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ON FUNDAMENTAL APPROXIMATIVE ABSOLUTE NEIGHBORHOOD RETRACTS

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ABSTRACT. In this paper we define and study a class of compacta under the name of Fundamental Approximative Absolute Neighborhood Retracts. This class includes Borsuk's Fundamental Absolute Neighborhood Retracts and Approximative Absolute Neighborhood Retracts in the sense of M. H. Clapp as proper subclasses. We also introduce the notion of q-strong movability and prove that Fundamental Approximative Absolute Neighborhood Retracts coincide with q-strongly movable compacta.

0. **Introduction.** In 1953 H. Noguchi [18] introduced the notions of ε-AR-sets and ε-ANR-sets and proved that some properties of compact AR's and ANR's were also valid for these classes of spaces. In 1968 A. Gmurczyk [11] and A. Granas [13] found more properties of AR's and ANR's which can be transferred to Noguchi's spaces and they called these sets approximative absolute retracts (or shortly AAR-sets) and approximative absolute neighborhood retracts (AANR_N-sets) giving up Noguchi's denotations which seem to be less convenient.

In 1971 M. H. Clapp [8] continued the research on Noguchi's spaces and introduced a generalization of the notion of AANR_N-set. The larger class of spaces obtained (the AANRC-sets) still possesses all the properties of ε-ANR's considered by Noguchi. In a second paper A. Gmurczyk [12] established a relation between the notion of an approximative retract and the Borsuk's notion of a fundamental retract [2] proving, in particular, that AAR's are fundamental absolute retracts and AANR_N's are fundamental absolute neighborhood retracts.

In 1974 S. A. Bogatyĭ [1] defined the notion of internal movability and proved that AANR_C-sets are internally movable. V. A. Kalinin [15] generalized Noguchi's and Clapp's results to the case of bicompacta.

In [5] K. Borsuk studied some properties of a class of metric compacta called NE-sets. This class coincides with that of AANR_C's.

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The interest in AANR's seems to be still alive. In the Conference on Shape Theory and Geometric Topology held in Dubrovnik (Yugoslavia) in 1981 Z. Čerin gave a talk entitled "ANR's and AANR's revisited" stating some interesting new properties of these spaces; in particular, he gave a movability condition characterizing $AANR_C$-sets. S. Mardešić introduced in a recent paper [17] a class of spaces called approximate polyhedra. In the compact metric case these spaces agree with $AANR_C$'s and K. Borsuk's NE-sets.

In this paper we define and study a class of compacta under the name of Fundamental Approximative Absolute Neighborhood Retracts (or shortly FAANR-sets). This class includes the classes of FANR-sets and $AANR_C$-sets as proper subclasses. In Section 1 we state the basic definitions and some properties of FAANR-sets. In Section 2 we characterize FAANR-sets as quasi-strongly movable compacta and in Section 3 we prove that a different notion of FAANR is equivalent to some classical notions of the theory of shape.

We assume as known the basic notions of the theory of shape. The reader can find them in [4], [10] and [16].

All spaces to be dealt with in this paper are metrizable and even, in certain cases, a metric d will be considered. In this paper ,compactum means compact metric space and the notation $M \in \mathrm{AR}(M \in \mathrm{ANR})$ means that M is an absolute retract (absolute neighborhood retract) for metrizable spaces.

1. FAANR-sets

DEFINITION 1.1. Let X' be a compactum lying in $M \in \mathrm{AR}$, X a subcompactum of X' and ε a positive number. A fundamental sequence $\mathbf{r} = \{r_k, X', X\}_{M,M}$ is said to be an *ε-fundamental retraction of X' into X (relatively to M)* if $r_k(x) \in X$ and $d(r_k(x), x) < \varepsilon$ for every $x \in X$ and $k = 1, 2, \ldots$

DEFINITION 1.2. Let X' be a compactum lying in $M \in \mathrm{AR}$ and X a closed subset of X'. X is a *fundamental approximative retract of X' (rel. M)* if for each positive number $\varepsilon > 0$ there is an ε-fundamental retraction of X' into X. (rel. M). The subset X is a *fundamental approximative neighborhood retract of X' (rel. M)* if for each positive number ε there is a closed neighborhood $U = U(\varepsilon)$ of X in X' and an ε-fundamental retraction of U into X (rel M.).

