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INVARIANT SUBSPACES IN THE BIDISC AND COMMUTATORS

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Abstract

Let M be an invariant subspace of $L^2(T^2)$ on the bidisc. V_1 and V_2 denote the multiplication operators on M by coordinate functions z and w, respectively. In this paper we study the relation between M and the commutator of V_1 and V_2^*. For example, M is studied when the commutator is self-adjoint or of finite rank.

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1. Introduction

We let T^2 be the torus that is the cartesian product of 2 unit circles in $\mathbb{C}$. The usual Lebesgue spaces, with respect to the Haar measure m of T^2, are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{T^2} f(z, w)\bar{z}^j \bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative.

A closed subspace M of L^2 is said to be invariant if

$$zM \subset M \qquad \text{and} \qquad wM \subset M.$$

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One can ask for a classification or an explicit description (in some sense) of all invariant subspaces of L^2, but this seems out of reach. In a previous paper [8], the author studied the relation between M and the structure of $M \ominus wM$. An important role was played by invariant subspaces of the form FN, where F is unimodular and $H^2 \subseteq N \subseteq$ the closure of $\bigcup_{n=0}^{\infty} \bar{z}^n H^2$.

Such invariant subspaces are related to those invariant subspaces studied previously in [2, 4, 3, 1]. However the condition on $M \ominus wM$ in [8] is a little unnatural. In this paper we will find natural conditions on M which imply that M is of the form FN.

Given an invariant subspace M of L^2, V_1 and V_2 denote the restriction of multiplications by z and w on M, respectively. Put

$$A_n = V_1^n V_2^* - V_2^* V_1^n \quad (n \geq 1)$$

and write $A = A_1$. In this paper we will describe invariant subspaces of L^2 when $A = 0$ or $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \not\supseteq wM$. Mandrekar [6] described M when $A = 0$ and M is in H^2. In fact, Theorem 2 in [6] is a corollary of (2) of Theorem 5 in [8] that was proved independently. In Section 2 we study invariant subspaces under several hypotheses on the restriction of V_1 to the kernel of V_2^*. All but one are previously known results [8]. In Section 3, A_n $(n \geq 1)$ is studied using results of Section 2. In Section 4 invariant subspaces are studied when A is of finite rank. In Section 5 we try to prove that if A is selfadjoint then $A = 0$. In Section 6 we give several examples to which the results of the previous sections can be applied.

We define several subspaces of L^2 which will be used later. Let $C(T^2)$ be the spaces of complex-valued continuous functions.

(i) $\mathscr{H}_1$ or $\mathscr{H}_2$ is the set of f (in L^2) with Fourier series:

$$\sum_{\substack{j \geq 0 \\ k=0}} a_{jk} z^j w^k \qquad \text{or} \qquad \sum_{\substack{k \geq 0 \\ j=0}} a_{jk} z^j w^k,$$

respectively. Put $\mathscr{A}_j = \mathscr{H}_j \cap C(T^2)$ for $j = 1, 2$.

(ii) $\mathscr{L}_1$ or $\mathscr{L}_2$ is the set of f (inL^2) with Fourier series:

$$\sum_{k=0} a_{jk} z^j w^k \qquad \text{(no restriction on } j)$$

or

$$\sum_{j=0} a_{jk} z^j w^k \qquad \text{(no restriction on } k),$$

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respectively. Put $\mathscr{C}_j = \mathscr{L}_j \cap C(T^2)$ for $j = 1, 2$.

(iii) $\mathbb{H}_1$ or $\mathbb{H}_2$ is the set of f (in $\mathscr{L}^2$) with Fourier series:

$$\sum_{k\geq 0} a_{jk} z^j w^k \qquad \text{(no restriction on } j)$$

or

$$\sum_{j\geq 0} a_{jk} z^j w^k \qquad \text{(no restriction on } k)$$

respectively. Put $\mathscr{B}_j = \mathbb{H}_j \cap C(T^2)$ for $j = 1, 2$.

