CLEFT EXTENSIONS FOR A HOPF ALGEBRA $k_q[X, X^{-1}, Y]$

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The concept of cleft extensions, or equivalently of crossed products, for a Hopf algebra is a generalization of Galois extensions with normal basis and of crossed products for a group. The study of these subjects was founded independently by Blattner-Cohen-Montgomery [**1**] and by Doi-Takeuchi [**4**]. In this paper, we determine the isomorphic classes of cleft extensions for a infinite dimensional non-commutative, non-cocommutative Hopf algebra $k_q[X, X^{-1}, Y]$, which is generated by a group-like element X and a $(1,X)$-primitive element Y. We also consider the quotient algebras of the cleft extensions.

Throughout we work over a field k. Algebra, Hopf algebra, linear and $\otimes$ mean k–algebra, Hopf algebra over k, k–linear and $\otimes_k$, respectively.

1. Preliminaries. In this section, we recall some fundamental definitions and results on cleft extensions.

Let H be a Hopf algebra with coalgebra structure Δ, ε. Fix an algebra C.

A right H–comodule algebra A (with H–comodule structure $\rho : A \to A \otimes H$) is called an *H–cleft extension over C* [**2**, p. 41], if A contains C as coinvariant subalgebra; that is, $C = \{a \in A \mid \rho(a) = a \otimes 1\}$, and if there exists a right H–comodule map $\phi : H \to A$ which is invertible in the convolution algebra $\mathrm{Hom}(H, A)$ [**10**, p. 69]. In this case, ϕ can be chosen so as to be unitary ($\phi(1) = 1$) [**4**, p. 813]. A unitary invertible H–comodule map $H \to A$ is called a section [**3**, p. 3056]. We call a pair (A, ϕ) of an H–cleft extension A/C and a section ϕ a *cleft system for H over C*.

A *morphism* (*isomorphism*) $f: A \to A'$ between H–extensions over C means a morphism (isomorphism) of H–comodule algebras such that $f(c) = c$ for all $c \in C$. Denote by

$$\mathrm{Cleft}(H, C)$$

the set of isomorphic classes of H–cleft extensions over C.

LEMMA 1.2. *Let $f: A \to A'$ be a morphism of H–extensions over C. If A/C is an H–cleft extension, then A'/C is also an H–cleft extension and f is an isomorphism.*

Proof. See the proof of [**8**, Lemma 1.3]

A cleft system (A, ϕ) can be characterized as a crossed product. Explicitly, if (A, ϕ) is a cleft system, set

$$h \rightharpoonup c = \sum \phi(h_{(1)}) c \phi^{-1}(h_{(2)}), \qquad (c \in C, h \in H)$$
$$\sigma(h, g) = \sum \phi(h_{(1)}) \phi(g_{(1)}) \phi^{-1}(h_{(2)} g_{(2)}), \qquad (h, g \in H)$$

then $(\rightharpoonup, \sigma)$ is a crossed system for H over C, and one can form a crossed product $C\#_\sigma H$ which is an H–extension over C with structure map $id \otimes \Delta : C\#_\sigma H \to C\#_\sigma H \otimes H$ [**4**], [**1**].

In this case, $C\#_\sigma H \to A$, $c\#h \mapsto c\phi(h)$, is an isomorphism of H–extensions over C. Conversely, if $(\rightharpoonup, \sigma)$ is a crossed system, $C\#_\sigma H$ is the corresponding crossed product, then $id \otimes \Delta : C\#_\sigma H \to C\#_\sigma H \otimes H$ makes $C\#_\sigma H$ into an H–cleft extension over C, and $\phi : H \to C\#_\sigma H$, $h \mapsto 1\#h$, is a section. See [**2**] and [**3**]. These give a $1-1$ correspondence between the isomorphic classes of cleft systems and the crossed systems (both for H over C).

An H–cleft extension A/C is said to be *twisted* (respectively, *smashed*), if there exists a section ϕ such that $\phi(H) \subset A^C$ (respectively, ϕ is an algebra map), where A^C is the centralizer of C in A. See [**3**, p. 3056, p. 3059].

Throughout, the boldface letters **N**, **Z**, stand for nonnegative integers, all integers respectively. $U(R)$ denotes the group of units in an algebra R.

2. Cleft Extensions for $k_q[X, X^{-1}, Y]$. Let $k\{X, Y, Z\}$ be the non-commutative free algebra on three variables. Then $k\{X, Y, Z\}$ has a bialgebra structure determined by

$$\begin{aligned} \Delta(X) &= X \otimes X, & \varepsilon(X) &= 1, \\ \Delta(Y) &= 1 \otimes Y + Y \otimes X, & \varepsilon(Y) &= 0, \\ \Delta(Z) &= Z \otimes Z, & \varepsilon(Z) &= 1. \end{aligned}$$

See [**10**, p. 89] or [**8**, p. 4543]. Now let $0 \neq q \in k$, then the two-sided ideal generated by $XZ-1$, $ZX-1$, $YX-qXY$, is a bi-ideal, and we have $Z = X^{-1}$ in the quotient bialgebra, denote by $k_q[X, X^{-1}, Y]$ the quotient bialgebra. $k_q[X, X^{-1}, Y]$ has an antipode determined by

$$S(X) = X^{-1}, \quad S(X^{-1}) = X, \quad S(Y) = -YX^{-1}.$$

For convenience, we write H_∞ for $k_q[X, X^{-1}, Y]$.

LEMMA 2.1.
(1) *H_∞ has a k–basis $\{X^nY^m, n \in \mathbf{Z}, m \in \mathbf{N}\}$,*
(2) $\Delta(X^nY^m) = \sum_{i=0}^{m} \binom{m}{i} X^nY^i \otimes X^{n+i}Y^{m-i}$, *$n \in \mathbf{Z}$, $m \in \mathbf{N}$, where $\binom{m}{i}_q$ denote the q–binomial coefficients* (cf. [7, p. 74]).

Proof. Easy.

THEOREM 2.2. *Let $C \subset A$ be an H_∞–extension. Then A is H_∞–cleft if and only if there exist elements x and y in A with $x \in U(A)$ such that*

$$\rho(x) = x \otimes X \quad \text{and} \quad \rho(y) = 1 \otimes Y + y \otimes X.$$

If this is the case, we have:
(1) *The map $\phi : H_\infty \to A$, $\phi(X^nY^m) = x^ny^m (n \in \mathbf{Z}, m \in \mathbf{N})$, is a section. The inverse is given by*

$$\phi^{-1}(X^nY^m) = (-1)^m q^{m(m-1)/2} y^m x^{-(n+m)}, \qquad n \in \mathbf{Z}, m \in \mathbf{N}.$$

(2) *A is a free left C–module with a basis $\{x^ny^m, n \in \mathbf{Z}, m \in \mathbf{N}\}$.*
(3) $(yx - qxy)x^{-2} \in C$.

