Canad. Math. Bull. Vol. **26** (3), 1983

FPF RINGS AND SOME CONJECTURES OF C. FAITH

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ABSTRACT. A left FPF ring is a ring R such that every finitely generated faithful left R-module generates the category of left R-modules. It is shown that such rings split into $R = A \oplus B$, where A is a two sided ideal, and A contains the left singular ideal of R as an essential submodule. If R is FPF on both sides B is two sided too, and R is the product of A and B. An example shows this is the best possible and that right FPF does not imply left FPF.

Introduction. The purpose of this note is to prove one theorem and then to produce an example which shows that this is the best one can do. The example is also seen to provide some insight into a number of conjectures of C. Faith, [3].

A ring is called left FPF if every faithful finitely generated left module is a generator of the category of left R-modules. We are able to show, by way of an example, that left FPF does not imply right FPF.

THEOREM [2, Theorem 10, p. 184]. *Any left FPF ring R splits into a product $R_1 \times R_2$ where R_1 is semiprime and if N is the nilradical of R, $N \subset R_2$, and for each two sided ideal I of R_2, $I \cap N = 0$ implies $I = 0$. In case R is commutative, N is contained in the singular ideal. It is the non commutative version of the last statement we investigate next.*

The left singular ideal of a left FPF ring.

THEOREM. *If R is left* FPF, *then there exists a two sided ideal A which is a direct summand of R as a left ideal such that $Z_l(R)$ is essential in A and R/A is left* FPF *and nonsingular on both sides. Moreover, if R is also right* FPF, *then $R = A \oplus A^{\perp}$ as rings.*

We start with some notation. Let S be a subset of an R-module. We set ${}^{\perp}S = \{r \in R : rs = 0 \text{ for all } s \in S\}$. Similarly for $S^{\perp}$. A left ideal L is essential in R, $L \, \Delta_e R$, if $L \cap H = 0$ implies $H = 0$ for any left ideal H of R. We take

Received by the editors July 24 1980, and, in final revised form, July 2, 1982.
Subject classification 16A 48, 16A 36
Key words, Faithful generators, FPF, Singular ideal, self-injective rings.

$Z(M)=\{x\in M: {}^{\perp}x\ \Delta_e R\}$, for an R-module m. If $Z(M)=0$ we say M is non-singular.

In order to prove the theorem we need a pair of lemmas.

LEMMA 1. *Let R be any ring. Let A be an ideal (two sided) in R which is closed as a left ideal i.e. $Z({}_R(R/A))=0$. If $A^{\perp}+A=R$ then ${}^{\perp}(A^{\perp})=A$.*

Proof. We know ${}^{\perp}(A^{\perp})\supset A$ so suppose B is a left ideal such that $B\subset {}^{\perp}(A^{\perp})$ and $B\cap A=0$. We have $1=a+y$ with $a\in A$ and $y\in A^{\perp}$. Since $B\subset {}^{\perp}(A^{\perp})$, $B(A^{\perp})=0$, so $b=ba\in A\cap B=0$. Hence $B=0$. Since A is closed, ${}^{\perp}(A^{\perp})=A$.

LEMMA 2. *Let R be left* FPF. *Let $A=\{x\in R: x+Z_l(R)\in Z({}_R(R/Z(R))\}$. Let D be a left ideal such that $D\cap A=0$. Then, D contains no non-trivial square zero left ideals.*

Proof. It is well known and not too difficult to see that A is closed. Suppose $B\subset D$ and $B^2=0$. Let C be a left ideal maximal w.r.t. $C\cap {}^{\perp}B=0$. If H is a two sided ideal and $H\subset C$, $HB=0$, since $HB\subset B\subset {}^{\perp}B$ and $HB\subset C$. This says $H\subset {}^{\perp}B$ so $H=0$. But then R/C is faithful, hence a generator. In particular R/C must generate B. Now ${}^{\perp}B$ embeds as an essential submodule of R/C under the natural map. A map from R/C to B is given by an element b of B such that $Cb=0$ and clearly $C+{}^{\perp}B$ is in the kernal of any such map, so $b\in Z(R)$. Then $B\subset Z(R)$, so $B=0$.

