$ is such that $\hat{d}(\overline{C}^{w^*}, C) > r$. Fix $x^{**} \in \overline{C}^{w^*}$ such that $d(x^{**}, C) > r$. By the Hahn–Banach Separation Theorem there exist $x^{***} \in X^{***}$ with $\|x^{***}\| = 1$ and $s \in \mathbb{R}$ such that

$$x^{***}(x^{**}) > s + r > s > \sup_{x \in C} x^{***}(x). \tag{2.1}$$

We will construct by induction two sequences (x_n) in C and (x_n^*) in B_{E^*} such that the following conditions are satisfied for each $n \in \mathbb{N}$:

- (i) $x^{**}(x_n^*) > s + r$;
- (ii) $x_n^*(x_m) < s$ for $m < n$;
- (iii) $x_m^*(x_n) > s + r$ for $m \leq n$.

By (2.1) and the Goldstine Theorem we can choose x_1^* satisfying (i). Now suppose that $n \in \mathbb{N}$ is such that x_m^* for $m \leq n$ and x_m for $m < n$ satisfy (i)–(iii). Using that (i) holds for x_1^*, \dots, x_n^* and that $x^{**} \in \overline{C}^{w^*}$, we can choose $x_n \in C$ satisfying (iii). Further, by (2.1) and the Goldstine Theorem we can find $x_{n+1}^* \in B_{E^*}$ satisfying (i) and (ii). This completes the construction.

By passing to subsequences we may assume that $\lim_n x_n^*(x_m)$ exists for all $m \in \mathbb{N}$ and that $\lim_m x_n^*(x_m)$ exists for all $n \in \mathbb{N}$ and (ii) and (iii) are satisfied. By taking further subsequences we may assume also that the limits $\lim_n \lim_m x_n^*(x_m)$ and $\lim_m \lim_n x_n^*(x_m)$ exist and that again (ii) and (iii) are satisfied. By the construction we get

$$\lim_n \lim_m x_n^*(x_m) \geq s + r \quad \text{and} \quad \lim_m \lim_n x_n^*(x_m) \leq s,$$

hence $\gamma(C) \geq r$. This completes the proof. \square

We note that in the second part of the proposition above we only have to use the convexity of C to prove the inequality $\hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C)$; the rest of the inequalities hold for an arbitrary bounded set. But for non-convex sets only the first part is interesting. This is witnessed by the following example which shows in particular the failure of the inequality $\hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C)$ if C is not convex.

Example 2.3. Let $E = c_0$ or $E = \ell_p$ for some $p \in (1, \infty)$. Let $H = \{e_n : n \in \mathbb{N}\}$, where e_n is the canonical n th basic vector. Then H is relatively weakly compact; hence, $\hat{d}(\overline{H}^{w^*}, E) = \text{ck}_E(H) = \text{Ja}_E(H) = \gamma(H) = 0$. However,

$$\text{Ja}(H) = \text{ck}(H) = \hat{d}(\overline{H}^{w^*}, H) = 1.$$

Proof. As the sequence (e_n) weakly converges to 0, H is relatively weakly compact. This finishes the proof of the first part. Moreover, \overline{H}^{w^*} is in fact the weak closure of H in E and is equal to $H \cup \{0\}$. Thus, clearly $\hat{d}(\overline{H}^{w^*}, H) = \text{ck}(H) = 1$. Finally, to show $\text{Ja}(H) \geq 1$, consider $x^* \in E^*$ represented by the sequence $(-1/2^n)_{n=1}^\infty$ in the respective sequence space. Then $\sup x^*(H) = 0$ and the only point in \overline{H}^{w^*} at which the supremum is attained is 0. The observation that $d(0, H) = 1$ completes the proof. \square

We remark that for non-convex H it is more natural to consider the quantity $\hat{d}(\overline{H}^{w^*}, \text{co } H)$ instead of using $\hat{d}(\overline{H}^{w^*}, H)$ (see §4). Similar versions of other quantities can be studied as well.

3. Quantitative versions of James's Theorem

This section is devoted to the proof of the main results of this paper. In the course of the proof we use a proof of James's Compactness Theorem due to Pryce in [14].

Theorem 3.1. *Let E be a Banach space and $H \subset E$ a bounded subset. Then*

$$\frac{1}{2}\gamma(H) \leq \text{Ja}_E(H).$$

Proof. Assume that $\gamma(H) > r$ for some $r > 0$. We denote by

- (i) F the space of all norm continuous positive homogenous real-valued functions on E , i.e. continuous functions $f: E \rightarrow \mathbb{R}$ satisfying $f(\alpha x) = \alpha f(x)$, $\alpha \geq 0$ and $x \in E$,
- (ii) $p(f) = \sup f(H)$, $f \in F$,
- (iii) $P(f) = \sup |f|(H)$, $f \in F$.

Then p is a sublinear functional and P is a seminorm on F .