Proof. See [**5**, Theorem 3.2].

Let (A, ϕ) be a cleft system for H_∞ over C, $x = \phi(X)$, $y = \phi(Y)$. Then x and y have properties described in Theorem 2.2. Set

$$\alpha(c) = xcx^{-1}, \delta(c) = [y, c]x^{-1}, c \in C, \text{ and } \gamma = (yx - qxy)x^{-2},$$

then we have the following result.

LEMMA 2.3.
(1) *$\alpha : C \to C$ is an algebra automorphism.*
(2) *$\delta : C \to C$ is a (1,α)-derivation, that is, a linear endomorphism such that*

$$\delta(cc') = \delta(c)\alpha(c') + c\delta(c'), \quad c, c' \in C.$$

(3) $\delta\alpha(c) - q\alpha\delta(c) = \gamma\alpha^2(c) - \alpha(c)\gamma$.

Proof. It is a straightforward verification. If $(\rightharpoonup, \sigma)$ is the crossed system induced from (A, ϕ), then $\alpha(c) = X \rightharpoonup c$, $\delta(c) = Y \rightharpoonup c$, and $\gamma = (\sigma(Y, X) - q\sigma(X, Y))\sigma(X, X)^{-1}$. See [**5**], [**8**].

DEFINITION 2.4. *Let $\alpha, \delta =\in$ End (C), $\gamma \in C$. The 3-tuple (α, δ, γ) is called an H_∞–cleft datum over C, if the three conditions in Lemma 2.3 are satisfied. We denote the set of all such data by*

$$\mathcal{D} = \mathcal{D}(H_\infty, C)$$

Now let $\underline{d} = (\alpha, \delta, \gamma)$ be an H_∞–cleft datum over C. Define $F_n, n \in \mathbf{Z}$, as follows:

$$\begin{cases} F_0 = 0, F_n = \gamma + q\alpha(F_{n-1}), n > 0, \\ F_n = -q^n\alpha^n(F_{-n}), n < 0. \end{cases} \tag{a}$$

LEMMA 2.5.
(1) $F_n = F_{n-1} + q^{n-1}\alpha^{n-1}(\gamma), \forall n \in \mathbf{Z}$.
(2) $F_{n+m} = F_n + q^n\alpha^n(F_m)$, $\forall n, m \in \mathbf{Z}$.

Proof. (1) We first prove it for $n \geq 0$ by induction on n. It is clear that (1) holds for $n = 0$ and 1. Now suppose that $n > 1$ and $F_{n-1} = F_{n-2} + q^{n-2}\alpha^{n-2}(\gamma)$, then

$$\begin{aligned} F_n &= \gamma + q\alpha(F_{n-1}) = \gamma + q\alpha(F_{n-2} + q^{n-2}\alpha^{n-2}(\gamma)) \\ &= \gamma + q\alpha(F_{n-2}) + q^{n-1}\alpha^{n-1}(\gamma) = F_{n-1} + q^{n-1}\alpha^{n-1}(\gamma). \end{aligned}$$

Thus (1) holds for all $n \geq 0$.

Next, let $n < 0$, then

$$\begin{aligned} F_n &= -q^n\alpha^n(F_{-n}) = -q^{n-1}\alpha^{n-1}(q\alpha(F_{-n})) \\ &= -q^{n-1}\alpha^{n-1}(F_{-n+1} - \gamma) && \text{(by (a))} \\ &= -q^{n-1}\alpha^{n-1}(F_{-(n-1)}) + q^{n-1}\alpha^{n-1}(\gamma) \\ &= F_{n-1} + q^{n-1}\alpha^{n-1}(\gamma) && \text{(by (a)).} \end{aligned}$$

(2) If $m = 0$, it is trivial. If $m = 1$, this is the case (1). Now let $m > 1$, and suppose that $F_{n+m-1} = F_n + q^n\alpha^n(F_{m-1})$ holds for all $n \in \mathbf{Z}$. Then

$$\begin{aligned} F_{n+m} &= F_{n+m-1} + q^{n+m-1}\alpha^{n+m-1}(\gamma) && \text{(by (1))} \\ &= F_n + q^n\alpha^n(F_{m-1}) + q^{n+m-1}\alpha^{n+m-1}(\gamma) && \text{(by induction hypothesis)} \\ &= F_n + q^n\alpha^n(F_{m-1} + q^{m-1}\alpha^{m-1}(\gamma)) \\ &= F_n + q^n\alpha^n(F_m) && \text{(by (1)).} \end{aligned}$$

Next, let $m < 0$, then

$$\begin{aligned} F_n &= F_{n+m+(-m)} \\ &= F_{n+m} + q^{n+m}\alpha^{n+m}(F_{-m}) && \text{(by the case } m \geq 0) \\ &= F_{n+m} - q^n\alpha^n(-q^m\alpha^m(F_{-m})) \\ &= F_{n+m} - q^n\alpha^n(F_m) && \text{(by (a)).} \end{aligned}$$

Hence $F_{n+m} = F_n + q^n\alpha^n(F_m)$, and so (2) holds.

LEMMA 2.6.
$\delta\alpha^n(c) = q^n\alpha^n\delta(c) + F_n\alpha^{n+1}(c) - \alpha^n(c)F_n, \forall c \in C, n \in \mathbf{Z}$.

Proof. One can prove it for $n \geq 0$ by induction on n. If $n < 0$, then

$$\begin{aligned} \delta\alpha^n(c) &= \alpha^n(\alpha^{-n}\delta(\alpha^n(c))) = q^n\alpha^n(q^{-n}\alpha^{-n}\delta(\alpha^n(c))) \\ &= q^n\alpha^n(\delta\alpha^{-n}(\alpha^n(c)) - F_{-n}\alpha^{-n+1}(\alpha^n(c)) + \alpha^{-n}(\alpha^n(c))F_{-n}) && \text{(by the case of } n \geq 0) \\ &= q^n\alpha^n\delta(c) - q^n\alpha^n(F_{-n})\alpha^{n+1}(c) + q^n\alpha^n(c)\alpha^n(F_{-n}) \\ &= q^n\alpha^n\delta(c) + F_n\alpha^{n+1}(c) - \alpha^n(c)F_n. && \text{(by (a))} \end{aligned}$$

Now we can form an H_∞–extension of C for the cleft datum $\underline{d} = (\alpha, \delta, \gamma)$ as follows.