2. The restriction of V_1 to Ker V_2^*

Let M be an invariant subspace of L^2. Put

$$S_1 = M \ominus wM \quad \text{and} \quad S_2 = M \ominus zM.$$

$S_1 = \text{Ker } V_2^*$ and $S_2 = \text{Ker } V_1^*$. In this section we derive a new result and results of the previous paper [8], which will be used in this paper.

PROPOSITION 1. *Let M be an invariant subspace of L^2. V_2^* is a one-to-one operator if and only if $M = \chi_{E_1} F \mathbb{H}_2 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_j} is a characteristic function of Borel set E_j on T^2, $\chi_{E_1} \in \mathscr{L}_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.*

The proof is in [3] and [7, page 164-165] since V_2^* is one-to-one if and only if $S_1 = \{0\}$.

PROPOSITION 2. *Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.*

(1) $V_1(\text{Ker } V_2^*) = \text{Ker } V_2^*$ *if and only if $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where F is unimodular, χ_{E_1} is a nonzero function in $\mathscr{L}_1$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.*

(2) $V_1(\text{Ker } V_2^*) \subsetneq \text{Ker } V_2^*$ *if and only if $M = FH^2$ for some unimodular function F.*

PROOF. Theorem 5 in [8] shows that $zS_1 = S_1$ (or $zS_1 \subsetneq S_1$) if and only if M has the form: $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ in (1) (or the form : $M = FH^2$ in (2), respectively). This implies the proposition because $\text{Ker } V_2^* = S_1$.

By Propositions 1 and 2, we are interested in an invariant subspace such that $\operatorname{Ker} V_2^*$ is not invariant under V_1.

PROPOSITION 3. *Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.*

(1) *There exists a nonzero function f in $\operatorname{Ker} V_2^*$ such that $V_1^n f$ belongs to* $\operatorname{Ker} V_2^*$ *for any integer n ($V_1^n = V_1^{*(-n)}$ when $n < 0$) if and only if $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_1} is a non-zero function in $\mathscr{L}_1$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.*

(2) *There exists a function f in $\operatorname{Ker} V_2^*$ such that $V_1^n f$ belongs to* $\operatorname{Ker} V_2^*$ *for any $n \geq 0$ and $V_1^\ell f$ is not in* $\operatorname{Ker} V_2^*$ *for some $\ell < 0$ if and only if $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in $\mathbb{H}_1$, and F is unimodular.*

PROOF. Part (1), under the hypothesis that $|f| > 0$ a.e., and (2), were proved in [8, Theorem 6]. We will prove (1) in general. Put $M_1 = \bigcap_{n\geq 0} w^n M$ then $M = (\sum_{n\geq 0} \oplus w^n S_1) \oplus M_1$. Let D be the largest closed subspace of S_1 with $zD \subseteq D$. If we let $D_3 = S_1 \ominus D$, $D_2 = \bigcap_{n\geq 0} z^n D$ and $D_1 = D \ominus D_2$ then

$$M = \left(\sum_{n\geq 0} \oplus w^n D_1\right) \oplus \left(\sum_{n\geq 0} \oplus w^n D_2\right) \oplus \left(\sum_{n\geq 0} \oplus w^n D_3\right) \oplus M_1.$$

Since $zD_2 = D_2$, by [3] and [7, pp. 164-165]

$$\left(\sum_{n\geq 0} \oplus D_2 w^n\right) \oplus M_1 = \chi_{E_1} F_1 \mathbb{H}_1 \oplus \chi_{E_2} F_2 \mathbb{H}_2 \oplus \chi_{E_3} L^2$$