Let $(f_i) \subset B_{E^*}$ and $(z_j) \subset H$ be sequences such that

$$\lim_i \lim_j f_i(z_j) - \lim_j \lim_i f_i(z_j) > r$$

and all the limits involved exist. By omitting finitely many elements of (f_i) we may assume that

$$\lim_j f_i(z_j) - \lim_j \lim_i f_i(z_j) > r, \quad i \in \mathbb{N}. \tag{3.1}$$

Hence, for every $i \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that

$$f_i(z_j) - \lim_i f_i(z_j) > r, \quad j \geq j_0.$$

Let X stand for the linear span of $\{f_i: i \in \mathbb{N}\}$. As X is separable in the seminorm P and the functionals f_i are equicontinuous for the norm on E , it follows from [14, Lemma 2] that we can suppose without loss of generality that

$$p\left(f - \liminf_i f_i\right) = p\left(f - \limsup_i f_i\right) \quad \text{for all } f \in X. \tag{3.2}$$

We denote

$$K_n = \text{conv}\{f_i: i \geq n\}, \quad n \in \mathbb{N},$$

and thus we obtain

$$F \supset E^* \supset X \supset K_1 \supset K_2 \supset \dots$$

By the proof of [14, Lemma 3] and bearing in mind the inequality (3.1), we obtain

$$p\left(f - \liminf_i f_i\right) > r, \quad f \in K_1. \tag{3.3}$$

Next we quote [14, Lemma 4].

Claim 3.2. *Let Y be a linear space, let ρ, β, β' be strictly positive numbers, let p be a sublinear functional on Y , let $A \subset Y$ be a convex set and let $u \in Y$ satisfy*

$$\inf_{a \in A} p(u + \beta a) > \beta \rho + p(u).$$

Then there exists $a_0 \in A$ such that

$$\inf_{a \in A} p(u + \beta a_0 + \beta' a) > \beta' \rho + p(u + \beta a_0).$$

This claim will be used to prove the following one, which is a mild strengthening of [14, Lemma 5]. Let us fix an arbitrary $r' \in (0, r)$.

Claim 3.3. *Let (β_n) be a sequence of strictly positive numbers. Then there exists a sequence (g_n) in F such that $g_n \in K_n$ for $n \in \mathbb{N}$ and*

$$p\left(\sum_{i=1}^n \beta_i (g_i - \liminf_j f_j)\right) > \beta_n r' + p\left(\sum_{i=1}^{n-1} \beta_i (g_i - \liminf_j f_j)\right), \quad n \in \mathbb{N}. \quad (3.4)$$

Proof. The construction proceeds by induction. Let $f_0 = \liminf_j f_j$.

If $n = 1$, we use Claim 3.2 for $u = 0$, $\beta = \beta_1$, $\beta' = \beta_2$, $\rho = r'$ and $A = K_1 - f_0$. By (3.3),

$$\begin{aligned} \inf_{g \in A} p(u + \beta g) &= \inf_{g \in A} \beta p(g) \\ &= \beta_1 \inf_{f \in K_1} p\left(f - \liminf_j f_j\right) \\ &> \beta_1 r' \\ &= \beta_1 r' + p(u), \end{aligned}$$

and hence Claim 3.2 gives the existence of $g_1 \in K_1$ satisfying

$$\inf_{f \in K_1} p(\beta_1(g_1 - f_0) + \beta_2(f - f_0)) > \beta_2 r' + p(\beta_1(g_1 - f_0)).$$

This finishes the first step of the construction.

Assume now that we have found $g_i \in K_i$, $i = 1, \dots, n-1$, for some $n \in \mathbb{N}$, $n \geq 2$, such that

$$\inf_{f \in K_{n-1} - f_0} p\left(\sum_{i=1}^{n-1} \beta_i (g_i - f_0) + \beta_n f\right) > \beta_n r' + p\left(\sum_{i=1}^{n-1} \beta_i (g_i - f_0)\right).$$

We use Claim 3.2 with $u = \sum_{i=1}^{n-1} \beta_i (g_i - f_0)$, $\beta = \beta_n$, $\beta' = \beta_{n+1}$, $\rho = r'$ and $A = K_n - f_0$. Since $K_n \subset K_{n-1}$, inductive hypothesis gives

$$\inf_{f \in A} p(u + \beta f) \geq \inf_{f \in K_{n-1} - f_0} p(u + \beta f) > \beta_n r' + p(u).$$

By Claim 3.2, there exists $g_n \in K_n$ such that

$$\inf_{f \in A} p\left(\sum_{i=1}^n \beta_i (g_i - f_0) + \beta_{n+1} f\right) > \beta_{n+1} r' + p\left(\sum_{i=1}^{n-1} \beta_i (g_i - f_0) + \beta_n (g_n - f_0)\right).$$

This completes the inductive construction.