(1) Let $B_{\underline{d}}$ be the skew Laurent polynomial algebra $C[x, x^{-1}, \alpha]$ on one variable x, that is $B_{\underline{d}} = \{\sum_{i=m}^{n} c_i x^i \mid m, n \in \mathbf{Z}, m \leq n, c_i \in C\}$ with the multiplication determined by $xc = \alpha(c)x$ for all $c \in C$.

(2) Define $\overline{\alpha} : B_{\underline{d}} \to B_{\underline{d}}, cx^n \mapsto cq^n x^n, c \in C, n \in \mathbf{Z}$, then $\overline{\alpha}$ is an algebra automorphism.

(3) Define $\overline{\delta} : B_{\underline{d}} \to B_{\underline{d}}, cx^n \mapsto (cF_n + \delta(c))x^{n+1}, c \in C, n \in \mathbf{Z}$, then $\overline{\delta}$ is an $\overline{\alpha}$-derivation of $B_{\underline{d}}$ by the following Lemma 2.7.

(4) Define $A_{\underline{d}}$ to be the Ore extension $B_{\underline{d}}[y, \overline{\alpha}, \overline{\delta}]$ with one variable y attached to the data $(B_{\underline{d}}, \overline{\alpha}, \overline{\delta})$ (cf. [7, Theorem I.7.1]), then $A_{\underline{d}}$ is a free left C–module with a basis $\{x^n y^m, n \in \mathbf{Z}, m \in \mathbf{N}\}$.

(5) Define

$$\rho_{\underline{d}} : A_{\underline{d}} \to A_{\underline{d}} \otimes H_\infty, cx^n y^m \mapsto \sum_{i=0}^{m} \binom{m}{i}_q cx^n y^i \otimes X^{n+i} Y^{m-i}, c \in C, n \in \mathbf{Z}, m \in \mathbf{N},$$

$$\phi_{\underline{d}} : H_\infty \to A_{\underline{d}}, X^n Y^m \mapsto x^n y^m, n \in \mathbf{Z}, m \in \mathbf{N}.$$

Lemma 2.7. *Let $B_{\underline{d}}, \overline{\alpha}, \overline{\delta}$ be as above. Then $\overline{\delta}$ is an $\overline{\alpha}$-derivation of $B_{\underline{d}}$.*

Proof. Note that $B_{\underline{d}} = \oplus_{n \in \mathbf{Z}} Cx^n$ as a k–vector space, hence $\overline{\delta}$ is well-defined. Now for any $c, c' \in C, n, m \in \mathbf{Z}$,

$$\begin{aligned}
\overline{\delta}((cx^n)(c'x^m)) &= \overline{\delta}(c\alpha^n(c')x^{n+m}) \\
&= (c\alpha^n(c')F_{n+m} + \delta(c\alpha^n(c')))x^{n+m+1} && \text{(by Definition of } \overline{\delta}) \\
&= (c\alpha^n(c')F_{n+m} + c\delta\alpha^n(c') + \delta(c)\alpha^{n+1}(c'))x^{n+m+1} && \text{(by Definition 2.4)} \\
&= (c\alpha^n(c')(F_n + q^n\alpha^n(F_m)) + c\delta\alpha^n(c') \\
&\quad + \delta(c)\alpha^{n+1}(c'))x^{n+m+1} && \text{(by Lemma 2.5(2))} \\
&= (cq^n\alpha^n(c'F_m) + c(\delta\alpha^n(c') + \alpha^n(c')F_n) \\
&\quad + \delta(c)\alpha^{n+1}(c'))x^{n+m+1} \\
&= cq^n x^n c' F_m x^{m+1} + c(q^n\alpha^n\delta(c') + F_n\alpha^{n+1}(c'))x^{n+m+1} \\
&\quad + \delta(c)\alpha^{n+1}(c')x^{n+m+1} && \text{(by Lemma 2.6)} \\
&= cq^n(x^n c' F_m + \alpha^n\delta(c')x^n)x^{m+1} \\
&\quad + (cF_n + \delta(c))\alpha^{n+1}(c')x^{n+m+1} \\
&= cq^n x^n(c'F_m + \delta(c'))x^{m+1} + (cF_n + \delta(c))x^{n+1}c'x^m \\
&= \overline{\alpha}(cx^n)\overline{\delta}(c'x^m) + \overline{\delta}(cx^n)(c'x^m).
\end{aligned}$$

Theorem 2.8. *Let $A_{\underline{d}}, \rho_{\underline{d}}, \phi_{\underline{d}}$ be as before. Then*
(1) $\rho_{\underline{d}}$ makes $A_{\underline{d}}$ into an H_∞–extension over C.
(2) $\phi_{\underline{d}}$ is a section, the inverse is given by

$$\phi_{\underline{d}}^{-1}(X^n Y^m) = (-1)^m q^{m(m-1)/2} y^m x^{-(n+m)}, n \in \mathbf{Z}, m \in \mathbf{N},$$

consequently, $(A_{\underline{d}}, \phi_{\underline{d}})$ is a cleft system for H_∞ over C.

Proof. (1) Set

$$\rho : B_{\underline{d}} \to A_{\underline{d}} \otimes H_\infty, cx^n \mapsto cx^n \otimes X^n, c \in C, n \in \mathbf{Z},$$

then ρ is well-defined. One can easily check that ρ is an algebra map, $\rho(c) = c \otimes 1$, $\rho(\overline{\alpha}(cx^n)) = \overline{\alpha}(cx^n) \otimes X^n$ and $\rho(\overline{\delta}(cx^n)) = \overline{\delta}(cx^n) \otimes X^{n+1}$ for all $c \in C, n \in \mathbf{Z}$. Let $\xi = 1 \otimes Y + y \otimes X \in A_{\underline{d}} \otimes H_\infty$, then

$$\begin{aligned}\xi\rho(cx^n) &= (1\otimes Y + y\otimes X)(cx^n\otimes X^n)\\ &= cx^n\otimes YX^n + ycx^n\otimes X^{n+1}\\ &= cx^n\otimes q^nX^nY + (\overline{\alpha}(cx^n)y+\overline{\delta}(cx^n))\otimes X^{n+1}\\ &= \overline{\alpha}(cx^n)\otimes X^nY + \overline{\alpha}(cx^n)y\otimes X^{n+1} + \overline{\delta}(cx^n)\otimes X^{n+1}\\ &= (\overline{\alpha}(cx^n)\otimes X^n)(1\otimes Y + y\otimes X) + \overline{\delta}(cx^n)\otimes X^{n+1}\\ &= \rho(\overline{\alpha}(cx^n))\xi + \rho(\overline{\delta}(cx^n)).\end{aligned}$$