REMARK 1.1. It is easy to see that if X is a closed subset of a compactum X' lying in $M \in \mathrm{AR}$, then for every pair (Y', Y) lying in a space $N \in \mathrm{AR}$ and homeomorphic to (X', X), the set Y is a fundamental approximative retract (fundamental approximative neighborhood retract) of Y' (rel. N) if and only if X is a fundamental approximative retract (resp. fundamental approximative neighborhood retract) of X' (rel. M). It follows that the specification of the space $M \in \mathrm{AR}$ in the definition of a fundamental approximative (neighborhood) retract is superfluous and will be omitted. Since each compactum is

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homeomorphic to a subset of the Hilbert cube Q, we may limit our consideration to the closed subsets of Q.

DEFINITION 1.3. A compactum X is a *fundamental approximative absolute neighborhood retract* (or shortly an FAANR) provided that for every homeomorphism h mapping X onto a subset $Y = h(X)$ of another compactum Y', Y is a fundamental approximative neighborhood retract of Y'.

It is clear that the classes of FANR-sets and AANR_C-sets are contained in the class of FAANR-sets. However, Example 1.1 below is an FAANR-set but it is neither an AANR_C nor an FANR.

The next theorem gives an useful characterization of FAANR-sets.

THEOREM 1.1. *A compactum X is an* FAANR-*set if and only if it is homeomorphic to a fundamental approximative neighborhood retract of the Hilbert cube Q.*

Proof. We are going to prove the sufficiency of the condition, the necessity being obvious.

Suppose that X is embedded in a compactum X'. We must prove that X is a fundamental approximative neighborhood retract of X'. We can assume by Remark 1.1 that X' lies in Q.

Let h be a homeomorphism mapping X onto a fundamental approximative neighborhood retract Y of Q and $\hat{h}: Q \to Q$, $\widehat{h^{-1}}: Q \to Q$ continuous extensions of h and h^{-1} respectively. For a given $\varepsilon > 0$ let us take $\delta > 0$ such that $d(h^{-1}(x), h^{-1}(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Since Y is a fundamental approximative neighborhood retract of Q, there exists a δ-fundamental retraction $\mathbf{r}: V \to Y$, where V is a closed neighborhood of Y in Q. Consider a closed neighborhood U of X in X' such that $\hat{h}(U) \subset V$. Then $\mathbf{r}' = \{r'_k = \widehat{h^{-1}} \cdot r_k \cdot \hat{h}, k = 1, 2, \ldots\}$ is a fundamental sequence from U into X (rel. Q) and it is easy to see that $r'_k(x) \in X$ for every $x \in X$. Moreover, from the fact that $d(r_k h(x), h(x)) < \delta$ for every point $x \in X$ it follows that

$$d(r'_k(x), x) = d(h^{-1} r_k h(x), h^{-1} h(x)) < \varepsilon$$

for every $x \in X$. Then $\mathbf{r}'$ is an ε-fundamental retraction of U into X and we can conclude that X is a fundamental approximative neighborhood retract of X'.

It should be noted that when the dimension of X is finite, Q can be replaced in Theorem 1.1 by a Euclidean space of sufficiently high dimension.

EXAMPLE 1.1. It is easy to see that if we consider a sequence of disjoint Warsaw circles $X_1, X_2, \ldots$ lying in the plane E^2 and converging to a point a belonging to $E^2 - \bigcup X_i$, then the compactum $X = \{a\} \cup \bigcup X_i$ is an FAANR-set but X is neither an AANR_C nor an FANR.

Let $\mathscr{C}$ be a class of compacta. The class $\mathscr{C}$ approximately dominates the

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compactum X (see [6]) provided for every $\varepsilon>0$ there exists an $Y\in\mathscr{C}$ and there exist maps $f:X\to Y$, $g:Y\to X$ such that gf is ε-close to the identity (i.e. $d(gf(x),x)<\varepsilon$ for every point $x\in X$). If $\mathscr{C}$ is the class of FAANR-sets we have

THEOREM 1.2. *If the compactum X is approximately dominated by the class of* FAANR's, *then X is an* FAANR.