We have obtained elements $g_n \in K_n$, $n \in \mathbb{N}$, such that

$$\inf_{g \in K_n} p\left(\sum_{i=1}^n \beta_i (g_i - f_0) + \beta_{n+1} (g - f_0)\right) > \beta_{n+1} r' + p\left(\sum_{i=1}^n \beta_i (g_i - f_0)\right).$$

Since $g_{n+1} \in K_{n+1} \subset K_n$, this yields

$$p\left(\sum_{i=1}^n \beta_i(g_i - f_0) + \beta_{n+1}(g_{n+1} - f_0)\right) > \beta_{n+1}r' + p\left(\sum_{i=1}^n \beta_i(g_i - f_0)\right).$$

This finishes the proof. □

Let $\beta_i > 0, i \in \mathbb{N}$, be chosen such that

$$\lim_n \frac{1}{\beta_n} \sum_{i=n+1}^\infty \beta_i = 0.$$

Let (g_n) be a sequence provided by Claim 3.3. Since for every $n \in \mathbb{N}$ we have that $g_n \in K_n \subset B_{E^*}$, we can select a weak*-cluster point $g_0 \in B_{E^*}$ of (g_n) . By [14, Lemma 6], we have the following observation.

Claim 3.4. For any $f \in X, p(f - g_0) = p(f - \liminf_n f_n)$.

By Claim 3.4, we can replace $\liminf_j f_j$ by g_0 in (3.4) and get the following inequalities

$$p\left(\sum_{i=1}^n \beta_i(g_i - g_0)\right) > \beta_n r' + p\left(\sum_{i=1}^{n-1} \beta_i(g_i - g_0)\right), \quad n \in \mathbb{N}. \tag{3.5}$$

We set $M = \sup\{\|x\| : x \in H\}$ and remark that $\|g_i - g_0\| \leq 2, i \in \mathbb{N}$.

We set $g = \sum_{i=1}^\infty \beta_i(g_i - g_0)$. Let $u \in \overline{H}^{w^*}$ be an arbitrary point satisfying $g(u) = \sup g(H)$. Then, for any $n \in \mathbb{N}$, from (3.5) we get

$$\begin{aligned} \sum_{i=1}^n \beta_i(g_i - g_0)(u) &= g(u) - \sum_{i=n+1}^\infty \beta_i(g_i - g_0)(u) \\ &\geq p(g) - 2M \sum_{i=n+1}^\infty \beta_i \\ &\geq p\left(\sum_{i=1}^n \beta_i(g_i - g_0)\right) - p\left(\sum_{i=1}^n \beta_i(g_i - g_0) - g\right) - 2M \sum_{i=n+1}^\infty \beta_i \\ &\geq p\left(\sum_{i=1}^n \beta_i(g_i - g_0)\right) - 4M \sum_{i=n+1}^\infty \beta_i \\ &> \beta_n r' + p\left(\sum_{i=1}^{n-1} \beta_i(g_i - g_0)\right) - 4M \sum_{i=n+1}^\infty \beta_i \\ &\geq \beta_n r' + \sum_{i=1}^{n-1} \beta_i(g_i - g_0)(u) - 4M \sum_{i=n+1}^\infty \beta_i. \end{aligned}$$

Hence,

$$(g_n - g_0)(u) \geq r' - 4M \frac{1}{\beta_n} \sum_{i=n+1}^\infty \beta_i, \quad n \in \mathbb{N},$$

which gives

$$\liminf_n (g_n - g_0)(u) \geq r'. \quad (3.6)$$

Let $v \in E$ be arbitrary. Then $g_0(v) \geq \liminf_n g_n(v)$, which along with (3.6) gives

$$\begin{aligned} r' &\leq \liminf_n g_n(u) - \liminf_n g_n(v) + g_0(v - u) \\ &\leq -\liminf_n (g_n(v) - g_n(u)) + g_0(v - u) \leq 2\|v - u\|. \end{aligned}$$

By the definition of $\text{Ja}_E(H)$ it follows that $\text{Ja}_E(H) \geq \frac{1}{2}r'$. Since r satisfying $\gamma(H) > r$ and $r' \in (0, r)$ are arbitrary, we conclude that $\text{Ja}_E(H) \geq \frac{1}{2}\gamma(H)$. \square

As a consequence of Theorem 3.1 we obtain that all measures of non-compactness that we have considered in this paper are equivalent. In other words, *all classical approaches used to study weak compactness in Banach spaces (Tychonoff's Theorem, Eberlein's Theorem, Grothendieck's Theorem and James's Theorem) are qualitatively and quantitatively equivalent.*

Corollary 3.5. *Let E be a Banach space.*

(i) *Let $H \subset E$ be a bounded set. Then the following inequalities hold true:*

$$\frac{1}{2}\gamma(H) \leq \text{Ja}_E(H) \leq \text{ck}_E(H) \leq \hat{d}(\overline{H}^{w^*}, E) \leq \gamma(H).$$

(ii) *Let $C \subset E$ be a bounded convex set. Then the following inequalities hold true:*

$$\begin{aligned} \text{ck}_E(C) &\leq \hat{d}(\overline{C}^{w^*}, E) \\ &\leq \text{Ja}_E(C) \leq \text{Ja}(C) \leq \text{ck}(C) \leq \hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C). \end{aligned}$$

Proof. This result follows from Proposition 2.2 and Theorem 3.1. \square

The fact that the measures of weak non-compactness $H \mapsto \hat{d}(\overline{H}^{w^*}, E)$, γ and ck_E are equivalent can be found in [2, 7] with very different approaches.

