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CHARACTER DEGREES AND CLT-GROUPS

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Let G be a finite group and let k be a field. We determine the smallest possible rank of a free kG-module that contains submodules of every possible dimension. As an application, we obtain various criteria for the wreath product of two finite groups to be a CLT-group.

1. RESULTS

A CLT-group is a finite group G of order n, say, having the property that for each divisor d of n, there exists a subgroup of index d in G. Note that it is sufficient to have this condition for all prime powers d dividing n.

Clearly, a CLT-group has Hall p'-subgroups for all primes p, and hence it is soluble. Conversely, if one is interested in proving that a soluble group is a CLT-group, then one has to consider F_pG-modules M and try to find "many" submodules of M which yield "enough" subgroups of p-power index (in the split extension of M by G, say).

Let k be a field and let M be a kG-module. Call M a *CLT-module* (*for* G) if for all integers d satisfying $0 \leqslant d \leqslant \dim_k M$, there exists a kG-submodule U of M with $\dim_k U = d$. The property of being a CLT-module clearly depends on the dimensions of its irreducible constituents and on its Loewy structure. For example, if G is perfect and x denotes the smallest dimension of a nontrivial irreducible kG-module, then the direct sum of less than $x-1$ copies of kG does not contain any submodule of dimension $x-1$. For applications to CLT-groups, we shall be interested in the case when char k is prime to $|G|$, and then the above example is, in some sense, worst possible in view of the following.

THEOREM A. *Let G be a finite group and let k be a field with char k prime to $|G|$. If x denotes the smallest dimension of a G-faithful kG-module, then the direct sum of $x-1$ copies of kG is a CLT-module.*

We will apply the above result to a local version of CLT-groups introduced in [4]. Indeed, G is called a p-CLT-group if for all powers p^a dividing $|G|$, there is a subgroup of index p^a. Most natural examples are transitive groups of degree p. Note that a p-CLT-group has a Hall p'-subgroup.

Our next result provides a method of constructing p-CLT-groups from given ones. Note that the hypothesis on G in the following holds for transitive groups of prime degree p.

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THEOREM B. *Let G be a p-CLT-group and assume that $O_{p'}(G) = 1$. If P is a p-group, then the wreath product $P \wr G$ is a p-CLT-group.*

Our methods also yield some more detailed information on the class R (see p.185 of [**1**]) of all groups G with the property that for every nilpotent group N, the wreath product $N \wr G$ is a CLT-group. Recall that G is called rational if all its irreducible complex characters are rational valued (see [**3**]).

THEOREM C. *Let G be a rational CLT-group. Then $G \in R$.*

Our final result gives some information on the structural properties of groups in R. It improves on Theorems 3(a) and 4(c) of [**1**].

THEOREM D. *Every soluble group G can be embedded as a subgroup of some group H in R. Furthermore if $\pi(G) \cap \{2, 3\} \neq \emptyset$, then we can take $\pi(H) = \pi(G)$. Conversely if $L \in R$ and $L \neq 1$, then $\pi(L) \cap \{2, 3\} \neq \emptyset$.*

Notation and terminology. All groups will be finite and all modules will be finitely generated. We shall use the notation $|G|$ for the order of a group G, $\pi(G)$ for the set of primes dividing $|G|$, and $\mathbb{N}$ for the non-negative integers $\{0, 1, 2, \ldots\}$. If p is a prime, then F_p will denote the field with p elements, and if $n \in \mathbb{N}$, the direct sum of n copies of a module M will be denoted by M^n. When r is a positive integer, $\mathbb{Z}_r$ will indicate the cyclic group of order r.

2. Proofs

For Theorem A, we need some information concerning the distribution of the irreducible character degrees of our group G. The first result shows that they cannot increase too quickly.

