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Extreme point properties of fixed-point sets

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We consider a semigroup S acting as affine continuous maps on a compact convex set X. F denotes the corresponding set of fixed points. Let $\text{ex}X$ and $\text{ex}F$ denote the corresponding sets of extreme points. If X is a simplex, conditions are given which ensure that when $x \in F$, the maximal measure representing x is invariant under S. We also prove $\text{ex}F = F \cap \text{ex}X$ under conditions involving extreme amenability of S. Topological properties of $\text{ex}F$ are also studied.

1. Definitions and notation

S will denote a semigroup and L_∞ the space of all bounded real valued functions on S. For $s \in S$ and $f \in L_\infty$, $l_s f \in L_\infty$ is defined by: $l_s f(t) = f(st)$. L denotes a closed subspace of L_∞, containing the constant functions and such that $l_s L \subseteq L$ for each $s \in S$. By a mean μ on L is meant an element $\mu \in L^*$ such that $\mu \geq 0$ and $\mu(1) = 1$ (hence $\|\mu\| = 1$). Such a mean $\mu \in L^*$ is called left invariant if $l_s^* \mu = \mu$ for all $s \in S$. When such a μ exists, L is said to be left amenable. If L is an algebra and a left invariant μ exists which is also multiplicative (that is, $\mu(fg) = \mu(f)\mu(g)$ for $f, g \in L$), L is said to be extremely left amenable. S is said to be (extremely) left amenable if

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L_∞ is (extremely) left amenable. M denotes the set of means on L - it is a compact convex set in the weak* topology.

X shall denote a compact convex set situated in some Hausdorff locally convex topological vector space E . $A(X)$ denotes all affine continuous real valued functions on X . $K(X)$ denotes all convex continuous real valued functions on X . A representation of S as a semigroup of continuous affine maps on X is assumed to be given. We define F to be $\{x \in X : sx = x \text{ for all } s \in S\}$, and it is the fixed point set associated with the given representation of S . F is a compact convex subset of X . ex$X$ and ex$F$ denote the sets of extreme points of X and F respectively.

Given $z \in X$, define $T_z : A(X) \to L_\infty$ by $(T_z f)(s) = f(sz)$ for $f \in A(X)$, $s \in S$. T_z is a positive linear operator of norm 1 . If $z \in X$ is such that $T_z(A(X)) \subseteq L$, T_z^* maps L^* into $A(X)^*$ - in fact if μ is a mean on L , $T_z^*\mu$ will be a mean on $A(X)$ and corresponds to an evaluation given by a point of X ([1]). Hence if $T_z(A(X)) \subseteq L$, T_z^* can be regarded as mapping the means on L into X .

For information on Choquet simplices etc., see [16]. For K-analytic and K-Borel sets see [13].

2. The extremely amenable case

LEMMA 2.1 ([1]). *Let* μ *be a left invariant mean on* L *and* $z \in X$ *be such that* $T_z(A(X)) \subseteq L$. *Then* $T_z^*\mu \in F$. *If* S *is left amenable,* F *is non void.*

Proof. For each $f \in A(X)$ and each $s \in S$ we have:

$$f(s \cdot T_z^*\mu) = \mu(T_z(f \circ s)) = \mu\left[l_s(T_z f)\right] = \mu(T_z f) = f(T_z^*\mu) .$$

Since $A(X)$ separates points of X , $s(T_z^*\mu) = T_z^*\mu$ for all $s \in S$; that is, $T_z^*\mu \in F$.

The last statement is due to Day [5], and follows from $T_z(A(X)) \subseteq L_\infty$ for all z .

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For a given compact Hausdorff space T, $p(T)$ shall denote all regular Borel probability measures on T. We define

$$I = \{\mu \in p(X) : \mu \circ s^{-1} = \mu, \text{ all } s \in S\} .$$

I is a weak* compact convex subset of $p(X)$, and F is non void if and only if I is non void. Given $v \in X$ let $S_v = \{s \in S : sv \in \mathrm{ex}X\}$.

Define $S_0 = \bigcap_{v \in \mathrm{ex}X} S_v$. S_0 is a subsemigroup of S. A subset S_1 of S is said to be left near thick if, given $t \in S$, there is $s \in S_1$ such that $ts \in S_1$.