Thus by the following Lemma 2.9, there is a unique algebra map $\overline{\rho}: A_{\underline{d}} = B_{\underline{d}}[y,\overline{\alpha},\overline{\delta}] \to A_{\underline{d}}\otimes H_\infty$ such that $\overline{\rho}(y) = 1\otimes Y + y\otimes X$ and the restriction of $\overline{\rho}$ on $B_{\underline{d}}$ is equal to ρ. In this case,

$$\begin{aligned}\overline{\rho}(cx^ny^m) &= \overline{\rho}(cx^n)\overline{\rho}(y^m) = \rho(cx^n)\overline{\rho}(y)^m\\ &= (cx^n\otimes X^n)(1\otimes Y + y\otimes X)^m\\ &= \sum_{i=0}^{m}\binom{m}{i}_q cx^ny^i\otimes X^{n+i}Y^{m-i},\ c\in C, n\in\mathbf{Z}, m\in\mathbf{N},\end{aligned}$$

hence $\overline{\rho} = \rho_{\underline{d}}$, and so $\rho_{\underline{d}}$ is an algebra map.

Next, it is clear that $(id\otimes\varepsilon)\rho_{\underline{d}} = id$. So as to prove the equation $(\rho_{\underline{d}}\otimes id)\rho_{\underline{d}} = (id\otimes\Delta)\rho_{\underline{d}}$, note that each side of it is an algebra map from $A_{\underline{d}}$ to $A_{\underline{d}}\otimes H_\infty\otimes H_\infty$, and that $A_{\underline{d}}$ is generated by C, x, x^{-1} and y as an algebra. Therefore it suffices to prove $(\rho_{\underline{d}}\otimes id)\rho_{\underline{d}}(a) = (id\otimes\Delta)\rho_{\underline{d}}(a)$ for $a = x, a = x^{-1}, a = y$ and for all $a\in C$, but it is an easy verification. Finally, it is clear that the coinvariant subalgebra of $A_{\underline{d}}$ is C.

(2) It follows immediately from Theorem 2.2 since

$$\rho_{\underline{d}}(x) = x\otimes X \quad\text{and}\quad \rho_{\underline{d}}(y) = 1\otimes Y + y\otimes X.$$

LEMMA 2.9. *Let R, E be algebras, $f: R\to E$ an algebra map, $\alpha,\delta\in$ End(R), $\xi\in E$. Assume that α is an algebra map, δ is an α–derivation. If the 4–tuple (f,α,δ,ξ) satisfies:*

$$\xi f(r) = f(\alpha(r))\xi + f(\delta(r)), \forall r\in R,$$

then there exists a unique algebra map $\overline{f}: R[t,\alpha,\delta]\to E$ such that $\overline{f}(t) = \xi$ and the restriction of $\overline{f}$ on R is f, where $R[t,\alpha,\delta]$ is the Ore extension attached to the data (R,α,δ).

Proof. Let $\beta_{n,m}$ be the linear endomorphism of R defined as the sum of all $\binom{n}{m}$ possible compositions of m copies of α and of $n-m$ copies of δ, then

$$t^nr = \sum_{m=0}^{n}\beta_{n,m}(r)t^m, \quad n\geq 0, r\in R.$$

See [**7**, Corollary I.7.4(7.9)]. Similarly, one can prove by induction on n that the relation

$$\xi^nf(r) = \sum_{m=0}^{n} f(\beta_{n,m}(r))\xi^m$$

holds for all $r \in R, n \geq 0$. Set

$$\bar{f}: R[t, \alpha, \delta] \to E, \sum_{i=0}^{n} r_i t^i \mapsto \sum f(r_i)\xi^i,$$

then $\bar{f}$ is a well-defined linear map. Now for any $r, r' \in R, n, n' \geq 0$,

$$\begin{aligned}\bar{f}((rt^n)(r't^{n'})) &= \bar{f}\left(\sum\nolimits_{m=0}^{n} r\beta_{n,m}(r')t^{m+n'}\right)\\ &= \sum\nolimits_{m=0}^{n} f(r\beta_{n,m}(r'))\xi^{m+n'}\\ &= \sum\nolimits_{m=0}^{n} f(r)f(\beta_{n,m}(r'))\xi^m\xi^{n'}\\ &= f(r)\xi^n f(r')\xi^{n'} = \bar{f}(rt^n)\bar{f}(r't^{n'}),\end{aligned}$$

hence $\bar{f}$ is an algebra map. It is clear that $\bar{f}(t) = \xi$ and $\bar{f}(r) = f(r)$ for all $r \in R$. The uniqueness is trivial.

By Lemma 2.3, for any cleft system (A, ϕ), there is a corresponding cleft datum (α, δ, γ) defined as before Lemma 2.3. In particular, for a given cleft datum $\underline{d} = (\alpha, \delta, \gamma)$ one can easily prove that the cleft datum induced from the cleft system $(A_{\underline{d}}, \phi_{\underline{d}})$ is exactly the given cleft datum $\underline{d}$.

THEOREM 2.10. *Let (A, ϕ) be a cleft system, $\underline{d} = (\alpha, \delta, \gamma)$ the corresponding cleft datum. Then $A \cong A_{\underline{d}}$ as H_∞–extensions over C.*

Proof. Let $x_1 = \phi(X)$, $y_1 = \phi(Y)$, then $\alpha(c) = x_1 c x_1^{-1}$, $\delta(c) = [y_1, c]x_1^{-1}$, $\gamma = (y_1x_1 - qx_1y_1)x_1^{-2}$ and x_1, y_1 have properties described in Theorem 2.2.

Note that the Larent polynomial algebra $k[X, X^{-1}]$ is a Hopf subalgebra of H_∞. Set

$$B = \rho^{-1}(A \otimes k[X, X^{-1}]),$$

where ρ is the structure map of the comodule algebra A, then B is a subalgebra of A and $\rho(B) \subset B \otimes k[X, X^{-1}]$. It follows that B is a $\mathbf{Z}$–graded algebra and $B_n = \{b \in B | \rho(b) = b \otimes X^n\} = \{a \in A | \rho(a) = a \otimes X^n\}, n \in \mathbf{Z}$. One can easily check that $B_n = Cx_1^n$, $n \in \mathbf{Z}$. Since $\alpha(c) = x_1 c x_1^{-1}$, $x_1 c = \alpha(c)x_1$ in B, hence the linear map

$$B_{\underline{d}} \to B, \quad cx^n \mapsto cx_1^n, c \in C, n \in \mathbf{Z},$$

is an algebra isomorphism which induces an algebra injection from $B_{\underline{d}}$ to A by the composition