Proof. Let $\varepsilon>0$ be given. Then, we have an $Y\in$ FAANR and maps $f:X\to Y$, $g:Y\to X$ such that $d(gf(x),x)<\varepsilon/2$ for every $x\in X$. Take a $\delta>0$ such that $d(g(y),g(z))<\varepsilon/2$ whenever $d(y,z)<\delta$. Since $Y\in$ FAANR there is a closed neighborhood V of Y in Q and a δ-fundamental retraction $\mathbf{s}:V\to Y$. Extend f to a map $\hat{f}:U\to V$, where U is a closed neighborhood of X in Q and consider fundamental sequences $\hat{\mathbf{f}}$ and $\mathbf{g}$ generated by $\hat{f}$ and g respectively. Then $\mathbf{r}=\mathbf{g}\cdot\mathbf{s}\cdot\hat{\mathbf{f}}$ is a fundamental sequence from U into X and from the fact that $d(s_kf(x),f(x))<\delta$ for every $x\in X$ it follows that $d(gs_kf(x),gf(x))<\varepsilon/2$. Hence $d(r_k(x),x)=d(gs_kf(x),x)\le d(gs_kf(x),\ gf(x))+d(gf(x),x)<\varepsilon$ for every $x\in X$. Moreover, it is clear that $r_k(x)\in X$ if $x\in X$. Therefore $\mathbf{r}$ is an ε-fundamental retraction and we may conclude that $X\in$ FAANR.

Every closed approximative retract of an AANR_C is itself an AANR_C (see [8]). The following corollary of Theorem 1.2 provides a similar result for FAANR's.

COROLLARY 1.1. *Suppose X is a closed approximative retract of an* FAANR *X'. Then X is an* FAANR.

Proof. Let $\varepsilon>0$ be given. The composition $r_\varepsilon\cdot i$ of the inclusion $i:X\to X'$ and an ε-retraction ([8]) $r_\varepsilon:X'\to X$ is ε-close to the identity 1_X. By Theorem 1.2 $X\in$ FAANR.

It should be noted that, in general, it is not true that a fundamental retract of an FAANR is an FAANR (see Remark 2.1).

A map $f:X\to Y$ between compacta is approximately right invertible (ARI) (see [6]) provided there is a sequence of maps $h_n;Y\to X$ and a null sequence ε_n of positive reals such that fh_n is ε_n-close to 1_Y. The following corollary can be easily obtained from Theorem 1.2.

COROLLARY 1.2. *If $f:X\to Y$ is an* ARI *map of an* FAANR *X into Y, then Y is also an* FAANR.

The next corollary is similar to Clapp's Theorem 2.3 [8].

COROLLARY 1.3. *Let X be a compactum. Suppose that for each positive integer n, X contains a subset X_n such that X_n is an* FAANR *and there is a mapping $r_n:X\to X_n$ such that for $x\in X$ $d(r_n(x),x)<1/n$. Then X is an* FAANR.

Proof. Immediate from Theorem 1.2.

It follows from Corollary 1.3 that if we consider a sequence of pointed Warsaw circles (X_1, a), $(X_2, a)\cdots$ lying in the plane E^2, such that $X_n \cap X_m = \{a\}$ if $n \neq m$ and for every $\varepsilon > 0$ the condition diameter $(X_k) < \varepsilon$ for almost all k is satisfied, then $X = \bigcup X_k$ is an FAANR.

A fundamental sequence $\mathbf{f} = \{f_k, X, Y\}$ is said to be *internal* if $f_k(X) \subset Y$ for every k. The following theorem gives a necessary and sufficient condition for a compactum to be an FAANR (compare [8], Theorem 2.2).

THEOREM 1.3. *A compactum X is an* FAANR *if and only if it has the property that for any $\varepsilon > 0$, given an internal fundamental sequence $\mathbf{f} = \{f_k\}$ from a closed subset Y of a compactum Y' into X, there is a closed neighborhood $V = V(\varepsilon)$ of Y in Y' and a fundamental sequence $\hat{\mathbf{f}} = \{\hat{f}_k, V, x\}$ such that the restriction of $\hat{\mathbf{f}}$ to Y (i.e. the fundamental sequence $\{\hat{f}_k, Y, X\}$ is an internal fundamental sequence and $d(\hat{f}_k(y), f_k(y)) < \varepsilon$ for every $y \in Y$ and $k = 1, 2, \ldots$*