LEMMA 2.2. *Let X be a Choquet simplex and suppose that S_0 is left near thick in S. Let $x \in X$ and $\mu \in p(X)$ be its (unique) representing maximal measure. Then if $x \in F$, $\mu \in I$; and if $x \in \mathrm{ex}F$, $\mu \in \mathrm{ex}I$.*

Proof. By Theorem 1 of Jellett [12], if $s(\mathrm{ex}X) \subseteq \mathrm{ex}X$ and $\mu \in p(X)$ is maximal, then $\mu \circ s^{-1}$ is maximal. Hence if $x \in F$ and μ is its representing maximal measure, $\mu \circ s^{-1}$ is maximal and represents $sx = x$, provided $s \in S_0$. By the uniqueness of representing maximal measures, $\mu = \mu \circ s^{-1}$ for all $s \in S_0$.

Now if $t \in S$ choose $s \in S_0$ such that $ts \in S_0$. Then

$$\mu \circ t^{-1} = \mu \circ s^{-1} \circ t^{-1} = \mu \circ (ts)^{-1} = \mu$$

as $ts \in S_0$; that is, $\mu \in I$.

If now $x \in \mathrm{ex}F$, let μ be its representing maximal measure. Let $\mu_1, \mu_2 \in I$ and $0 < \lambda < 1$ be such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. If $f \in K(X)$, μ maximal implies $\mu(\bar{f}-f) = 0$. Since $\bar{f} - f \geq 0$, we have $\mu_1(\bar{f}-f) = \mu_2(\bar{f}-f) = 0$. Hence for each $f \in K(X)$, $\mu_1(\bar{f}) = \mu_1(f)$ and $\mu_2(\bar{f}) = \mu_2(f)$. It now follows (Phelps [17], p. 64) that μ_1 and μ_2 are maximal. Let x_1, x_2 respectively be the resultants of μ_1, μ_2. Because $\mu_1, \mu_2 \in I$, $x_1, x_2 \in F$, and $x = \lambda x_1 + (1-\lambda)x_2$. Since $x \in \mathrm{ex}F$,

$x = x_1 = x_2$. By uniqueness of representing maximal measures, $\mu = \mu_1 = \mu_2$. Hence $\mu \in \mathrm{ex}I$.

LEMMA 2.3. *Let* L *be left amenable. Let* $\mu \in I$ *be such that, for each* $f, g \in C(X)$ *, the function on* S *defined by* $s \to \mu((f \circ s)g)$ *belongs to* L *. Then the following are equivalent:*

(1) $\mu \in \mathrm{ex}I$;

(2) *for each left invariant mean* m *on* L *and* $f, g \in C(X)$,
$m\{s \to \mu((f \circ s)g)\} = \mu(f)\mu(g)$;

(3) *for some mean* m *on* L *we have* $m\{s \to \mu((f \circ s)g)\} = \mu(f)\mu(g)$ *for all* $f, g \in C(X)$.

Proof. This can be adapted from [4], Theorem 2.1.

THEOREM 2.4. *Suppose that:*

(1) L *is left amenable;*

(2) *each element of* S *has a right zero;*

(3) X *is a Choquet simplex;*

(4) S_0 *is left near thick in* S ;

(5) *for* $\mu \in I$ *and* $f, g \in C(X)$ *, the function* $s \to \mu((f \circ s)g)$ *belongs to* L .

Then $F \cap \mathrm{ex}X = \mathrm{ex}F$.

Proof. We obviously have $\mathrm{ex}X \cap F \subseteq \mathrm{ex}F$. Conversely let $x \in \mathrm{ex}F$ and let $\mu \in p(X)$ be its representing maximal measure. By Lemma 2.2, $\mu \in \mathrm{ex}I$ and Lemma 2.3 gives that for each left invariant mean m on L , and $f, g \in C(X)$, $m\{s \to \mu((f \circ s)g)\} = \mu(f)\mu(g)$. By (2) choose $t \in S$ such that $st = t$. Then $\mu((f \circ s)g) = \mu((fg) \circ t) = \mu(fg)$. Then for $f, g \in C(X)$, $\mu(fg) = \mu(f)\mu(g)$. Hence there is $x_0 \in X$ such that $f(x_0) = \mu(f)$ for $f \in C(X)$; that is, $\mu = \varepsilon_{x_0}$. Because μ was chosen to be maximal, $x_0 \in \mathrm{ex}X$ (Phelps [17], p. 8); and for $f \in A(X)$, $f(x) = \mu(f) = f(x_0)$. Hence $x = x_0 \in \mathrm{ex}X$ so that $\mathrm{ex}F \subseteq F \cap \mathrm{ex}X$. We

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deduce $\text{ex}F = F \cap \text{ex}X$.