$$f: B_{\underline{d}} \to B \hookrightarrow A.$$

Now one can prove in A that (cf. [**8**, Lemma 2.11])

$$y_1x_1^n = q^n x_1^n y_1 + F_n x_1^{n+1}, n \in \mathbf{Z},$$

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where F_n is defined as in (a) by α, γ. Hence in A we have

$$\begin{aligned} y_1 f(cx^n) &= y_1 c x_1^n = [y_1, c]x_1^n + c y_1 x_1^n \\ &= \delta(c)x_1^{n+1} + c(q^n x_1^n y_1 + F_n x_1^{n+1}) \\ &= c q^n x_1^n y_1 + (cF_n + \delta(c))x_1^{n+1} \\ &= f(cq^n x^n)y_1 + f((cF_n + \delta(c))x^{n+1}) \\ &= f(\overline{\alpha}(cx^n))y_1 + f(\overline{\delta}(cx^n)), \; c \in C, n \in \mathbf{Z}, \end{aligned}$$

it follows from Lemma 2.9 that f can be uniquely extended to $A_{\underline{d}}$, that is,

$$\bar{f}: A_{\underline{d}} \to A, \; cx^n \mapsto cx_1^n, \; y \mapsto y_1, \; c \in C, n \in \mathbf{Z},$$

is an algebra map. Clearly, $\bar{f}$ is a morphism of H_∞–extension over C, and is bijective by Lemma 1.1 or Theorem 2.2(2).

THEOREM 2.11. *Let A be an algebra containing C as a subalgebra. Then $C \subset A$ is an H_∞–cleft extension if and only if there exists a cleft datum $\underline{d}$ such that $A/C \cong A_{\underline{d}}/C$, that is, there is an algebra isomorphism $f: A \to A_{\underline{d}}$ with $f(c) = c$ for all $c \in C$.*

Proof. It follows from Theorem 2.8 and 2.10.

THEOREM 2.12. *Let $\underline{d} = (\alpha, \delta, \gamma)$ and $\underline{d}' = (\alpha', \delta', \gamma'_{\,})$ be H_∞– cleft data over C. Then $A_{\underline{d}} \cong A_{\underline{d}'}$ as H_∞– extensions over C if and only if there exist $a \in U(C)$ and $b \in C$ such that*

$$\begin{cases} (1) & \alpha'(c) = a\alpha(c)a^{-1}, \\ (2) & \delta'(c) = (\delta(c) + b\alpha(c) - cb)a^{-1}, \\ (3) & \gamma' \;\; = (a\gamma + b\alpha(a) + \delta(a) - qa\alpha(b))(a\alpha(a))^{-1}. \end{cases} \tag{b}$$

Proof. Assume that $f: A_{\underline{d}'} \to A_{\underline{d}}$ is an isomorphism of H_∞–extensions over C. Set

$$a = f(x')x^{-1}, \quad b = (f(y') - y)x^{-1},$$

then one can prove that $a \in U(C)$, $b \in C$, and the conditions (1)–(3) hold.

Conversely, suppose that there exist $a \in U(C)$ and $b \in C$ such that the conditions (1)–(3) hold. Set

$$f: B_{\underline{d}'} = C[x', x'^{-1}, \alpha'] \to A_{\underline{d}}, \; cx'^n \mapsto c(ax)^n, \; c \in C, n \in \mathbf{Z},$$

one can easily prove that f is an algebra map with $f(c) = c, f(x') = ax$, $c \in C$. We claim that

$$(y + bx)f(r) = f(\overline{\alpha'}(r))(y + bx) + f(\overline{\delta'}(r)) \tag{c}$$

holds for any $r \in B_{\underline{d}'}$. First, if the relation (c) holds for $r_1, r_2 \in B_{\underline{d}'}$ then

$$
\begin{aligned}
(y+bx)f(r_1r_2) &= (y+bx)f(r_1)f(r_2)\\
&= (f(\overline{\alpha'}(r_1))(y+bx)+f(\overline{\delta'}(r_1)))f(r_2)\\
&= f(\overline{\alpha'}(r_1))(y+bx)f(r_2)+f(\overline{\delta'}(r_1)r_2)\\
&= f(\overline{\alpha'}(r_1))(f(\overline{\alpha'}(r_2))(y+bx)+f(\overline{\delta'}(r_2)))+f(\overline{\delta'}(r_1)r_2)\\
&= f(\overline{\alpha'}(r_1)\overline{\alpha'}(r_2))(y+bx)+f(\overline{\alpha'}(r_1)\overline{\delta'}(r_2)+\overline{\delta'}(r_1)r_2)\\
&= f(\overline{\alpha'}(r_1r_2))(y+bx)+f(\overline{\delta'}(r_1r_2)),
\end{aligned}
$$

that is, the relation (c) holds for r_1r_2. Next, a straightforward verification shows that the relation (c) holds for $r = x', r = x'^{-1}$, and for all $r \in C$. Finally, since $B_{\underline{d}'}$ is generated by C, x', x'^{-1} as an algebra, the relation (c) holds for all $r \in B_{\underline{d}'}$. It follows from Lemma 2.9 that there is a unique algebra map $\bar{f}: A_{\underline{d}'} \to A_{\underline{d}}$ such that $\bar{f}(c) = c,\ c \in C, \bar{f}(x') = ax$, and $\bar{f}(y') = y + bx$. One can easily check that $\bar{f}$ is a morphism of H_∞–extensions over C. By Lemma 1.1, $\bar{f}$ must be bijective.

LEMMA 2.13. *Let $\underline{d} = (\alpha, \delta, \gamma)$ be a cleft datum for H_∞ over $C, a \in U(C), b \in C$. Define $\alpha', \delta', \gamma'$ by (b) in Theorem 2.12. The $\underline{d}' = (\alpha', \delta', \gamma')$ is also a cleft datum for H_∞ over C.*

Proof. Consider $A_{\underline{d}}$, set $x_1 = ax, y_1 = y + bx$, then $x_1 \in U(A_{\underline{d}}), y_1 \in A_{\underline{d}}$, and $\rho_{\underline{d}}(x_1) = x_1 \otimes X$, $\rho_{\underline{d}}(y_1) = 1 \otimes Y + y_1 \otimes X$. Define

$$\phi : H_\infty \to A_{\underline{d}},\ X^nY^m \mapsto x_1^n y_1^m,\ n \in \mathbf{Z}, m \in \mathbf{N},$$

then by Theorem 2.2, ϕ is also a section for $A_{\underline{d}}$. A straightforward computation shows that $\alpha'(c) = x_1cx_1^{-1}, \delta'(c) = [y_1, c]x_1^{-1}$ and $\gamma' = (y_1x_1 - qx_1y_1)x_1^{-2}$. Hence $\underline{d}' = (\alpha', \delta', \gamma')$ is exactly the cleft datum induced from the left system $(A_{\underline{d}}, \phi)$.