In § 5 we offer several examples showing that in the corollary above any of the inequalities may become equalities and that most of them may become strict.

Corollary 3.6. *Let E be a Banach space and let $C \subset E$ be a closed convex bounded subset. Then C is weakly compact provided $\text{Ja}_E(C) = 0$ (i.e. if for every $\varepsilon > 0$ and every $x^* \in X^*$ there is $x^{**} \in \overline{C}^{w^*}$ such that $x^{**}(x^*) = \sup x^*(C)$ and $d(x^{**}, E) \leq \varepsilon$).*

4. Relationship to the quantitative version of Krein's Theorem

Let E be a Banach space and let $C \subset E$ be a bounded convex set. Then $\text{ext } \overline{C}^{w^*}$, the set of extreme points of \overline{C}^{w^*} , is a boundary for \overline{C}^{w^*} . Therefore, the following inequalities are obvious:

$$\left. \begin{aligned} \hat{d}(\overline{\text{ext } \overline{C}^{w^*}}^{w^*}, C) &\geq \hat{d}(\text{ext } \overline{C}^{w^*}, C) \geq \text{Ja}(C), \\ \hat{d}(\overline{\text{ext } \overline{C}^{w^*}}^{w^*}, E) &\geq \hat{d}(\text{ext } \overline{C}^{w^*}, E) \geq \text{Ja}_E(C). \end{aligned} \right\} \tag{4.1}$$

These inequalities enable us to prove the following statement.

Corollary 4.1. *Let E be a Banach space and let $H \subset E$ be a bounded set. Then the following inequalities hold:*

- (i) $\hat{d}(\overline{\text{co } H}^{w^*}, E) \leq 2\hat{d}(\overline{H}^{w^*}, E);$
- (ii) $\hat{d}(\overline{\text{co } H}^{w^*}, \text{co } H) \leq 2\hat{d}(\overline{H}^{w^*}, \text{co } H).$

Proof. Set $C = \overline{\text{co } H}$. Then $\text{ext } \overline{C}^{w^*} \subset \overline{H}^{w^*}$, so the inequalities follow from (4.1) and Corollary 3.5. □

We remark that assertion (i) was proved in [7] and independently in [8] and [5]. In [8, 10] some examples are given which show that the inequality is optimal, i.e. the equality can take place if the quantities are non-zero. However, these examples do not work for assertion (ii). Hence, the following problem seems to be natural.

Question 4.2. Let E be a Banach space and let $H \subset E$ be a bounded set. Is it true that

$$\hat{d}(\overline{\text{co } H}^{w^*}, \text{co } H) = \hat{d}(\overline{H}^{w^*}, \text{co } H)?$$

In [7], Corollary 4.1 (i) is called a quantitative version of Krein's Theorem. Krein's Theorem asserts that a closed convex hull of a weakly compact set is again weakly compact. This is the case when the quantities are 0. In view of this assertion (ii) may also be called a quantitative version of Krein's Theorem. An interesting phenomenon is that there are examples showing that the inequality (i) is sharp but we do not know whether the inequality (ii) is sharp. Both examples showing sharpness of (i) are of similar nature: a set H is constructed in a space E_0 such that

$$\hat{d}(\overline{\text{co } H}^{w^*}, \text{co } H) = \hat{d}(\overline{H}^{w^*}, \text{co } H) = 1.$$

Then the space E_0 is enlarged in a clever way to E such that $\hat{d}(\overline{\text{co } H}^{w^*}, E)$ equals 1 but $\hat{d}(\overline{H}^{w^*}, E)$ decreases to $\frac{1}{2}$. If the space E is enlarged even further, the quantity $\hat{d}(\overline{\text{co } H}^{w^*}, E)$ will also decrease to $\frac{1}{2}$ and it will no longer be a counter-example. Hence, a possible counter-example showing sharpness of (ii) should be of a quite different nature.

Moreover, one can show (although it is not obvious) that the answer to the above question is positive if H is norm-separable. This is another indication of a great difference between (i) and (ii) as the example from [10] is norm-separable (see Example 5.6).

5. Examples

In this section we collect examples showing the sharpness of some of the inequalities that are collected in Corollary 3.5. We remark that unless all the quantities are zero, at least one of the inequalities must be strict. We stress again that the examples in this section show in particular that any of the inequalities may become equalities and that most of them may become strict.

Example 5.1. Let $E = c_0$ and $C = B_E$. Then $\gamma(C) = 1$ and $\text{Ja}_E(C) = 1$. Hence, all other quantities are also equal to 1.

Proof. The equality $\gamma(C) = 1$ follows from [13, Example 2.7 and Theorem 2.8]. To show that $\text{Ja}_E(C) \geq 1$, take $x^* \in E^*$ represented by the sequence $(1/2^n)_{n=1}^\infty$ in ℓ_1 . The only element of $\overline{C}^{w^*} = B_{\ell_\infty}$ at which x^* attains its supremum on C is the constant sequence $(1)_{n=1}^\infty$ whose distance from E is clearly 1. The rest now follows from Corollary 3.5. \square

Example 5.2. Let $E = \ell_1$ and $C = B_E$. Then $\gamma(C) = 2$ and $\hat{d}(\overline{C}^{w^*}, C) = 1$. Hence, all other quantities are equal to 1.