Now to the proof of the theorem. Let A be given by the equation $A/Z(R)=Z({}_R(R/Z(R))$. We first claim $A+A^{\perp}=R$. To see this take B to be a left ideal maximal w.r.t. $A\cap B=0$. Then, because R is left FPF, $R=A^{\perp}+RB^{\perp}$ because $R/A\oplus R/B$ is faithful and $A^{\perp}+RB^{\perp}$ is the trace of $R/A\oplus R/B$ in R. Suppose $Bx=0$. Then $(A+B)x\in A$. Because of the choice of B, $A+B$ is an essential left ideal so $(x+A)\in Z({}_R(R/A))=0$ by the choice of A. So $x\in A$. But then $B^{\perp}\subset A$, so $RB^{\perp}\subset A$ and so $R=A^{\perp}+A$. If we write $1=a+a_1$ where $a\in A$ and $a_1\in A^{\perp}$, then $a^2=a$ and $A=Ra$ so A is a direct summand as a left ideal. This is the first part of the theorem.

If we take $R=Ra\oplus R(1-a)$ and use Lemma 2 we see that R/A is semiprime. If we can show R/A is FPF, then R/A is nonsingular by [7, Thms 1 and 3]. Let M be a finitely generated faithful left R/A module. Form $A\oplus M$ as R-modules (give M the natural R structure). Now ${}^{\perp}M=A$ so ${}^{\perp}(A\oplus M)={}^{\perp}A\cap A$. Using lemma 1 we see that $0={}^{\perp}R={}^{\perp}(A+A^{\perp})={}^{\perp}A\cap {}^{\perp}(A^{\perp})={}^{\perp}A\cap A$. So $A\oplus M$ is finitely generated and faithful over R. But trace $A\subset A$ since $Z_l(R)$ is essential in A; so M must generate R/A as an R-module, because $Z_l(R/A)=0$ as an R-module (as an R/A module too). It follows that M generates R/A as an R/A module, also. So R/A is left FPF.

In order to prove the last statement we have seen that $A\cap {}^{\perp}A=0$. But this implies that ${}^{\perp}A+{}^{\perp}({}^{\perp}A)=R$ when R is right FPF. Now if $B={}^{\perp}A\cap {}^{\perp}({}^{\perp}A)$,

$B \cap A = 0$ and $B^2 = 0$, so by Lemma 2, $B = 0$. So ${}^{\perp}A \oplus {}^{\perp}({}^{\perp}A) = R$ as two sided ideals. It follows that ${}^{\perp}A$ is generated by a central idempotent. Now take $R/{}^{\perp}A = \bar{R}$. The ring $\bar{R}$ is FPF. Let $\bar{A}$ be the image of A in $\bar{R}$. Then ${}^{\perp}(\bar{A}) = 0$ in $\bar{R}$ and $\bar{A} = \bar{R}\bar{a}$. So $\bar{A}$ generates $\bar{R}$ and hence is $\bar{R}$, since trace $\bar{A} = \bar{A}$. It follows that $A = {}^{\perp}({}^{\perp}A)$ and $R = {}^{\perp}A \oplus A$.

We now give an example of a left FPF ring for which the closure of the left singular ideal is not a two sided summand. Let T be a commutative self-injective von Neumann regular ring which is not semi-simple. Let M be an essential maximal ideal and $S = T/M$. Let $\mathcal{D} = \mathrm{End}_T(S)$. Now T is a V-ring and hence S is injective over T and $\mathcal{D}$ is a division ring. Set $R = \left\{\begin{pmatrix} t & s \\ 0 & d \end{pmatrix} : t \in T, s \in S, d \in \mathcal{D}\right\}$. By letting T act on the left of S and $\mathcal{D}$ on the right of S we make R into a ring in the usual way. We claim R is the desired example. First, R is left self injective. To see this notice that there are only two types of left ideas: Those which contain $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and those for which $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ acts as a left annihilator. Using this and Baer's criterion one easily shows that R is left self-injective. The left singular ideal of R is $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ and its closure is $\begin{pmatrix} 0 & S \\ 0 & \mathcal{D} \end{pmatrix}$. Each left ideal of R contains a two sided ideal which is an essential submodule so if M is a finitely generated faithful left R-module with generators $m_1, m_2, \ldots, m_k$, we have $\bigcap_{i=1}^{k} {}^{\perp}m_i = 0$. It follows that R embeds in a direct sum of copies of M and since R is left self-injective this embedding splits, i.e. R is left FPF. As we have seen R is not right FPF by the theorem.

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