By the proof of Theorem 2.12, we know that if $f: A_{\underline{d}'} \to A_{\underline{d}}$ is an isomorphism of H_∞–extensions over C, where $\underline{d} = (\alpha, \delta, \gamma)$ and $\underline{d}' = (\alpha', \delta', \gamma')$ are cleft data, then there exist $a \in U(C)$ and $b \in C$ such that

$$f(x') = ax, \quad f(y') = y + bx,$$

and $\underline{d}, \underline{d}'$, a and b satisfy the three relations in (b). Now let $\underline{d}'' = (\alpha'', \delta'', \gamma'')$ be another cleft datum, if $g : A_{\underline{d}''} \to A_{\underline{d}'}$ is also an isomorphism of H_∞–extensions determined by a pair $(s, t) \in U(C) \times C$, that is

$$g(x'') = sx', \quad g(y'') = y' + tx',$$

then the composition $fg : A_{\underline{d}''} \to A_{\underline{d}}$ is determined by the pair $(sa, ta + b)$, that is,

$$(fg)(x'') = sax, \quad (fg)(y'') = y + (ta + b)x.$$

The group $U(C)$ acts on the additive group C by the right multiplication. So we have the group $U(C) \ltimes C$ of semi-direct product with the multiplication (cf. [**8**, p. 4553])

$$(s \ltimes t)(a \ltimes b) = sa \ltimes (ta + b).$$

Thus from the above discussion and Theorem 2.12 we have

COROLLARY 2.14.
(1) *$U(C) \ltimes C$ acts on the set $\mathcal{D}$ from the left with the action*

$$\underline{d'} = (a \ltimes b)\underline{d}$$

defined by (b).
(2) *Suppose $d, \underline{d'} \in \mathcal{D}$. Then $A_{\underline{d}} \cong A_{\underline{d'}}$, if and only if $\underline{d}$ and $\underline{d'}$ are $U(C) \ltimes C$–equivalent.*

THEOREM 2.15. *$\underline{d} \mapsto A_{\underline{d}}$ gives a 1–1 correspondence between the set $U(C) \ltimes C \setminus \mathcal{D}(H_\infty, C)$ of $U(C) \ltimes C$–orbits in $\mathcal{D}(H_\infty, C)$ and the set $Cleft(H_\infty, C)$ of isomorphic classes of H_∞–cleft extensions over C.*

Proof. It follows by Theorem 2.10 and corollary 2.14(2).

Note that for any $\gamma \in C$, $(1, 0, \gamma)$ is a cleft datum if and only if $\gamma \in Z(C)$, the center of C. Let α be an algebra automorphism of C, δ a $(1, \alpha)$–derivation of C, then $(\alpha, \delta, 0)$ is a cleft datum if and only if $\delta\alpha = q\alpha\delta$.

Let $\underline{d} = (\alpha, \delta, \gamma) \in \mathcal{D}$, then $\phi_{\underline{d}}(H_\infty) \subset A_{\underline{d}}^C$ if and only if $\alpha = 1$ and $\delta = 0$; $\phi_{\underline{d}}$ is an algebra map if and only if $\gamma = 0$. Thus by a method similar to [**8**, Prop. 2.24], we have

PROPOSITION 2.16. *Let $\underline{d} = (\alpha, \delta, \gamma) \in \mathcal{D}$. Then*
(1) *$A_{\underline{d}}$ is twisted if and only if there exist $a \in U(C)$, $b \in C$ such that*

$$\alpha(c) = aca^{-1}, \quad \delta(c) = [b, c]a^{-1}, \quad c \in C.$$

(2) *$A_{\underline{d}}$ is smashed, if and only if there exist $a \in U(C)$, $b \in C$ such that $\gamma = q\alpha(b) + (\delta(a) - ab)\alpha(a)^{-1}$.*

3. Quotient Algebra of $A_{\underline{d}}$. In this section, we write $A(\underline{d}, C)$ for $A_{\underline{d}}$, $\underline{d} \in \mathcal{D}(H_\infty, C)$.

Let s be a multiplicative set in C. Then s *satisfies the right Ore condition* $cs \bigcap dC$ is nonempty for all $c \in C$ and $d \in s$, while s is *right reversible* if $dc = 0, c \in C, d \in s$ implies $cd' = 0$ for some $d' \in s$. A *right Ore set* is any multiplicative set satisfying the right Ore condition, while a *right denominator set* is any right reversible right Ore set [**6**, p. 144].

Let s be a right Ore set in C, set

$$t_s(C) = \{c \in C \mid cd = 0 \text{ for some } d \in s\},$$

then $t_s(C)$ is an ideal of C.

Let s be a right denominator set of C, then there exists a *right quotient algebra* Cs^{-1} of C with respect to s [**6**, Theorem 9.7], that is, there is an algebra map $\theta : C \to Cs^{-1}$ such that:
(a) $\theta(d)$ is a unit of Cs^{-1} for all $d \in s$.
(b) Each element of Cs^{-1} has the form $\theta(c)\theta(d)^{-1}$ for some $c \in C, d \in s$.
(c) $\ker\theta = t_s(C)$.

If s consists of regular elements of C, then $\ker\theta = t_s(C) = 0$. In this case, C is a subalgebra of Cs^{-1} regarding θ as embedding, and each element of Cs^{-1} takes the form cd^{-1} for some $c \in C$ and $d \in s$.

Throughout the following, assume that s is a right denominator set in C, and that $\theta : C \to Cs^{-1}$ is a right quotient algebra of C with respect to s.