Proof. It is clear that $\hat{d}(\overline{C}^{w^*}, C) \leq 1$. Further, the inequality $\gamma(C) \geq 2$ is witnessed by sequences (x_n) and (x_n^*) , where x_n is the n th canonical basic vector of ℓ_1 and $x_n^* \in B_{\ell_\infty}$ is defined by

$$x_n^*(m) = \begin{cases} 1, & m \leq n, \\ -1, & m > n. \end{cases}$$

The rest follows from Corollary 3.5. \square

Example 5.3. Let $E = C([0, \omega])$ and $C = \{x \in E : 0 \leq x \leq 1 \text{ and } x(\omega) = 0\}$. Then $\hat{d}(\overline{C}^{w^*}, E) = \frac{1}{2}$ and $\text{Ja}(C) = 1$. Hence, $\text{Ja}_E(C) = \text{ck}_E(C) = \frac{1}{2}$ and $\text{ck}(C) = \hat{d}(\overline{C}^{w^*}, C) = \gamma(C) = 1$.

Proof. Note that E^* is canonically identified with $\ell_1([0, \omega])$ and E^{**} with $\ell_\infty([0, \omega])$.

To show that $\hat{d}(\overline{C}^{w^*}, E) \leq \frac{1}{2}$ we observe that the constant function $\frac{1}{2}$ belongs to E and that $C \subset \frac{1}{2} + \frac{1}{2}B_E$. Thus, $\overline{C}^{w^*} \subset \frac{1}{2} + \frac{1}{2}B_{E^{**}}$.

Further, consider the element $x^* \in E^* = \ell_1([0, \omega])$ given by $x^*(n) = 1/2^n$ for $n < \omega$ and $x^*(\omega) = 0$. Then the only element of \overline{C}^{w^*} at which x^* attains its supremum on C is $\chi_{[0, \omega)}$, whose distance to C is clearly equal to 1. Thus, $\text{Ja}(C) \geq 1$.

The rest follows from Corollary 3.5. \square

Example 5.4. Let $E = C_0([0, \omega_1])$ and $C = \{x \in E : 0 \leq x \leq 1\}$. Then $\hat{d}(\overline{C}^{w^*}, E) = 1$ and $\text{ck}(C) = \frac{1}{2}$. Hence, $\text{Ja}_E(C) = \text{Ja}(C) = \text{ck}_E(C) = \frac{1}{2}$ and $\hat{d}(\overline{C}^{w^*}, C) = \gamma(C) = 1$.

Proof. First note that the dual E^* can be identified with $\ell_1([0, \omega_1])$ and the second dual E^{**} with $\ell_\infty([0, \omega_1])$.

To show that $\hat{d}(\overline{C}^{w^*}, E) \geq 1$ we note that the constant function 1 belongs to \overline{C}^{w^*} and its distance to E is 1.

Next we will show that $\text{ck}(C) \leq \frac{1}{2}$. Let (x_n) be any sequence in C . There is some $\alpha < \omega_1$ such that $x_n|_{(\alpha, \omega_1)} = 0$ for each $n \in \mathbb{N}$. As the interval $[0, \alpha]$ is countable, there is a subsequence (x_{n_k}) which converges pointwise on $[0, \omega_1]$. The limit is an element of $\ell_\infty([0, \omega_1]) = E^{**}$. Denote the limit by x^{**} . Then the sequence (x_{n_k}) weak* converges to x^{**} . Thus, in particular, $x^{**} \in \text{clust}_{E^{**}}((x_n))$. Set $x = \frac{1}{2}\chi_{[0, \alpha]}$. Then $x \in C$ and $\|x^{**} - x\| \leq \frac{1}{2}$ (as $0 \leq x^{**} \leq 1$ and $x^{**}|_{(\alpha, \omega_1)} = 0$). The inequality $\text{ck}(C) \leq \frac{1}{2}$ now follows.

The rest follows from Corollary 3.5. □

Example 5.5. Let $E = C([0, \omega_1])$ and $C = \{x \in E : 0 \leq x \leq 1 \text{ and } x(\omega_1) = 0\}$. Then $\hat{d}(\overline{C}^{w^*}, E) = \text{ck}(C) = \frac{1}{2}$ and $\hat{d}(\overline{C}^{w^*}, C) = 1$. Hence, $\text{Ja}_E(C) = \text{Ja}(C) = \text{ck}_E(C) = \frac{1}{2}$ and $\gamma(C) = 1$.

Proof. We start similarly to the proof of Example 5.3; note that E^* is canonically identified with $\ell_1([0, \omega_1])$, and E^{**} with $\ell_\infty([0, \omega_1])$.

To show that $\hat{d}(\overline{C}^{w^*}, E) \leq \frac{1}{2}$, notice that the constant function $\frac{1}{2}$ belongs to E and that $C \subset \frac{1}{2} + \frac{1}{2}B_E$. Thus, $\overline{C}^{w^*} \subset \frac{1}{2} + \frac{1}{2}B_{E^{**}}$.