where $\chi_{E_j} \in \mathscr{L}_j$ $(j = 1, 2)$, $\chi_{E_1} + \chi_{E_3} \leq 1$ a.e. and $\chi_{E_2} + \chi_{E_3} \geq 1$ a.e. If there exists a nonzero function f in $\operatorname{Ker} V_2^*$ such that $z^n f$ belongs to $\operatorname{Ker} V_2^*$ for any integer n, then χ_{E_1} is nonzero. Since $\chi_{E_1}(\bar{F}_1 M \ominus \mathbb{H}_1)$ is invariant under multiplication of w, $\chi_{E_1}(\bar{F}_1 M \ominus \mathbb{H}_1) = \{0\}$ and hence $\chi_{E_1} M = \chi_{E_1} F_1 \mathbb{H}_1$. Since $\chi_{E_1} M \subset M$, $(1 - \chi_{E_1})M \subset M$ and $M = \chi_{E_1} M \oplus (1 - \chi_{E_1})M$. We can prove $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. For $z\mathscr{A}_1(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and

$$\left[z\mathscr{A}_1(1 - \chi_{E_1})\right] = (1 - \chi_{E_1})\mathscr{L}_1 \ni \bar{z}(1 - \chi_{E_1}).$$

Therefore $\bar{z}(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and hence $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. Hence

$$(1 - \chi_{E_1})M = \chi_{E_4} F_4 \mathbb{H}_1 + \chi_{E_3} L^2$$

where $\chi_{E_4} \in \mathscr{L}_1$, $\chi_{E_4} + \chi_{E_3} \leq 1$ a.e. and F_4 is unimodular. Thus M has the form $\chi_{E'} F \mathbb{H}_1 + \chi_{E''} L^2$ for some unimodular F where $\chi_{E'} = \chi_{E_1} + \chi_{E_4}$ and $\chi_{E''} = \chi_{E_3}$.

3. Commutator of V_1^n and V_2^*

Put $A_n = V_1^n V_2^* - V_2^* V_1^n$ ($n \geq 1$) and write $A = A_1$. The following trivial lemmas are important.

LEMMA 1. $A_n V_2 = 0$ *for any* $n \geq 1$ *and hence* $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supseteq wM$.

LEMMA 2. *For any* f *in* $\operatorname{Ker} V_2^*$, $f \in \operatorname{Ker} A_n$ *if and only if* $z^n f \in \operatorname{Ker} V_2^*$.

PROOF. When $f \in \operatorname{Ker} V_2^*$, if $f \in \operatorname{Ker} A_n$ then $V_2^* V_1^n f = 0$ and hence $z^n f \in \operatorname{Ker} V_2^*$. Conversely if $z^n f \in \operatorname{Ker} V_2^*$ then $A_n f = V_1^n V_2^* f = 0$.

In general A_n is a nonzero operator. The structure of M is simple when $A = 0$ or $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$. The following theorems, which make this precise, are corollaries of [8, Theorem 5].

THEOREM 4. *Let* M *be an invariant subspace of* L^2 *with* $A = 0$. *Then one and only one of the following occurs.*

(1) $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ *where* χ_{E_1} *is in* $\mathscr{L}_1$, $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. *and* F *is unimodular.*

(2) $M = \chi_{E_1} F \mathbb{H}_2 + \chi_{E_2} L^2$ *where* χ_{E_1} *is in* $\mathscr{L}_2$, $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. *and* F *is unimodular.*

(3) $M = FH^2$ *for some unimodular function.*

Conversely, if (1), (2) *or* (3) *holds for an invariant subspace* M *of* L^2, *then* $A = 0$.

PROOF. If M has the form (1) then $S_2 = \{0\}$ and hence $A^* = 0$ because $A^* V_1 = 0$. Therefore $A = 0$. If M has the form (2) then $S_1 = \{0\}$ and hence $A = 0$ because $AV_2 = 0$ by Lemma 1. If M has the form (3) then $zS_1 \subseteq S_1$ and hence $V_1 \operatorname{Ker} V_2^* \subseteq \operatorname{Ker} V_2^*$. Therefore $A = 0$ on $\operatorname{Ker} V_2^*$ and $A = 0$ because $AV_2 = 0$. Conversely suppose $A = 0$. Then $zS_1 \subseteq S_1$. If $S_1 = 0$, then by Proposition 1, M has the form (2). If $S_1 \neq 0$, then (since $zS_1 \subseteq S_1$) Proposition 2 implies that M has either the form (1) or the form (3).