LEMMA 3.1. *Assume $\alpha : C \to C$ is an algebra automorphism. If $\alpha(s) = s$, then there exists a unique algebra automorphism α_0 of Cs^{-1} such that $\alpha_0\theta = \theta\alpha$. Furthermore, if $\delta : C \to C$ is a $(1, \alpha)$– derivation then there exists a unique $(1, \alpha_0)$— derivation δ_0 of Cs^{-1} such that $\delta_0\theta = \theta\delta$.*

Proof. Assume α is an algebra automorphism of C with $\alpha(s) = s$. One can easily check that the composition $C \xrightarrow{\alpha} C \xrightarrow{\theta} Cs^{-1}$ also makes Cs^{-1} into a right quotient algebra of C with respect to s. It follows by [**6**, Corollary 9.5] that there exists a unique algebra isomorphism $\alpha_0 : Cs^{-1} \to Cs^{-1}$ such that $\alpha_0\theta = \theta\alpha$. Furthermore, assume δ is a $(1,\alpha)$– derivation of C. We claim that $t_s(C)$ is δ–stable, i.e. $\delta(t_s(C)) \subset t_s(C)$. In fact, let $c \in t_s(C)$ then $cd = 0$ for some $d \in s$. Therefore

$$0 = \delta(cd) = \delta(c)\alpha(d) + c\delta(d).$$

However $c\delta(d) \in t_s(C)$ since $t_s(C)$ is an ideal of C, so $c\delta(d)d' = 0$ for some $d' \in s$, and hence $\delta(c)\alpha(d)d' = 0$. But $\alpha(d)d' \in s$, so $\delta(c) \in t_s(C)$, and then it follows that $t_s(C)$ is δ– stable. Now if δ_0 is a $(1, \alpha_0)$– derivation of Cs^{-1} with $\delta_0\theta = \theta\delta$, then for any $c \in C, d \in s$, we have

$$\begin{aligned}
0 = \delta_0(1) = \delta_0(\theta(d)\theta(d)^{-1}) &= \delta_0(\theta(d))\alpha_0(\theta(d)^{-1}) + \theta(d)\delta_0(\theta(d)^{-1}) \\
&= \theta(\delta(d))(\alpha_0\theta(d))^{-1} + \theta(d)\delta_0(\theta(d)^{-1}) \\
&= \theta(\delta(d))\theta(\alpha(d))^{-1} + \theta(d)\delta_0(\theta(d)^{-1}),
\end{aligned}$$

therefore, $\delta_0(\theta(d)^{-1}) = -\theta(d)^{-1}\theta(\delta(d))\theta(\alpha(d))^{-1}$, and so

$$\begin{aligned}
\delta_0(\theta(c)\theta(d)^{-1}) &= \delta_0(\theta(c))\alpha_0(\theta(d)^{-1}) + \theta(c)\delta_0(\theta(d)^{-1}) \\
&= \theta(\delta(c))\theta(\alpha(d))^{-1} - \theta(c)\theta(d)^{-1}\theta(\delta(d))\theta(\alpha(d))^{-1}.
\end{aligned}$$

It follows that δ_0 must be unique. As to existence, let us define $\delta_0 \in \mathrm{End}(Cs^{-1})$ by

$$\delta_0(\theta(c)\theta(d)^{-1}) = \theta(\delta(c))\theta(\alpha(d))^{-1} - \theta(c)\theta(d)^{-1}\theta(\delta(d))\theta(\alpha(d))^{-1}, \quad c \in C, d \in s.$$

A tedious and standard verification shows that δ_0 is well defined, and is a $(1, \alpha_0)$–derivation of C_s. Clearly, $\delta_0\theta = \theta\delta$.

LEMMA 3.2. *Let $\underline{d} = (\alpha, \delta, \gamma) \in \mathcal{D}(H_\infty, C)$ with $\alpha(s) = s, \alpha_0, \delta_0$ as in Lemma 3.1, $\gamma_0 = \theta(\gamma)$ in Cs^{-1}. Then $\underline{d_0} = (\alpha_0, \delta_0, \gamma_0)$ is a cleft datum for H_∞ over Cs^{-1}, i.e. $\underline{d_0} \in \mathcal{D}(H_\infty, Cs^{-1})$.*

Proof. By Definition 2.4 and Lemma 3.1, we only have to prove that the equation $\delta_0\alpha_0(p) - q\alpha_0\delta_0(p) = \gamma_0\alpha_0^2(p) - \alpha_0(p)\gamma_0$ holds for any $p \in Cs^{-1}$. In fact, let $p = \theta(c)\theta(d)^{-1}$, $c \in C, d \in s$, then

$$\begin{aligned}
&\delta_0\alpha_0(\theta(c)\theta(d)^{-1}) - q\alpha_0\delta_0(\theta(c)\theta(d)^{-1})\\
&= \delta_0(\theta(\alpha(c))\theta(\alpha(d))^{-1}) - q[\theta(\alpha\delta(c))\theta(\alpha^2(d))^{-1} - \theta(\alpha(c))\theta(\alpha(d))^{-1}\theta(\alpha\delta(d))\theta(\alpha^2(d))^{-1}]\\
&= \theta(\delta\alpha(c))\theta(\alpha^2(d))^{-1} - \theta(\alpha(c))\theta(\alpha(d))^{-1}\theta(\delta\alpha(d))\theta(\alpha^2(d))^{-1}\\
&\quad - q[\theta(\alpha\delta(c))\theta(\alpha^2(d))^{-1} - \theta(\alpha(c))\theta(\alpha(d))^{-1}\theta(\alpha\delta(d))\theta(\alpha^2(d))^{-1}]\\
&= \theta(\delta\alpha(c) - q\alpha\delta(c))\theta(\alpha^2(d))^{-1} - \theta(\alpha(c))\theta(\alpha(d))^{-1}\theta(\delta\alpha(d) - q\alpha\delta(d))\theta(\alpha^2(d))^{-1}\\
&= \theta(\gamma\alpha^2(c) - \alpha(c)\gamma)\theta(\alpha^2(d))^{-1} - \theta(\alpha(c))\theta(\alpha(d))^{-1}\theta(\gamma\alpha^2(d) - \alpha(d)\gamma)\theta(\alpha^2(d))^{-1}\\
&= \gamma_0\alpha_0^2(\theta(c)\theta(d)^{-1}) - \alpha_0(\theta(c)\theta(d)^{-1})\gamma_0.
\end{aligned}$$

THEOREM 3.3. *Let* $\underline{d} = (\alpha, \delta, \gamma) \in \mathcal{D}(H_\infty, C)$ *with* $\alpha(s) = s$ *and* $\underline{d_0} = (\alpha_0, \delta_0, \gamma_0) \in \mathcal{D}(H_\infty, Cs^{-1})$ *as in Lemma 3.2. Then* s *is also a right denominator set of* $A(\underline{d}, C)$, *and* $A(\underline{d_0}, Cs^{-1})$ *is a right quotient algebra of* $A(\underline{d}, C)$ *with respect to* s; i.e. $A(\underline{d_0}, Cs^{-1}) \cong A(\underline{d}, C)s^{-1}$.