Proof. Let $\mathbf{f}: Y \to X$ be an internal fundamental sequence, where Y is a closed subset of the compactum Y'. If $X \in$ FAANR, then for $\varepsilon > 0$ given there is a neighborhood $U \in$ ANR of X in Q and an ε-fundamental retraction $\mathbf{r}: U \to X$. The fundamental sequence $\mathbf{j} \cdot \mathbf{f}$ obtained as composition of $\mathbf{f}$ and a fundamental sequence $\mathbf{j}: X \to U$ generated by the inclusion admits an extension $\mathbf{f}'$ to a closed neighborhood V of Y in Y' (see Borsuk [4] p. 261). It is immediate to verify that $\hat{\mathbf{f}} = \mathbf{r} \cdot \mathbf{f}'$ fulfils the required properties.

Conversely if X satisfies the condition of the theorem and $\varepsilon > 0$ is given, take $Y = X$, $Y' = Q$ and $\mathbf{f}$ generated by 1_X. Then $\hat{\mathbf{f}}$ is an ε-fundamental retraction and $X \in$ FAANR.

2. **Quasi-strong movability.** A compactum X lying in a space $M \in$ AR is said to be strongly movable in M (see Borsuk [4], p. 263) if for every neighborhood U of X in M there is a neighborhood U_0 of X in M such that for every neighborhood U' of X (in M) there is a homotopy $\phi: U_0 \times I \to U$ satisfying the following two conditions;

1) $\phi(x, 0) = x$ and $\phi(x, 1) \in U'$ for every point $x \in U_0$.

2) $\phi(x, 1) = x$ for every $x \in X$.

In [4], p. 264 Borsuk proves that FANR-spaces are the same as strongly movable compacta, Z. Čerin has recently introduced ([6] and [7]) the notions of $\mathscr{C}_p$-e-movability and $\mathscr{C}$-e-movability proving that a compactum X is an ANR (AANR_C) iff X is $\mathscr{C}_p$-e-movable (resp. $\mathscr{C}$-e-movable). We are now going to introduce a movability condition that we call quasi-strong-movability (shortly q-strong movability).

DEFINITION 2.1. Let X be a compactum lying in $M \in$ AR. X is *q-strongly movable* (*in M*) provided for every neighborhood U of X in M and for every

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$\varepsilon>0$ there exists a neighborhood U_0 of X in M such that for every neighborhood U' of X in M there is a homotopy $\phi: U_0 \times I \to U$ such that

1) $\phi(x, 0)=x$ and $\phi(x, 1) \in U'$ for every point $x \in U_0$
2) $\phi(x, 1) \in X$ and $d(\phi(x, 1), x)<\varepsilon$ for every $x \in X$.

In the conditions of Definition 2.1 we say that U_0 realizes the ε-strong movability of X in U.

It can be easily proved that q-strong movability has an absolute character, not depending on the space $M \in \mathrm{AR}$ in which X is embedded.

The next theorem gives another characterization of FAANR-sets. All compacta are assumed to lie in Q.

THEOREM 2.1. *A metric compactum X is an* FAANR-*set if and only if it is q-strongly movable.*

Proof. Let us suppose first that $X \in \mathrm{FAANR}$ and consider an $\varepsilon>0$ and $U \in \mathrm{ANR}$, closed neighborhood of X in Q. There exists a $\delta>0$ such that any two δ-near maps $f, g: A \to U$ defined on an arbitrary space are homotopic ([14]). Since $X \in \mathrm{FAANR}$, there is an ε'-fundamental retraction $\mathbf{r}=\{r_k, W_0, X\}$, where $\varepsilon'=\min\{\varepsilon, \delta\}$ and W_0 is a closed neighborhood of X in Q. Then

$$(1) \qquad r_{k_0|W_0} \simeq r_{k|W_0} \quad \text{in } U \text{ for } \quad k \geq k_0.$$

Since $r_{k_0|X}$ and the identity 1_X are ε'-near (and hence homotopic) there exists a closed neighborhood $U_0 \subset W_0$ of X in Q such that

$$(2) \qquad r_{k_0|U_0} \simeq 1_{U_0} \quad \text{in } U.$$

Now let U' be an arbitrary neighborhood of X in Q. Since $\mathbf{r}$ is a fundamental sequence there is a $k \geq k_0$ such that $r_k(U_0) \subset U'$ and it follows from (1), (2) and the fact that $\mathbf{r}$ is an ε'-fundamental retraction that

$$r_{k|U_0} \simeq r_{k_0|U_0} \simeq 1_{U_0} \quad \text{in } U, \quad r_k(X) \subset X \quad \text{and} \quad d(r_k(x), x)<\varepsilon' \leq \varepsilon$$

for every $x \in X$. In consequence, X is q-strongly movable.