LEMMA 1. *Let G be a group and let k be a field. Let x be the dimension of some G-faithful kG-module and let $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_r$ be the dimensions of the irreducible kG-modules. Then $d_{i+1} \leqslant x d_i$ for all i.*

PROOF: Let $V_1, V_2, \ldots, V_r$ be the distinct irreducible kG-modules with $\dim_k V_i = d_i$ $(1 \leqslant i \leqslant r)$, let $a \in \mathbb{N}$ with $1 \leqslant a \leqslant r-1$, and let U be a faithful kG-module of dimension x. For $n \in \mathbb{N}$, let $U^{\otimes n}$ denote the tensor product of U with itself n-times with G acting diagonally. By [**6**, Theorem 1] or [**2**, Theorem III.2.16], there exist $s, t \in \mathbb{N}$ with $t > a$ such that V_t is *not* a composition factor of $U^{\otimes s}$ if $i > a$, but V_i *is* a composition factor of $U^{\otimes s+1}$. Thus $U^{\otimes s+1}$ has a series whose factors are kG-modules of the form $V_j \otimes U$ with $j \leqslant a$, and it follows that V_t is a composition factor of $V_j \otimes U$ for some $j \leqslant a$. Therefore $d_t \leqslant x d_j$, hence $d_{a+1} \leqslant x d_a$ and Lemma 1 is proved. ∎

PROOF OF THEOREM A: Let $V_1, V_2, \ldots, V_r$ be the distinct irreducible kG-modules, let $d_i = \dim_k V_i$, and let δ_i be the number of times V_i occurs in kG $(1 \leqslant i \leqslant r)$. Assume that $d_i \leqslant d_{i+1}$ $(1 \leqslant i \leqslant r-1)$. We now establish the following claim: *if* $1 \leqslant a \leqslant r$ *and* $0 \leqslant y < d_a$, *then there exist* $e_i \in \mathbb{N}$ *with* $e_i \leqslant x-1$ $(1 \leqslant i < a)$ *such that*

$$y = \sum_{i=1}^{a-1} e_i d_i.$$

We prove this by induction on a; clearly the result is true if $a = 1$, so we assume that $a > 1$. Note that $d_a - d_{a-1} \leqslant (x-1)d_{a-1}$ by Lemma 1, hence

$$y - (d_{a-1} - 1) \leqslant (x-1)d_{a-1}.$$

Choose $e_{a-1} \in \mathbb{N}$ minimal such that $y - (d_{a-1} - 1) \leqslant e_{a-1}d_{a-1}$ (so $0 \leqslant e_{a-1} \leqslant x-1$). Then

$$0 \leqslant y - e_{a-1}d_{a-1} \leqslant d_{a-1} - 1$$

(the left inequality holds because of the minimality of e_{a-1}). By induction on a, we may write

$$y - e_{a-1}d_{a-1} = \sum_{i=1}^{a-2} e_i d_i$$

with $0 \leqslant e_i \leqslant x-1$, and the claim is established.

We now show that if $0 \leqslant l \leqslant (x-1)|G|$, then kG^{x-1} has a submodule of dimension l. This is clear if $l = (x-1)|G|$, so we assume that $l < (x-1)|G|$. Now choose $b \in \mathbb{N}$ such that

$$0 \leqslant l - \sum_{i=b+1}^{r} (x-1)\delta_i d_i - md_b < d_b$$

for some $m \in \mathbb{N}$ with $m < (x-1)\delta_b$. By the claim, there exist $e_i \in \mathbb{N}$ with $e_i \leqslant x-1$ such that

$$l - \sum_{i=b+1}^{r} (x-1)\delta_i d_i - md_b = \sum_{i=1}^{b-1} e_i d_i.$$

If $V = \oplus_{i=b+1}^{r} V_i^{(x-1)\delta_i} \oplus V_b^m \oplus \oplus_{i=1}^{b-1} V_i^{e_i}$, then V is a submodule of kG^{x-1} with dimension l, as required. ∎

Remark. Similar arguments show

THEOREM A$'$. *Let G be a perfect group and let k be any field. If x denotes the smallest dimension of a G-faithful kG-module, then kG^{x-1} is a CLT-module.*

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LEMMA 2. *Let p be a prime, let G be a p-CLT-group and let H be a Hall p'-subgroup of G. If F_pG is a CLT-module for H, then $W = P \wr G$ is a p-CLT-group for all p-groups P. Moreover, if G is a CLT-group, then so is W.*