Mandrekar [6] considered Theorem 4 when M is in H^2. Then since $\bigcap_{n=1}^{\infty} z^n H^2 = \bigcap_{n=1}^{\infty} w^n H^2 = \{0\}$, M has the form (3). Now we wish to consider invariant subspaces with $A \neq 0$.

THEOREM 5. *Let M be an invariant subspace of L^2 such that $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. Then one and only one of the following holds.*

(1) $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ *where χ_{E_1} is a nonzero function in $\mathscr{L}_1$, $\chi_{E_1} + \chi_{E_2} \le 1$ a.e. and F is unimodular.*

(2) $M = FN$ *where N is an invariant subspace which contains H^2 and is contained properly in $\mathbb{H}_1$, and F is unimodular.*

Conversely, if (1), (2) *or* (3) *holds for an invariant subspace M of L^2, then* $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

PROOF. Suppose $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. Then there exists $f \ne 0$ in $(\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n) \ominus wM$. In particular, $f \in \operatorname{Ker} V_2^*$. By Lemma 2, $z^n f \in \operatorname{Ker} V_2^*$ for $n \ge 0$, so by Proposition 3, M is of either the form (1) or the form (2). Conversely if M has the form (1), then $A = 0$ by Theorem 4. Then V_2^* commutes with V_1, so it commutes with every power of V_1; hence $A_n = 0$ for $n \ge 1$. Thus $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = M$. Since χ_{E_1} is a nonzero function, $M \ne wM$ because $M \ominus wM = \chi_{E_1} F \mathscr{L}_1$. Therefore $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. If M has the form (2) by (2) of Proposition 3 there exists a function f in $\operatorname{Ker} V_2^*$ such that $z^n f \in \operatorname{Ker} V_2^*$ for any $n \ge 0$. By Lemma 2, $f \in \operatorname{Ker} A_n$ while f is orthogonal to wM because $f \in \operatorname{Ker} V_2^*$. Therefore $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

4. Finite rank commutators

Theorem 4 describes those invariant subspaces with $A = 0$. Now we are interested in invariant subspaces in which A has finite rank. The following lemma was pointed out to the author by Professor K. Takahashi. It implies that if $A_n = 0$, then $A = 0$.

LEMMA 3. $V_1^* A_n = A_{n-1}$ *for $n > 1$ and hence* $\operatorname{Ker} A_n \subset \operatorname{Ker} A_{n-1}$.

The proof is clear.

PROPOSITION 6. *Let M be an invariant subspace of L^2.*

(1) *If* $\dim \operatorname{Ker} V_2^*$ *is finite then A_n is finite rank r_n,* $\sup_n r_n < \infty$, *and* $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = wM$.

(2) *Suppose* $\dim \operatorname{Ker} V_2^*$ *is infinite. If A_n is finite rank r_n and* $\sup_n r_n < \infty$ *then* $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

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PROOF. (1) By Lemma 1, A_n is finite rank r_n and $r_n \leq \dim \operatorname{Ker} V_2^*$. Using Lemma 2, if $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$, then there exists a nonzero function f such that $f \in M \ominus wM$ and $z^n f \in M \ominus wM$ for any $n \geq 0$, and this implies that $\dim \operatorname{Ker} V_2^*$ is infinite. (2) By Lemma 3 and hypothesis, $r_n \leq r_{n+1}$, so ultimately, r_n is constant. Also $r_n = \dim(\operatorname{Ker} A_n)^{\perp}$, while $(\operatorname{Ker} A_{n+1})^{\perp}$ contains $(\operatorname{Ker} A_n)^{\perp}$, so ultimately $(\operatorname{Ker} A_n)^{\perp}$ does not change with n. But then neither does $\operatorname{Ker} A_n$. In other words, $\operatorname{Ker} A_n = \operatorname{Ker} A_{n_0}$ if $n \geq n_0$, and hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = \operatorname{Ker} A_{n_0}$, for some $n_0 \geq 1$. Therefore if $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = wM$ then $\operatorname{Ker} A_{n_0} = wM$ and hence $(\operatorname{Ker} A_{n_0})^{\perp} = \operatorname{Ker} V_2^*$. Since $\dim \operatorname{Ker} V_2^*$ is infinite, this contradicts the hypothesis that A_{n_0} is of finite rank, and hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