Now let us prove the sufficiency of the movability condition. Let $\varepsilon>0$ be given and take a null sequence (ε_n) of positive numbers such that $\sum \varepsilon_n \leq \varepsilon$. Let us consider a decreasing sequence of neighborhoods of X in Q

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$$

such that

a) (V_i) is a base of neighborhoods of X in Q.
b) V_{n+1} realizes the ε_n-strong movability of X in V_n.

It follows that for every $i=1,2, \ldots$ there exists a homotopy

$$\phi_i: V_{i+1} \times I \to V_i$$

such that

$$\phi_i(x, 0) = x \quad \text{and} \quad \phi_i(x, 1) \in V_{i+2} \quad \text{for every } x \in V_{i+1},$$

$$\phi_i(x, 1) \in X \quad \text{and} \quad d(\phi_i(x, 1), x) < \varepsilon_i \quad \text{for every } x \in X.$$

Setting $r'_1(x) = \phi_1(x, 1)$ we get a map $r'_1: V_2 \to V_1$ such that $\operatorname{im} r'_1 \subset V_3$, $r'_1(X) \subset X$ and $d(r'_1(x), x) < \varepsilon_1$ for every $x \in X$.

Assume now that for an index k a map $r'_k: V_2 \to V_k$ is defined such that

$$\text{(3)} \qquad \operatorname{im} r'_k \subset V_{k+2}, \qquad r'_k(X) \subset X, \qquad r'_k \simeq r'_{k-1} \text{ in } V_k \quad (\text{for } k \geq 2)$$

and

$$d(r'_k(x), x) < \varepsilon_1 + \cdots + \varepsilon_k < \varepsilon \quad \text{for every point } x \in X.$$

Then $\phi_{k+1}(r'_k(x), 1)$ is defined on the set V_2 and setting

$$r'_{k+1}(x) = \phi_{k+1}(r'_k(x), 1)$$

we get a map $r'_{k+1}: V_2 \to V_{k+1}$ such that $\operatorname{im} r'_{k+1} \subset V_{k+3}$ and $r'_{k+1}(X) \subset X$. Moreover,

$$\begin{aligned} d(r'_{k+1}(x), x) &\leq d(r'_{k+1}(x), r'_k(x)) + d(r'_k(x), x) \\ &= d(\phi_{k+1}(r'_k(x), 1), r'_k(x)) + d(r'_k(x), x) < \varepsilon_1 + \cdots + \varepsilon_{k+1} \end{aligned}$$

for every $x \in X$ and the homotopy $\Phi: V_2 \times I \to V_{k+1}$ defined as composition of $r'_k \times id: V_2 \times I \to V_{k+2} \times I$ and $\phi_{k+1}: V_{k+2} \times I \to V_{k+1}$ joins in V_{k+1} the maps r'_k and r'_{k+1}.

Thus we get a sequence of maps $r'_k: V_2 \to V_k$, satisfying conditions (3). It follows that if we take arbitrary continuous extensions $r_k: Q \to Q$ of the maps r'_k and a closed neighborhood U of X contained in the interior of V_2 we get a fundamental sequence

$$\mathbf{r} = \{r_k, U, X\}$$

which is an ε-fundamental retraction. This completes the proof of the theorem.

In [9], p. 178, C. Cox gives an example of a compactum obtained as union of two movable compacta A and B, having one point in common, which is not movable. It is easy to verify that A and B are AANR_C-sets (see Bogatyĭ, [1], Remark 3). Therefore, it follows from Theorem 2.1 that, in general, it is not true that if $X, Y, X \cap Y \in \text{FAANR}$, then $X \cup Y \in \text{FAANR}$.