PROOF: Following [**1**, p.188], let $B(P)$ denote the base group of $P \wr G$ and let $L(G, p)$ denote the class of all p-groups having the property that $B(P)$ contains H-invariant subgroups of every possible order. Then $\mathbb{Z}_p \in L(G, p)$ because F_pG is a CLT-module for H. Since P has a normal series with factors isomorphic to $\mathbb{Z}_p$, an obvious generalisation of Theorem 1(b) of [**1**] shows that $P \in L(G, p)$, and hence $P \wr G$ is a p-CLT-group. The final sentence of the Lemma is now clear. ∎

PROOF OF THEOREM B: Write $|G| = p^a m$ where p is prime to m, and let H be a Hall p'-subgroup of G. From $O_{p'}(G) = 1$, we infer that G acts faithfully on the cosets of H, hence there exists a faithful F_pH-module of dimension p^a. Furthermore F_pG, viewed as an F_pH-module, splits into a direct sum of p^a copies of F_pH. Now Theorem A implies that F_pG is a CLT-module for H and the result follows from Lemma 2. ∎

Note that none of the hypotheses of Theorem B can be dispensed with. First, $\mathbb{Z}_2 \wr \mathbb{Z}_5$ is not a 2-CLT-group, so $O_{p'}(G) = 1$ is necessary. For the other hypothesis, let $p = 2$ and $G = PSL(2, 7)$. Then a Hall $2'$-subgroup H of G is isomorphic to the Frobenius group of order 21, and F_2G is a CLT-module for H. But $\mathbb{Z}_2 \wr G$ is not a 2-CLT-group because it does not contain any subgroup of index four.

For the proof of Theorem C, we need the following generalisation of [**4**, Proposition 7] to arbitrary fields.

THEOREM E. *Let G be a soluble group and let k be a splitting field for G. Then kG is a CLT-module for G.*

First we require a preparatory lemma (see Itô's result; Satz 17.10 on p.570 of [**5**]).

LEMMA 3. *Let A be an abelian normal subgroup of the group G. If k is a splitting field for G and V is an irreducible kG-module, then $\dim_k V \leqslant |G/A|$.*

PROOF: By enlarging k if necessary, we may assume that k is also a splitting field for A. If

$$0 = U_0 < U_1 < \ldots < U_n = kA$$

is a kA-composition series for kA, then $\dim_k U_{i+1}/U_i = 1$ for all i because k is a splitting field for A, and

$$0 = U_0 \otimes_{kA} kG < U_1 \otimes_{kA} kG < \ldots < U_n \otimes_{kA} kG = kG$$

is a kG-series for kG. Thus V is a composition factor of $U_{i+1} \otimes_{kA} kG/U_i \otimes_{kA} kG$ for some i, hence $\dim_k V \leqslant |G/A|$. ∎

PROOF OF THEOREM E: Let $p =$ char k. We use induction on $|G|$, the result certainly being true if $|G| = 1$. Let A be a minimal normal subgroup of G, so A is an elementary abelian q-group for some prime q. Let $\mathbf{a}$ denote the augmentation ideal of kA; thus $\{a - 1 \mid a \in A \setminus 1\}$ is a k-basis for $\mathbf{a}$. We have two cases to consider.

Case (i). $q = p$. Then for all $r \in \mathbb{N}$,

$$\mathbf{a}^r kG/\mathbf{a}^{r+1}kG \cong \mathbf{a}^r/\mathbf{a}^{r+1} \otimes_{kA} kG \cong (k[G/A])^s$$

where $s = \dim_k \mathbf{a}^r/\mathbf{a}^{r+1}$. But $\mathbf{a}^r = 0$ if r is large enough, hence kG has a filtration with modules isomorphic to $k[G/A]$. Since the result is true for G/A, it is also true for G.