COROLLARY 2.5. *A semigroup S is extremely left amenable if and only if S is left amenable and each element of S has a right zero.*

Proof. If S is extremely left amenable, Granirer [8] has shown that any finite subset of S has a common right zero. Hence each element of S has a right zero. Conversely, apply Theorem 2.4. For X take the means on L_∞ . Represent S as a semigroup of continuous affine maps on X by: $s \to l_s^*$, $s \in S$. $\text{ex}X$ is the set of multiplicative means on L_∞ and $l_s^*(\text{ex}X) \subseteq \text{ex}X$ for all $s \in S$ is clear. All the conditions of Theorem 2.4 hold so we may deduce that $\mu \in \text{ex}X$ exists such that $l_s^*\mu = \mu$ for all $s \in S$. But this simply says that S is extremely left amenable.

REMARKS. (1) In view of the characterizations in [8] and [13], p. 66, the result of the corollary is that a semigroup has the common fixed point property on compacta if and only if it has the individual fixed point property on any set and the common fixed point property (continuous affine maps) on compact convex sets. There are many semigroups in which each element has a right zero but which are not left amenable. For example, for each natural number n , let S_n be the semigroup of continuous maps from the closed unit sphere of Euclidean n-space into itself. The Brouwer fixed point theorem ensures that each element of S_n has a right zero. However, S_n is obviously not extremely left amenable. It follows that S_n is not left amenable, $n = 1, 2, 3, \ldots$.

(2) The conditions of Theorem 2.4 are fulfilled if S is extremely left amenable, X is a simplex and $s(\text{ex}X) \subseteq \text{ex}X$. The conclusion $\text{ex}F = F \cap \text{ex}X$ then extends the main result of Mitchell [15] and Theorem 6 (b) of Granirer [9].

3. Topological properties of ex$F$

The action of S on X is said to be weakly almost periodic if for each $f \in C(X)$, $\{f \circ s : s \in S\}$ is relatively weakly compact in $C(X)$. We introduce M_0 , the set of maximal measures in $p(X)$. Define $I_0 = I \cap M_0$. $r : p(X) \to X$ denotes the resultant map, and it is

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continuous. We give conditions which ensure that F is a Choquet simplex. We are also interested in the way in which the properties of $\text{ex}X$ influence those of $\text{ex}F$.

THEOREM 3.1. *Suppose that* X *is a Choquet simplex and that* S_0 *is left near thick in* S . *Then* F *is a Choquet simplex.*

Proof. The resultant map $r : I_o \to F$ is injective and by Lemma 2.2 it is also onto F . r is also affine so that F will be a Choquet simplex if I_o is a Choquet simplex. Let $\lambda, \mu \in \tilde{I}_o = \{\alpha\gamma : \alpha \geq 0, \gamma \in I_o\}$. Then $\lambda, \mu \in \tilde{I} = \{\alpha\gamma : \alpha > 0, \gamma \in I\}$ and I is a simplex (Phelps [17], p. 80). Let γ_0 be the greatest lower bound in $\tilde{I}$ of $\lambda, \mu \in \tilde{I}$. Then it follows from [17], p. 65, that γ_0 is maximal (since $\gamma_0 \leq \lambda$ in the usual ordering). So $\gamma_0 \in I_o$ and we immediately see that γ_0 is the greatest lower bound of λ, μ in I_o . Hence I_o is a simplex. By our earlier remarks, F is a simplex.

LEMMA 3.2. *Let* $\text{ex}X$ *be a K-Borel set in* X . *Then* M_o *is a K-Borel set in* $p(X)$. *(Of course* $p(X)$ *has the weak* topology in which it is a compact Hausdorff space.)*

Proof. We introduce the family A whose elements are Borel subsets of X . We define a Borel set A in X to be in A if and only if for each real α , $\{\mu \in p(X) : \mu(A) \geq \alpha\}$ is K-Borel in $p(X)$. If $A \subseteq X$ is compact then $\{\mu \in p(X) : \mu(A) \geq \alpha\}$ is compact in $p(X)$ for all real α . Hence A contains the compact subsets of X .