The inequality $\text{ck}(C) \leq \frac{1}{2}$ can be proved in the same way as in Example 5.4. In fact, it follows from that example, since $C_0([0, \omega_1])$ is isometric to $\{x \in E : x(\omega_1) = 0\}$, and hence our set C coincides with the set C from Example 5.4.

Finally, $\hat{d}(\overline{C}^{w^*}, C) \geq 1$ as $\chi_{[0, \omega_1]} \in \overline{C}^{w^*}$ and its distance from C is equal to 1.

The rest follows from Corollary 3.5. □

Example 5.6. There is a Banach space E and a closed convex bounded subset $C \subset E$ such that $\text{Ja}_E(C) = \frac{1}{2}$ and $\text{ck}_E(C) = \text{Ja}(C) = 1$. Hence,

$$\text{ck}(C) = \hat{d}(\overline{C}^{w^*}, E) = \hat{d}(\overline{C}^{w^*}, C) = \gamma(C) = 1.$$

Proof. We use the example from [10]. Therein a set $K_0 \subset [0, 1]^{\mathbb{N}}$ and a free ultrafilter u over \mathbb{N} are constructed such that (in particular) the following assertions are satisfied:

- (a) K_0 consists of finitely supported vectors and is closed in the topology of uniform convergence on \mathbb{N} but not in the pointwise convergence topology;
- (b) for each $x \in \bar{K}_0$ (the closure taken in the pointwise convergence topology) we have $\lim_u x(n) = 0$;
- (c) for each $x \in \bar{K}_0 \setminus K_0$ there are infinitely many $n \in \mathbb{N}$ such that $x(n) = 1$.

Let $E = \{x \in C(\beta\mathbb{N}) : x(u) = 0\}$. We remark that $\beta\mathbb{N}$ is canonically identified with the space of ultrafilters over \mathbb{N} , and hence we have $u \in \beta\mathbb{N}$. Let us consider the embedding $\kappa : \bar{K}_0 \rightarrow E$ defined by

$$\kappa(x)(p) = \lim_p x(n), \quad p \in \beta\mathbb{N}, \quad x \in \bar{K}_0.$$

By (b) it is a well-defined mapping with values in E . Let $B = \kappa(K_0)$. Then B is a bounded norm-closed subset of E . Set $C = \text{conv } B$.

It is proved in [10] that $\hat{d}(\overline{B}^{w*}, E) \leq \frac{1}{2}$. As \overline{B}^{w*} contains extreme points of \overline{C}^{w*} , by (4.1) we get $\text{Ja}_E(C) \leq \frac{1}{2}$.

In [10] it is proved that $\hat{d}(\overline{C}^{w*}, E) \geq 1$. We will show that even $\text{ck}_E(C) \geq 1$. To do this it is enough to observe that $C \subset \{x \in E : x|_{\beta\mathbb{N} \setminus \mathbb{N}} = 0\}$ (this follows from (a)). The latter space is isometric to c_0 . As c_0^* is separable, each element of \overline{C}^{w*} is a weak* limit of a sequence from C . It follows that $\text{ck}_E(C) = \hat{d}(\overline{C}^{w*}, E) \geq 1$.

By Corollary 3.5 it remains to prove that $\text{Ja}(C) \geq 1$. To do that let us first recall that the dual to E can be canonically identified with the space of all signed Radon measures on $\beta\mathbb{N} \setminus \{u\}$. This space can be decomposed as

$$E^* = \ell_1 \oplus_1 M(\beta\mathbb{N} \setminus (\mathbb{N} \cup \{u\})).$$

The second dual is then represented as

$$E^{**} = \ell_\infty \oplus_\infty M(\beta\mathbb{N} \setminus (\mathbb{N} \cup \{u\}))^*.$$

Denote by j the canonical embedding of E into E^{**} and by ρ the embedding $\rho: \ell_\infty \rightarrow E^{**}$ given by $\rho(x) = (x, 0)$ using the above representation. Now,

$$\rho(\ell_\infty) = \{x^{**} \in E^{**} : x^{**}(\mu) = 0 \text{ whenever } \mu \in M(\beta\mathbb{N} \setminus \{u\}) \text{ is such that } \mu|_{\mathbb{N}} = 0\}.$$

So $\rho(\ell_\infty)$ is weak* closed and, moreover, ρ is a weak*-to-weak* homeomorphism (ℓ_∞ being considered as the dual to ℓ_1).