COROLLARY 1. *Let M be an invariant subspace of H^2. If A_n has finite rank r_n and* $\sup_n r_n < \infty$ *then* $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

PROOF. If $\dim \operatorname{Ker} V_2^*$ is finite, by [8, Theorem 3] there exists a nonzero function g in L^{∞} such that $gM \subset M$ and $g \notin H^{\infty}$. By [1, Proposition 3] this implies that $M \not\subset H^2$, so $\dim \operatorname{Ker} V_2^*$ is infinite. Part (2) of Proposition 6 implies the corollary.

COROLLARY 2. *Let M be an invariant subspace with $\bigcap_{n=1}^{\infty} z^n M = \{0\}$. Then the following are equivalent:*

(1) $\dim \operatorname{Ker} V_2^* = \infty$, *$A_n$ is finite rank r_n and* $\sup_n r_n = r < \infty$*;*

(2) *$M = FN$ for some unimodular F and some invariant subspace with $N = K \oplus H^2 \subsetneq \mathbb{H}_1$. Moreover $N \ominus wN = (K \ominus wK) \oplus \mathscr{H}_1$ and if S is the largest closed subspace of finite codimension r of $N \ominus wN$ such that $zS \subset S$, then $S \supseteq \mathscr{H}_1$.*

PROOF. (1) implies (2). By (2) of Proposition 6, $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$ and hence by hypothesis and Theorem 5, M has the form FN. If we put $S' = \left[\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n\right] \ominus wM$, then by Lemma 2, S' is the largest closed subspace of $S_1 = M \ominus wM$ with $zS' \subset S'$, and by hypothesis $\dim(S_1 \ominus S') = r < \infty$. Put $S = \bar{F} S'$; then S is the desired subspace and (2) follows.

We prove (2) implies (1). By hypothesis of (2), FS is the largest closed subspace of S_1 with $zFS \subset FS$ and hence $FS = \left[\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n\right] \ominus wM$. This implies (1).

5. Selfadjoint commutators

In this section we will study the following conjecture: if $A = A^*$ then $A = 0$. Unfortunately we have not been able to resolve it. However we will give some partial answers.

LEMMA 4. $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. *If* $A = A^*$ *then* $A_n = V^{n-1}A$ *and hence* $\operatorname{Ker} A_n = \operatorname{Ker} A$.

PROOF. For any $n > 1$

$$\begin{aligned} A_n &= V_1^{n-1}(V_1V_2^*) - (V_2^*V_1^{n-1})V_1 \\ &= V_1^{n-1}(V_1V_2^*) - (V_1^{n-1}V_2^* - A_{n-1})V_1 \\ &= V_1^{n-1}A + A_{n-1}V_1 \end{aligned}$$

and hence $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $AV_1 = 0$ and hence $A_n = V_1^{n-1}A$.

