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LETTERS TO THE EDITOR

ON CONDITIONAL ORNSTEIN–UHLENBECK PROCESSES

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Abstract

It is well known that the law of a Brownian motion started from $x>0$ and conditioned never to hit 0 is identical with the law of a three-dimensional Bessel process started from x. Here we show that a similar description is valid for all linear Ornstein–Uhlenbeck Brownian motions. Further, using the same techniques, it is seen that we may construct a non-stationary Ornstein–Uhlenbeck process from a stationary one.

EXCESSIVE TRANSFORMS

Let $X=\{X_t, t\geqq 0\}$ be a linear, canonical Ornstein–Uhlenbeck process with the drift parameter $\gamma\in\mathbb{R}$, i.e. X is a diffusion with the generator

$$\mathcal{A}=\tfrac{1}{2}D^2-\gamma x D,$$

where $D=d/dx$. Introduce $Y(t)=(X_1^2(t)+X_2^2(t)+X_3^2(t))^{\frac{1}{2}}$, where X_i, $i=1, 2, 3$, are three independent copies of X. It is well known that $Y=\{Y_t, t\geqq 0\}$ is a time-homogeneous diffusion and has the generator

$$\mathcal{G}=\tfrac{1}{2}D^2+\left(\frac{1}{x}-\gamma x\right)D, \qquad x>0,$$

(see e.g. [4]).

Theorem 1. For $\alpha>0$ let $Y, Y_0=\alpha$, be a diffusion with the generator $\mathcal{G}$, and $\hat{X}, \hat{X}_0=\alpha$, an Ornstein–Uhlenbeck process killed when it (eventually) hits 0.

(i) For $\gamma\geqq 0$ let $\hat{Y}$ be the process obtained from Y by killing it exponentially with rate γ and conditioning it to hit 0. Then $\hat{X}$ and $\hat{Y}$ are identical in law.

(ii) For $\gamma\leqq 0$ let $\bar{X}$ be the process obtained from $\hat{X}$ by killing it exponentially with rate $-\gamma$ and conditioning it to drift out to ∞. Then $\bar{X}$ and Y are identical in law.

Remark. The case $\gamma=0$, i.e. there is no killing, is included in both (i) and (ii) above, and was first, perhaps, treated by McKean (see [2], and also [1], [5]). He interpreted this kind of conditioning as an h-transform of a three-dimensional Bessel process in the case (i) and of a killed Brownian motion in the case (ii), and explained excursions of Brownian motion in these terms. The theorem above may be considered as a generaliza-

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tion to this result of McKean. However, the proper generalization is simply that:

> the Ornstein–Uhlenbeck process conditioned not to hit 0 is a diffusion with the generator $s^{-1}\mathscr{A}(s\ \cdot)$, where s is the scale function for X.

Proof. As we remarked above, the conditioning described in the theorem should be interpreted as an excessive transform of the diffusion in question. We refer to [3], pp. 299–308 for details.

In the case $\gamma>0$ we note that $h(x)=1/x$ is the (essentially) unique decreasing and strictly positive solution to $\mathscr{G}u=\gamma u$. Therefore $\hat{Y}$ is a diffusion with the transition semigroup

$$\hat{P}_t f := \frac{1}{h} e^{-\gamma t} P_t(hf),$$

where P is the semigroup of Y. Using the definition of infinitesimal generator it is seen that $\hat{Y}$ has the generator

$$\hat{\mathscr{G}}f := \frac{1}{h}(\mathscr{G}-\gamma)(hf).$$

On $C^2[0,\infty)$ we have $\hat{\mathscr{G}}f=\mathscr{A}f$, and the boundary condition is killing. This completes the proof of the case (i).

In the case $\gamma<0$ we observe that $g(x)=x$ solves $\mathscr{A}u=-\gamma u$. The function g is increasing and strictly positive on $(0,\infty)$ and satisfies the boundary condition of killing, i.e. $\lim_{x\downarrow 0} g(x)=g(x)=0$. Therefore $\bar{X}$ is a diffusion with the generator

$$\bar{\mathscr{A}} := \frac{1}{g}(\mathscr{A}+\gamma)(gf).$$

On $C^2[0,\infty)$ we have $\bar{\mathscr{A}}f=\mathscr{G}f$, and the proof is complete.

Theorem 2. For $\gamma>0$ let X^+ and X^- be two Ornstein–Uhlenbeck processes with the drift parameters γ and $-\gamma$, respectively. Denote with $\tilde{X}$ the process obtained from X^+ by killing it exponentially with rate γ and conditioning it to drift out to $+\infty$ with probability $\frac{1}{2}$ and to $-\infty$ also with probability $\frac{1}{2}$. Then $\tilde{X}$ and X^- are identical in law.

Proof. It is well known that the positive, strictly increasing and decreasing solutions $\phi^{\uparrow}$ and $\phi^{\downarrow}$, respectively, to the equation $\mathscr{A}u=\gamma u$ $(\gamma>0)$ are $\phi^{\downarrow}(x)=\phi^{\uparrow}(-x)=\exp(\gamma x^2/2)D_{-1}(\sqrt{2\gamma}x)$, where

$$D_\upsilon(x)=\frac{\exp(-x^2/4)}{\Gamma(-\upsilon)}\int_0^\infty t^{-\upsilon-1}\exp(-xt-t^2/2)\,dt$$

is the parabolic cylinder function of order υ. Hence the conditioning is realized with the function

$$h(x)=\phi^{\uparrow}(x)+\phi^{\downarrow}(x)=C\cdot\exp(\gamma x^2),$$

where C is a constant, and the proof may now be concluded as above.

Remark. Denote by $\mathbb{P}_0^+$ and $\mathbb{P}_0^-$ the probability measures associated with $X^+, X_0^+=0$, and $X^-, X_0^-=0$, respectively. Of course, $\mathbb{P}_0^+$ and $\mathbb{P}_0^-$, when restricted to the σ-field $\mathscr{F}_t=\sigma\{\omega_s : s\le t\}$, are equivalent. From the proof above it is seen that

$$\left.\frac{d\mathbb{P}_0^-}{d\mathbb{P}_0^+}\right|_{\mathscr{F}_t}=\exp(-\gamma\,.\,t+\gamma\,.\,\omega_t^2)\qquad \mathbb{P}_0^+\text{-a.s.};$$

a result which is also easily deduced from the general (Cameron–Martin–Girsanov) expression for Radon–Nikodym derivative of two diffusion processes.

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References

[1] KNIGHT, F. (1969) Brownian local times and taboo processes. *Trans. Amer. Math. Soc.* **143,** 173–185.

[2] MCKEAN, JR., H. P. (1963) Excursions of a non-singular diffusion. *Z. Wahrscheinlichkeitsth.* **1,** 230–239.

[3] PITMAN, J. AND YOR, M. (1981) Bessel processes and infinitely divisible laws. In *Proc. Durham Conference 1980.* Lecture Notes in Mathematics **851,** Springer-Verlag, Berlin.

[4] SHIGA, T. AND WATANABE, S. (1973) Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrscheinlichkeitsth.* **27,** 37–46.

[5] WILLIAMS, D. (1974) Path decomposition and continuity of local time for one-dimensional diffusions. *Proc. London Math. Soc.* (3) **28,** 738–768.