Finally, $\rho|_{K_0} = (j \circ \kappa)|_{K_0}$ and hence $\overline{B}^{w*} = \rho(\overline{K}_0)$. Fix some $x \in \overline{K}_0 \setminus K_0$ and let $A \subset \mathbb{N}$ be infinite such that $x|_A = 1$. Such a set A exists due to (c). Enumerate $A = \{a_n : n \in \mathbb{N}\}$ and define an element $u \in \ell_1$ by

$$u(k) = \begin{cases} \frac{1}{2^{n+1}}, & k = a_n, \\ 0, & k \in \mathbb{N} \setminus A. \end{cases}$$

Furthermore, define the element $x^* \in E^*$ by $x^* = (u, 0)$ (using the above representation). Then $\|x^*\| = 1$, so $\sup x^*(C) \leq 1$. Moreover, $\rho(x)(x^*) = 1$; hence, $\sup x^*(C) = 1$. Let $x^{**} \in \overline{C}^{w*}$ be such that $x^{**}(x^*) = 1$. Then $x^{**} = (\rho(y), 0)$ for some $y \in \ell_\infty$. As $\|y\| \leq 1$, we get $y|_A = 1$. But then $d(y, c_0) = 1$; hence, $d(x^{**}, C) \geq 1$. So $\text{Ja}(C) \geq 1$ and the proof is completed. \square

The above examples show that any of the inequalities from Corollary 3.5 can be strict, with one possible exception which is described in the following problem.

Question 5.7. Let E be a Banach space and let $C \subset E$ be a bounded convex set. Is $\text{Ja}(C)$ then equal to $\text{ck}(C)$?

6. The case of a weak* angelic dual unit ball

In this section we collect several results saying that under some additional conditions some of the inequalities from Corollary 3.5 become equalities. The basic assumption

will be that the dual unit ball B_{E^*} is weak* angelic, i.e. that whenever $A \subset B_{E^*}$ and $x^* \in \overline{A}^{w^*}$, there is a sequence in A which weak* converges to x^* . Inspired by [7] we introduce the following quantity. If E is a Banach space and $H \subset E$ is a bounded subset, we set

$$\gamma_0(H) = \sup \left\{ \left| \lim_i \lim_j x_i^*(x_j) \right| : (x_j) \subset H, (x_i^*) \subset B_{E^*}, x_i^* \xrightarrow{w^*} 0 \right\},$$

assuming the limits involved exist. It is clear that $\gamma_0(H) \leq \gamma(H)$. In general γ_0 is not an equivalent quantity to the others. Indeed, if $E = \ell_\infty$ and $C = B_E$, then $\gamma_0(C) = 0$ by the Grothendieck property of E . But, in the case when B_{E^*} is angelic, we have the following.

Theorem 6.1. *Let E be a Banach space such that B_{E^*} is weak* angelic.*

(i) *Let $H \subset E$ be any bounded subset. Then we have*

$$\frac{1}{2}\gamma(H) \leq \gamma_0(H) = \text{Ja}_E(H) = \text{ck}_E(H) = \hat{d}(\overline{H}^{w^*}, E) \leq \gamma(H).$$

(ii) *Let $C \subset E$ be any bounded convex subset. Then the following inequalities hold true:*

$$\begin{aligned} \frac{1}{2}\gamma(C) &\leq \gamma_0(C) = \text{Ja}_E(C) = \text{ck}_E(C) = \hat{d}(\overline{C}^{w^*}, E) \\ &\leq \text{Ja}(C) \leq \text{ck}(C) \leq \hat{d}(\overline{C}^{w^*}, C) \leq \gamma(C). \end{aligned}$$

Proof. Part (ii) follows from (i) and Corollary 3.5. As for part (i), in view of Corollary 3.5 it is enough to prove that $\text{Ja}_E(H) \geq \gamma_0(H)$ and $\hat{d}(\overline{H}^{w^*}, E) \leq \gamma_0(H)$. The second inequality follows from [7, Proposition 14 (ii)].

The first inequality follows from the proof of Theorem 3.1. In fact, the angelicity assumption is not needed here. Let us indicate the necessary changes, as follows.

Suppose that $\gamma_0(H) > r$. The space F is not needed, but we define the sublinear functional p on E^* by $p(f) = \sup f(H)$ for $f \in E^*$. Fix a sequence (z_j) in H and (f_i) in B_{E^*} such that f_i weak* converge to 0 and $\lim_i \lim_j f_i(z_j) > r$ and all the limits involved exist. Without loss of generality suppose that for every $i \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have $f_i(z_j) > r$.

As $\limsup_i f_i = \liminf_i f_i = 0$, we get the assertion (3.2) without any calculation. We define K_n for $n \in \mathbb{N}$ in the same way. The assertion (3.3) then says that $p(f) > r$ for all $f \in K_1$. Fix any $r' < r$ and a sequence (β_n) of strictly positive numbers. Claim 3.3 now yields a sequence (g_n) with $g_n \in K_n$ such that

$$p\left(\sum_{i=1}^n \beta_i g_i\right) > \beta_n r' + p\left(\sum_{i=1}^{n-1} \beta_i g_i\right).$$

As the f_n weak* converge to 0, g_n weak* converge to 0 as well. Thus, $g_0 = 0$. Now, if the sequence (β_n) quickly converges to 0 (i.e. satisfies the same condition as in the original proof), we set $g = \sum_{i=1}^\infty \beta_i g_i$. Let $u \in \overline{H}^{w^*}$ be an arbitrary point with $g(u) = \sup g(H)$.