PROPOSITION 7. *Suppose M is an invariant subspace with $A = A^*$. Then*

(1) $A_n^2 = 0$ *for any* $n > 1$;

(2) *if A has finite rank r, then A_n is also of finite rank r for any $n > 1$;*

(3) *if* $\operatorname{Ker} V_1^* \bigcap \operatorname{Ker} V_2^* = \{0\}$, *then* $A = 0$;

(4) *if* $\operatorname{Ker} V_1^* \bigcap \operatorname{Ker} V_2^* \neq \{0\}$, *then* $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

PROOF. (1) If $A = A^*$ then $A_n^2 = V_1^{n-1}AV_1^{n-1}A = 0$ for $n > 1$ by Lemma 4 because $A^*V_1 = 0$. Part (2) is clear by Lemma 4. Since $AV_1 = AV_2 = 0$, $A = 0$ on $[zM + wM]$ (the closed linear span of $zM + wM$). If $\operatorname{Ker} V_1^* \bigcap \operatorname{Ker} V_2^* = \{0\}$, then $M = [zM + wM]$ and hence $A = 0$. Suppose $\operatorname{Ker} V_1^* \bigcap \operatorname{Ker} V_2^* \neq \{0\}$. If $zM \not\subset wM$ then $[zM + wM] \supsetneq wM$ and hence $\operatorname{Ker} A \supsetneq wM$. By Lemma 4, $\bigcap_{n=1}^{\infty} A_n \supsetneq wM$. The case $zM \subset wM$ does not occur (as pointed out to me privately by Professor K. Takahashi). For if $f \in \operatorname{Ker} V_1^* \bigcap \operatorname{Ker} V_2^*$, then $A = A^*$ implies $V_2^*V_1f = V_1^*V_2f$. If $zM \subseteq wM$, then $V_2^*V_1f = \bar{w}zf$, and hence

$$\|f\| = \|V_2^*V_1f\| = \|V_1^*V_2f\|.$$

Thus $V_1^*V_2f = \bar{z}wf$, and so $\bar{w}zf = \bar{z}wf$; hence $f = 0$. (Since otherwise $z^2 = w^2$ in a set of positive measure.)

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We do not know whether $A = A^*$ implies $A = 0$. However, there exist many invariant subspaces such that A is unitarily equivalent to A^* and $A \neq 0$ (see Example 2). Put $Uf(z, w) = f(w, z)$ for any f in L^2. Then U is a unitary operator on L^2 and U^2 is the identity operator I on L^2. Let M be an invariant subspace which is invariant under U. Then U is an isometry on M and $U^2 = I$ on M. Hence U can be assumed to be a unitary operator on M.

PROPOSITION 8. *Let M be an invariant subspace of L^2 which is invariant under U. Then $V_2 U = U V_1$ and $U A^* U = A$.*

6. Examples

In the previous sections, invariant subspaces M, satisfying $\bigcap_{n=0}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$, were important. In this section, we will give several examples of such invariant subspaces.

EXAMPLE 1. Suppose M is a non-trivial invariant subspace in H^2. Let R be the orthogonal projection in L^2 with range $H^2 \ominus M$, and let the operator J_z on $H^2 \ominus M$ be defined by $J_z f = R(zf)$. If J_z is of finite rank n then there exists an analytic polynomial p of z of degree n such that $p(S_z) = 0$, and hence $p(z)H^2 \subset M$. The inner part of $p(z)$ is a finite Blashke product $F = F(z)$ of degree m, and $m \neq 0$ because $M \neq H^2$. Since $\bar{F} H^2$ is in $\mathbb{H}_1$, $N = \bar{F} M$ lies between H^2 and $\mathbb{H}_1$. Then $M = FN$, $N = K \oplus H^2$ and $\dim K \ominus wK \leq m$. For

$$K \subset \sum_{j=0}^{\infty} \oplus (\bar{F} \mathscr{H}_1 \ominus \mathscr{H}_1) w^j$$