Now let (A_i) be a sequence in A such that $A_i \downarrow A$. Then $\{\mu \in p(X) : \mu(A) \geq \alpha\}$ is simply $\bigcap_1^\infty \left\{\mu \in p(X) : \mu(A_i) \geq \alpha\right\}$, which is K-Borel as $A_i \in A$ for all i . If (B_i) is a sequence in A with $B_i \uparrow B$, for each (m, n) define $K_{(m,n)} = \left\{\mu \in p(X) : \mu(B_m) \geq \alpha - \frac{1}{n}\right\}$. Since $\{\mu \in p(X) : \mu(B) \geq \alpha\}$ is just $\bigcap_{n=1}^\infty \left[\bigcup_{m=n}^\infty K_{(m,n)}\right]$, which is K-Borel as $B_i \in A$, we must have $B \in A$.

We have now shown that $\mathcal{A}$ contains compact sets and is closed under countable increasing unions and countable decreasing intersections. Because the compact sets are closed under finite unions and finite intersections it follows (in much the same way that Halmos [10] proves his Theorem B, p. 27) that $\mathcal{A}$ contains the smallest family closed under countable unions and intersections and containing the compact sets; that is, $\mathcal{A}$ contains the K-Borel sets. Hence for each K-Borel set A, $\{\mu \in p(X) : \mu(A) = 1\}$ is K-Borel in $p(X)$. Hence $\{\mu \in p(X) : \mu(\mathrm{ex}X) = 1\}$ is K-Borel in $p(X)$. However this set is just M_0 ([17], p. 122).

THEOREM 3.3. *Suppose that*

(1) *X is a Choquet simplex;*

(2) *L is left amenable;*

(3) *S_0 is left near thick in S;*

(4) *the action of S on X is weakly almost periodic;*

(5) *for $\mu \in I$ and $f, g \in C(X)$ the function $s \to \mu((f \circ s)g)$ belongs to L.*

Then if $\mathrm{ex}X$ *is compact,* $\mathrm{ex}F$ *is compact. If* $\mathrm{ex}X$ *is K-Borel in X,* $\mathrm{ex}F$ *is K-analytic in F.*

Proof. We may clearly assume that F is non void. We have $I_0 = I \cap M$ so clearly $\mathrm{ex}I \cap M_0 \subseteq \mathrm{ex}I_0$. However if $\mu \in \mathrm{ex}I_0$, $\mu_1, \mu_2 \in I$ and $0 < \lambda < 1$ with $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, then $\mu_1, \mu_2 \in M_0$ (since $\gamma \in p(X)$ is maximal if and only if $\gamma(\bar{f}-f) = 0$ for all $f \in K(X)$, [17], p. 64). $\mu \in \mathrm{ex}I_0$ now gives $\mu = \mu_1 = \mu_2$. Hence $\mathrm{ex}I_0 \subseteq \mathrm{ex}I \cap M_0$. We deduce that $\mathrm{ex}I_0 = \mathrm{ex}I \cap M_0$.

Now the proof of Theorem 4.5 of [4] can be modified so that when applied in this situation we deduce that $\mathrm{ex}I$ is compact. If $\mathrm{ex}X$ is compact, $M_0 = \{\mu \in p(X) : \mu(\mathrm{ex}X) = 1\}$ is compact, so that $\mathrm{ex}I_0$ is compact. Then $\mathrm{ex}F$, being the continuous image of $\mathrm{ex}I_0$ under the resultant map, is compact.

If $\mathrm{ex}X$ is K-Borel, Lemma 3.2 gives that M_0 is K-Borel. Hence $\mathrm{ex}I$ is K-Borel in I. $\mathrm{ex}F$ is now K-analytic, because it is the continuous image of exI_0 under the resultant map (see [2]).

REMARKS. The conditions of Theorem 3.3 are satisfied if G is an equi-continuous group of affine homeomorphisms of the simplex. In this case the rôle of L is played by the bounded almost periodic functions on G - this admits a left invariant mean by [11], p. 250.

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