Proof. Let $F_n (n \in \mathbf{Z})$ be as in (a) for $\underline{d}$ over C, then

$$\begin{cases}\theta(F_0) = 0, \quad \theta(F_n) = \gamma_0 + q\alpha_0(\theta(F_{n-1})), n > 0\\ \theta(F_n) = -q^n\alpha_0^n(\theta(F_{-n})), n < 0.\end{cases}$$

Thus by the structure of $A(\underline{d}, C)$ and $A(\underline{d_0}, Cs^{-1})$ as in §2, θ can be uniquely extended to an algebra map $\overline{\theta}$ from $A(\underline{d}, C)$ to $A(\underline{d_0}, Cs^{-1})$ such that $\overline{\theta}(x) = x$ and $\overline{\theta}(y) = y$. Clearly, $\overline{\theta}(d) = \theta(d)$ is a unit of $A(\underline{d_0}, Cs^{-1})$ for all $d \in s$. Note that $A(\underline{d}, C) = B_{\underline{d}}[y, \overline{\alpha}, \overline{\delta}]$ and $B_{\underline{d}} = C[x, x^{-1}, \alpha]$, $A(\underline{d}, C)$ is also a free right C–module with a basis $\{y^m x^n, n \in \mathbf{Z}, m \in \mathbf{N}\}$ since α and $\overline{\alpha}$ are automorphisms. Similarly, $A(\underline{d_0}, Cs^{-1})$ is a free right Cs^{-1}–module with a basis $\{y^m x^n, n \in \mathbf{Z}, m \in \mathbf{N}\}$. It follows that each element of $A(\underline{d_0}, Cs^{-1})$ has the form $\overline{\theta}(r)\overline{\theta}(d)^{-1}$ for some $r \in A(\underline{d}, C)$ and $d \in s$, and that $\ker\overline{\theta} = t_s(A(\underline{d}, C))$ by [**6**, Lemma 9.2(a)]. Hence $A(\underline{d_0}, Cs^{-1})$ is a right quotient algebra of $A(\underline{d}, C)$ with respect to s, and s is a right denominator set of $A(\underline{d}, C)$. This completes the proof.

Note that the algebra map $\overline{\theta} : A(\underline{d}, C) \to A(\underline{d_0}, Cs^{-1})$ is also a right H_∞–comodule map, i.e. $\rho_{\underline{d_0}}\overline{\theta} = (\overline{\theta} \otimes id)\rho_{\underline{d}}$. It can be easily seen that $\overline{\theta}\phi_{\underline{d}} = \phi_{\underline{d_0}}$ and $\phi_{\underline{d_0}}^{-1} = \overline{\theta}\phi_{\underline{d}}^{-1}$. Thus using crossed products, we can present Theorem 3.3 as follows.

THEOREM 3.4. *Let* $(\rightharpoonup, \sigma)$ *be a crossed system for* H_∞ *over* C. *If* $X \rightharpoonup s = s$, *then* $\rightharpoonup$ *can be uniquely extended to a weak action* $\rightarrow$ *of* H_∞ *on* Cs^{-1} *determined by*

$$X \rightarrow \theta(c)\theta(d)^{-1} = \theta(X \rightharpoonup c)\theta(X \rightharpoonup d)^{-1},$$

$$Y \rightarrow \theta(c)\theta(d)^{-1} = \theta(Y \rightharpoonup c)\theta(X \rightharpoonup d)^{-1} - \theta(c)\theta(d)^{-1}\theta(Y \rightharpoonup d)\theta(X \rightharpoonup d)^{-1},$$

and $(\rightarrow, \theta\sigma)$ *is a crossed system for* H_∞ *over* Cs^{-1}. *In this case,*

$$\overline{\theta} : C\#_\sigma H_\infty \to Cs^{-1}\#_{\theta\sigma}H_\infty \; c\#h \mapsto \theta(c)\#h, \quad c \in C, h \in H_\infty,$$

is a right H_∞*–comodule algebra map, and* $\overline{\theta}$ *makes* $Cs^{-1}\#_{\theta\sigma}H_\infty$ *into a right quotient algebra of* $C\#_\sigma H_\infty$ *with respect to* s; *that is,*

$$Cs^{-1}\#_{\theta\sigma}H_\infty \cong (C\#_\sigma H_\infty)s^{-1}.$$

Recall that C is an *integral domain* if the product of nonzero elements is always nonzero. An integral domain C is a *right Ore domain* if the set s of all nonzero elements of C satisfies the right Ore condition. In this case, Cs^{-1} exists, which is a division algebra, usually denote it by $Q(C)$. Thus as corollaries we have the following results.

COROLLARY 3.5. *Let $C\#_\sigma H_\infty$ be a crossed product. If C is a right Ore domain, then there exists a unique crossed product $Q(C)\#_\sigma H_\infty$ containing $C\#_\sigma H_\infty$ as a subalgebra, and $Q(C)\#_\sigma H_\infty = (C\#_\sigma H_\infty)s^{-1}$, where $s = C \setminus \{0\}$.*

COROLLARY 3.6. *Let A/C be an H_∞–cleft extension. If C is a right Ore domain, then $s = C \setminus \{0\}$ is a right denominator set of A, and As^{-1} is an H_∞–cleft extension over $Q(C)$.*

Note that the left versions of all above results still hold.

THEOREM 3.7. *Let A/C be an H_∞–cleft extension. Then*

(1) *If C is prime, then so is A.*

(2) *A is an integral domain if and only if C is an integral domain.*

(3) *A is right (respectively left) Noetherian if and only if C is right (respectively left) Noetherian.*

Proof. If A is an integral domain, it is clear that C is also an integral domain. By Theorem 2.10, we can regard $A = A(\underline{d}, C)$ for some cleft datum $\underline{d} = (\alpha, \delta, \gamma)$. If follows by the structure of $A(\underline{d}, C)$ or Theorem 2.2 that any element z of A can be uniquely expressed as a finite sum $z = \sum_{n\in\mathbf{Z}, m\in\mathbf{N}} c_{n,m}x^n y^m$, where $c_{n,m} \in C$ and almost all $c_{n,m} = 0$, and that $z \in C$ if and only if $c_{n,m} = 0$ for all $n \neq 0$ or $m \neq 0$. Henceby, if I is a right ideal of C, then IA is a right ideal of A and $IA = \oplus_{n\in\mathbf{Z}, m\in\mathbf{N}} Ix^n y^m$. So $IA \cap C = I$. On the other hand, since α and $\overline{\alpha}$ are algebra automorphisms, A is a free right C–module with the basis $\{y^m x^n, n \in \mathbf{Z}, m \in \mathbf{N}\}$. Thus a similar argument shows that if I is a left ideal of C then AI is a left ideal of A and $AI \cap C = I$. It follows that if A is right (left) Noetherian then C is also right (left) Noetherian. The rest follows from [**9**, Theorem 1.2.9 and 1.4.5].

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