REMARK 2.1. Quasi-strong movability fails to be a shape invariant. Consider in the plane the circle S_n with center 0 and radius $1 - 1/n$ for $n = 2, 3, \ldots$. Let A_n be the annulus determined by S_n and S_{n+1} and let $L_n \subset \operatorname{Int} A_n$ be a homeomorphic copy of R, which spirals to both S_n and S_{n+1}. Consider also the circle S with center 0 and radius 1 and the segment $T = \{(x, 0) \mid -2 \leq x \leq -1\}$. Then, the continuum $X = T \cup S \cup \bigcup S_n \cup \bigcup L_n$ is not quasi-strongly movable (compare with Mardešić's Example 2, [17]). However, X decomposes the plane in infinitely many components and, in consequence, it is shape equivalent to the bouquet of Warsaw circles described after Corollary 1.3, which is an FAANR.

According to Bogatyĭ ([1], Addendum) every movable compactum is a fundamental retract of an $AANR_C$. Since X is movable (see [3], p. 79) we conclude that it is not always true that a fundamental retract of an FAANR is an FAANR.

3. **A characteristic property of FANR's.** The definition of an FAANR-set given in Section 1 is parallel to Clapp's definition of an AANR-set. A notion of FAANR in the sense of Noguchi could also be considered. However, from Theorem 3.1 below it follows that this notion would be equivalent to Borsuk's notion of FANR-set. This theorem also implies that a possible notion of Fundamental Approximative Absolute Retract (FAAR-set) would be coincident with the well-known concept of Fundamental Absolute Retract (FAR-set).

DEFINITION 3.1. Let X be a closed subset of a compactum X'. X is said to be a *fundamental approximative neighborhood retract of X' in the sense of Noguchi* if there exists a closed neighborhood U of X in X' such that X is a fundamental approximative retract of U.

REMARK 3.1. Similarly to the considerations in Remark 1.1 we do not need to mention the space $M \in \mathrm{AR}$ where X' is supposed to lie. No generality is lost to assume $M = Q$.

THEOREM 3.1. *A compactum X is an* FAR-*space if and only if for every compactum X' containing X, the set X is a fundamental approximative retract of X'. X is an* FANR-*space if and only if for every compactum X' containing X, the set X is a fundamental approximative neighborhood retract of X' in the sense of Noguchi.*

Proof. If X is a fundamental approximative retract of every compactum containing it then, in particular, X is a fundamental approximative retract of the Hilbert cube Q.

Let U be a closed neighborhood of X in Q. We can assume that $U \in \mathrm{ANR}$, so there exists an $\varepsilon > 0$ such that any two ε-near maps $f, g : A \to U$ are homotopic. Consider an ε-fundamental retraction $\mathbf{r} = \{r_k, Q, X\}$ and take an index k such that $r_k(Q) \subset U$. Since $r_{k|X}$ and the identity 1_X are ε-near they are homotopic and by the homotopy extension theorem there exists a map $r : Q \to U$ such that $r_{|X} = 1_X$. Then, by Theorem 27.1, p. 110 of [3] X is a Fundamental Absolute Retract. The converse is obvious.

Suppose now that for every compactum X' containing X, the set X is a fundamental approximative neighborhood retract of X' in the sense of

Noguchi. Then, $X \in$ FAANR and it follows from Theorem 2.1 that X is movable. Let us see that X is an Absolute Weak Neighborhood Retract (AWNR, see Bogatyĭ [1] p. 97); then by Bogatyĭ's theorem stating that a movable AWNR is an FANR we will get the result. Since X is a fundamental approximative neighborhood retract of Q in the sense of Noguchi it is a δ-fundamental retract of a fixed neighborhood U_0 of X in Q for every $\delta > 0$. Let $U \in$ ANR be a closed neighborhood of X in Q and $\varepsilon > 0$ as before. There exists an ε-fundamental retraction $\mathbf{r} = \{r_k, U_0, X\}$ of U_0 into X. Take an index k such that $r_k(U_0) \subset U$. Then $r_{k|X}$ and 1_X are ε-near and, hence, homotopic in U. By the homotopy extension theorem there exists a map $r: U_0 \to U$ such that $r_{|X} = 1_X$. Therefore X is an AWNR-set and the proof is complete.

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