# Appendix B

## Asymptotic Analysis

Asymptotic analysis concerns the notion of the behaviour of functions, $f(x)$, as certain parameters go to their limiting values, usually zero or infinity. For convenience and without loss of generality, we will consider functions as their arguments go to infinity. Obviously the limit to any finite value $x_0$ can be obtained by taking $y = \frac{1}{(x-x_0)}$ to infinity.

We define

$$f(x) \sim g(x) \tag{B.1}$$

if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \to 1. \tag{B.2}$$

The binary relation of equivalence satisfies many obvious properties: for any smooth function $F(y)$, if $f(x) \sim g(x)$ then

$$F(f(x)) \sim F(g(x)). \tag{B.3}$$

This specifically is useful when applied to powers, $f \sim g$ implies

$$f^r \sim g^r \tag{B.4}$$

for any real number $r$. If $f(x) \sim g(x)$ and $a(x) \sim b(x)$ then

$$a(x)f \sim b(x)g(x). \tag{B.5}$$

Asymptotic analysis is most useful in the application of asymptotic expansions of functions. An asymptotic expansion of a function $f(x)$ is a series representation of a function that does not necessarily converge, and hence must be truncated at the expense of adding a remainder term. A very famous example of an asymptotic expansion is the Stirling approximation for the factorial, $N!$. The Stirling approximation is given by

$$\ln(\Gamma(z)) = z\ln z - z - \frac{1}{2}\ln z + \ln 2\pi + \sum_{n=1}^{N-1} \frac{B_{2n}}{2n(2n-1)x^{2n-1}} + R_N(z), \tag{B.6}$$

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where $B_n$ are the Bernoulli numbers with

$$R_N \leq \frac{|B_{2N|}}{2N(2N-1)|z|^{2N-1}} \tag{B.7}$$

for real $z$. The Bernoulli numbers behave as

$$B_{2N} = (-1)^{N+1}\frac{2(2N)!}{(2\pi)^{2N}}\zeta(2N). \tag{B.8}$$

The zeta function being bounded, we see that the Stirling approximation diverges.

A series expansion can be obtained for the factorial of a positive integer $N$ by expanding the $\Gamma$ function. Typically, the series expansion gives a very accurate approximation for the function that becomes maximally accurate after a certain number of terms in the expansion. At any finite truncation of the series, the remainder can be understood to be smaller than the subsequent term that has been dropped. Thus if we have a function $f(x)$ and its asymptotic series $g_1(x) + g_2(x) + \cdots$ then

$$f(x) - (g_1(x) + g_2(x) + \cdots + g_{k-1}(x)) \sim g_k(x) \tag{B.9}$$

for each $k$ up to a maximum $k_{\text{max}}$ which depends on $x$. For larger values of $x$, $k_{\text{max}}$ increases. But after this term, the expansion starts to diverge, and it is not a good approximation to the original function. Thus for the Stirling approximation, for a given $N$, we should sum a finite number of terms to obtain a good approximation to $N!$, that number fixed by the value of $N$. However, if we look at the subsequent terms in the expansion, we find that they start to increase, and eventually they increase so much that the series fails to converge. Truncating the series at a given term $k_{\text{max}}$ gives an approximation that is as small as the first term neglected, which can be very good approximation even though the asymptotic series does not converge.

We use the notation

$$f(x) - (g_1(x) + g_2(x) + \cdots + g_{k-1}(x)) = o(g_k(x)), \tag{B.10}$$

which generally in physics is translated as the difference

$$f(x) - (g_1(x) + g_2(x) + \cdots + g_{k-1}(x)) \tag{B.11}$$

is of the order of $g_k(x)$. However, there is a precise mathematical sense to this relation, it means that for every positive $\epsilon$ there exists a positive real number $X$ such that, for $x \geq X$,

$$f(x) - (g_1(x) + g_2(x) + \cdots + g_{k-1}(x)) \leq \epsilon g_k(x). \tag{B.12}$$

If $f(x) = o(g(x))$ and $g(x) \neq 0$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0. \tag{B.13}$$