Case (ii). $q \neq p$. Since $|A|$ is invertible in k, we may write

$$kG = \mathbf{a}kG \oplus k[G/A]$$

as kG-modules. We now use the sandwich technique as described in [**1**, p.186]. Suppose $0 \leqslant \lambda \leqslant |G|$. By Lemma 3, there exists a kG-submodule S_2 of $\mathbf{a}kG$ such that $0 \leqslant \lambda - \dim_k S_2 \leqslant |G/A|$. Using induction, there exists a kG-submodule S_1 of $k[G/A]$ with $\dim_k S_1 = \lambda - \dim_k S_2$. Then $S_1 \oplus S_2$ is a kG-submodule of kG and $\dim_k (S_1 \oplus S_2) = \lambda$, as required. ∎

PROOF OF THEOREM C: Let p be a prime and let H be a Hall p'-subgroup of G. Since G is rational, F_p is a splitting field for G [**3**, Lemma 2]. Moreover, G is soluble and so Theorem E implies that F_pG is a CLT-module for G. Therefore F_pG is a CLT-module for H and the result follows from Lemma 2. ∎

In the proof of Theorem D, we need a method for producing groups G with the property that kG is a CLT-module for G. The following lemma does what is required.

LEMMA 4. *Let A and G be groups, and let k be a field. If $|A| \geqslant |G|$ and kA is a CLT-module for A, then $k[A \times G]$ is a CLT-module for $A \times G$.*

PROOF: Let char $k = p$ (where p is a prime or 0), let $H = A \times G$ and let $n = |H|$. First suppose G is a p-group ($G = 1$ if $p = 0$). Then kG has a series with each factor isomorphic to the trivial module k, hence kH has a series of kH-modules with each factor isomorphic to kA (trivial G-action). Since kA contains submodules of all dimensions between 0 and $|A|$, the result follows in this case.

Now suppose G is not a p-group ($G \neq 1$ if $p = 0$). If $0 \leqslant d \leqslant n$, we need to prove that kH has a submodule of dimension d. Note that if N is a submodule of kH, then $\mathrm{Hom}_k(kH/N, k)$ (the dual kH-module of kH/N) is a submodule of kH with dimension $n - \dim_k N$, so we may assume that $d \leqslant n/2$. Let P be the indecomposable projective kG-module with simple quotient k, and write $kG = P \oplus Q$.

Then $kH = kA \otimes_k P \oplus kA \otimes_k Q$. Since k is a kG-submodule of P and kA has kA-submodules of all dimensions between 0 and $|A|$, it follows that $kA \otimes_k P$ has kH-submodules of all dimensions between 0 and $|A|$. In particular, $kA \otimes_k P$ has kH-submodules of all dimensions between 0 and $|G|$, because $|G| \leqslant |A|$. Now G is not a p-group, thus it contains a p'-subgroup $B \neq 1$, so we can write $kB = k \oplus U$ as kB-modules where $U \neq 0$. Then

$$kG = k \otimes_{kB} kG \oplus U \otimes_{kB} kG$$

and P is a direct summand of $k \otimes_{kB} kG$, hence $\dim_k Q \geqslant |G|/2$. Thus $d \leqslant \dim_k kA \otimes_k Q$ because $d \leqslant n/2$, so there is a kH-submodule M_1 of $kA \otimes_k Q$ with the property that $d - |G| \leqslant \dim_k M_1 \leqslant d$ since $\dim_k Q < |G|$. Now choose a kH-submodule M_2 of $kA \otimes_k P$ of dimension $d - \dim_k M_1$. Then $M_1 \oplus M_2$ is a kH-submodule of kH of dimension d, as required. ∎

PROOF OF THEOREM D: We wish to embed the soluble group G as a subgroup of some $H \in R$, with the property that $\pi(H) = \pi(G)$ if $\pi(G) \cap \{2, 3\} \neq \emptyset$. In view of [**5**, p.663], we may assume that G is a CLT-group. Let A be either a 2-group or a noncyclic 3-group, let $H = A \times G$, and assume that $|A| \geqslant |G|$. By the proof of Theorem 3(b) and 3(c) of [**1**], F_pA is a CLT-module for A for all primes p, hence F_pH is a CLT-module for H by Lemma 4. It now follows from Lemma 2 that $H \in R$, and so it remains to prove the last sentence of Theorem D.

Suppose $|L|$ is odd. Since $\mathbb{Z}_2 \wr L$ is a CLT-group, we see that F_2L must be a CLT-module for L, so let V be a submodule of dimension two. Now F_2L contains exactly one submodule of dimension one, hence V is not centralised by L and it follows that three divides $|L|$, as required.

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