By the final calculation we get $\liminf_n g_n(u) \geq r'$. If $v \in E$ is arbitrary, then $g_n(v) \rightarrow 0$, and thus

$$r' \leq \liminf g_n(u) - \lim g_n(v) = \liminf g_n(u - v) \leq \|u - v\|.$$

Thus, $\text{Ja}_E(H) \geq r'$, so $\text{Ja}_E(H) \geq \gamma_0(H)$. \square

We remark that the spaces from Examples 5.1–5.3 are separable and therefore they have weak* angelic unit ball. It follows that in Theorem 6.1 all the inequalities, with a possible exception of $\text{Ja}(C) \leq \text{ck}(C)$, may be strict. Note also that, under the weaker assumption of the Banach space E having Corson property \mathcal{C} , it has been proved in [2, Proposition 2.6] that for any bounded set $H \subset E$ we have $\text{ck}_E(H) = \hat{d}(\overline{H}^{w^*}, E)$.

The following theorem shows that all the quantities are equal in the very special case $E = c_0(\Gamma)$.

Theorem 6.2. *Let Γ be an arbitrary set and let $E = c_0(\Gamma)$.*

(i) *Let $H \subset E$ be a bounded set. Then we have*

$$\gamma_0(H) = \text{Ja}_E(H) = \text{ck}_E(H) = \hat{d}(\overline{H}^{w^*}, E) = \gamma(H).$$

(ii) *Let $C \subset E$ be a convex bounded subset. Then we have*

$$\gamma_0(C) = \text{Ja}_E(C) = \text{ck}_E(C) = \hat{d}(\overline{C}^{w^*}, E) = \text{Ja}(C) = \text{ck}(C) = \hat{d}(\overline{C}^{w^*}, C) = \gamma(C).$$

Proof. It is enough to prove $\gamma_0(H) \geq \gamma(H)$. If $\gamma(H) = 0$, this inequality is trivial. So, suppose that $\gamma(H) > 0$. Fix an arbitrary $r > 0$ such that $\gamma(H) > 0$. We find sequences $(x_i) \subset H$, $(x_j^*) \subset B_{E^*}$ and $\eta > 0$ such that

$$\lim_i \lim_j x_i^*(x_j) - \lim_j \lim_i x_i^*(x_j) > r(1 + \eta),$$

where all the limits involved exist. As B_{E^*} is weak* sequentially compact, by passing to a subsequence we may suppose that the sequence (x_i^*) weak* converges to some $x^* \in B_{E^*}$. Then

$$\lim_i \lim_j (x_i^* - x^*)(x_j) > r(1 + \eta).$$

We claim that

$$\limsup \|x_i^* - x^*\| \leq 1.$$

Suppose this is not the case. Then, up to passing to a subsequence, we may suppose that there is $\delta > 0$ such that $\|x_i^* - x^*\| \geq 1 + \delta$ for each $i \in \mathbb{N}$. To continue the proof we recall that E^* is canonically identified with $\ell_1(\Gamma)$ and that the weak* topology on bounded sets coincides with the pointwise convergence topology. Using this identification we can find a finite set $F \subset \Gamma$ such that

$$\sum_{\gamma \in \Gamma \setminus F} |x^*(\gamma)| < \frac{1}{3}\delta.$$

Further, as x_i^* weak* converges to x^* , there is $i_0 \in \mathbb{N}$ such that for each $i \geq i_0$ we have

$$\sum_{\gamma \in F} |x_i^*(\gamma) - x^*(\gamma)| < \frac{1}{3}\delta.$$

Fix any $i \geq i_0$. Then we have

$$\begin{aligned} \|x_i^*\| &\geq \sum_{\gamma \in \Gamma \setminus F} |x_i^*(\gamma)| \\ &\geq \sum_{\gamma \in \Gamma \setminus F} |x_i^*(\gamma) - x^*(\gamma)| - \sum_{\gamma \in \Gamma \setminus F} |x^*(\gamma)| \\ &= \|x_i^* - x^*\| - \sum_{\gamma \in F} |x_i^*(\gamma) - x^*(\gamma)| - \sum_{\gamma \in \Gamma \setminus F} |x^*(\gamma)| \\ &> 1 + \delta - \frac{1}{3}\delta - \frac{1}{3}\delta \\ &= 1 + \frac{1}{3}\delta. \end{aligned}$$

This is a contradiction.

So, omitting a finite number of elements, we can suppose that $\|x_i^* - x^*\| < 1 + \eta$ for all $i \in \mathbb{N}$. Set $y_i^* = (x_i^* - x^*)/(1 + \eta)$. Then $y_i^* \in B_{E^*}$, the sequence (y_i^*) weak* converges to 0 and

$$\lim_i \lim_j y_i^*(x_j) > r.$$

Thus, $\gamma_0(H) \geq r$ and the proof is completed. □

The equalities $\text{ck}_E(H) = \hat{d}(\overline{H}^{w^*}, E) = \gamma(H)$ in the case when $E = c_0$ and $H \subset E$ is a bounded subset also follow easily from [13, Theorem 2.8] (see also [1, Corollary 3.4.3]).

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