and hence $\dim K \ominus wK \leq \dim(\bar{F} \mathscr{H}_1 \ominus \mathscr{H}_1)$. By Theorem 5, $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$ and by Corollary 2, A_n is of finite rank r_n, and $\sup_n r_n \leq m$. Since $\dim(H^2 \ominus M) < \infty$ and $\dim(H^2 \ominus FH^2) = \infty$, $M \neq FH^2$ and hence $\dim K \ominus wK \neq 0$. By Corollary 2, $0 < \sup_n r_n \leq m$. For if $\sup_n r_n = 0$ then $(K \ominus wK) \oplus \mathscr{H}_1$ is an invariant subspace under the multiplication of z and $(K \ominus wK) \oplus \mathscr{H}_1 \subseteq \bar{F} \mathscr{H}_1$. By Beurling's theorem (cf [4, page 4])

$$(K \ominus wK) \oplus \mathscr{H}_1 = \bar{q} \mathscr{H}_1$$

and q is a nonconstant finite Blaschke product of degree $\leq m$ because $K \ominus wK \neq \{0\}$. Therefore $N = \bar{q} H^2$. If $q \neq F$ then $M = GH^2$ and $m > \ell$ (the degree of

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G) where $G = F\bar{q}$. Hence J_z is of finite rank ℓ. This contradiction implies that $0 \neq \sup_n r_n$. By Theorem 4, M does not have the form qH^2 for any unimodular q. This is given in [1, Corollary 2].

EXAMPLE 2. If M is an invariant subspace in H^2, of finite codimension n, then by Example 1, $M = F_1N_1 = F_2N_2$ where F_1 and F_2 are finite Blashke products of z and w, respectively, and $H^2 \subset N_j \subset \mathbb{H}_j$ $(j = 1, 2)$. Then both A and A^* are finite rank of degree $m \leq n$ and $m \neq 0$. By [8, Theorem 3], $\dim \operatorname{Ker} V_2^* = \dim \operatorname{Ker} V_1^* = \infty$. By Example 1, M does not have the form qH^2 for any unimodular q. Put $M = [zH^2 + wH^2]$: then M is of finite codimension 1. Moreover M is invariant under U. Hence A has rank one and $UA^*U = A$.

EXAMPLE 3. Let M be an invariant subspace of L^2. Invariant subspaces M satisfying $w^nM \supset zM$ for any $n \geq 1$, or $z^nM \supset wM$ for any $n \geq 1$, were studied in [3, 4, 7]. In general, if $wM \supset zM$, then $AV_1 = 0$, since $AV_2 = 0$ by Lemma 1, and because $V_2M \supset V_1M$. Hence by the first part of Lemma 4, $A_n = V_1^{n-1}A$. Thus $\operatorname{Ker} A_n = \operatorname{Ker} A$ for any $n \geq 1$. If $w^nM \supset zM$ for any $n \geq 1$, it is known (see [7]) that

$$M = q(\mathscr{H}_2 \oplus z\mathbb{H}_2)$$

or

$$M = \chi_{E_1}\mathbb{H}_2 \oplus \chi_{E_2}L^2$$

where q is unimodular, $\chi_{E_1} \in \mathscr{L}_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. Hence if $wM \neq M$ then $M \ominus wM = \{q\}$, $\dim \operatorname{Ker} V_2^* = 1$, $\operatorname{Ker} A_n = wM$ and A_n is of rank 1 for any $n \geq 1$. If $z^nM \supset wM$ for any $n \geq 1$, by Proposition 6 we have $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$ and $M \ominus wM = q(\mathscr{H}_1 \oplus \bar{z}\mathscr{H}_1)$ for some unimodular q. In [3], the authors considered the following generalizations of the above invariant subspaces: for any fixed $\ell \geq 1$ M satisfies $w^nM \supset z^\ell M$ for any $n \geq 1$; or $z^nM \supset w^\ell M$ for any $n \geq 1$. They described completely such invariant subspaces and showed that if $zM \neq M$ or $wM \neq M$, then $M = FN$, where F is unimodular, and $\mathbb{H}_1 \supset N \supset z^\ell\mathbb{H}_1$, or $\mathbb{H}_2 \supset N \supset w^\ell\mathbb{H}_2$. Hence if $zM \neq M$ and $z^nM \supset w^\ell M$, then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$.

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