A FAMILY OF CONFORMALLY ASYMMETRIC RIEMANN SURFACES

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Abstract. We give explicit examples of asymmetric Riemann surfaces (that is, Riemann surfaces with trivial conformal automorphism group) for all genera $g \geq 3$. The technique uses Schreier coset diagrams to construct torsion-free subgroups in groups of signature $(0; 2, 3, r)$ for certain values of r.

1. Introduction. Suppose Γ is a co-compact Fuchsian group acting properly discontinuously on the hyperbolic plane $\mathbb{H}^2$. The group Γ then has a presentation

$$\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_s \mid x_1^{m_1} = \ldots = x_s^{m_s} = \prod_{i=1}^{s} x_i \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle,$$

which is encapsulated by its signature $(g; m_1, m_2, \ldots, m_s)$, where g is the genus of the surface $\mathbb{H}^2/\Gamma$ and $m_1, m_2, \ldots, m_s$ are the orders of the maximal finite subgroups in Γ. In the case where $g = 0$ and $s = 3$, Γ is called a *triangle group*. The hyperbolic area of a fundamental region for the action of Γ on $\mathbb{H}^2$ is proportional to

$$\mu(\Gamma) = 2g - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right),$$

where, necessarily, $\mu(\Gamma) > 0$.

If Γ is torsion-free, so that it acts freely on $\mathbb{H}^2$ and has signature $(g; —)$, then $S = \mathbb{H}^2/\Gamma$ is a Riemann surface of genus $g \geq 2$. Its group of conformal automorphisms $\mathrm{Aut}(S)$ is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normaliser of Γ in $PSL(2, \mathbb{R})$, the orientation preserving conformal isometry group of the hyperbolic plane. In this paper we construct Riemann surfaces S for all genera $g \geq 3$ where $\mathrm{Aut}(S)$ is trivial, by searching for appropriate Γ amongst the subgroups of certain triangle groups. It is worth recalling that any genus 2 surface, being hyperelliptic, admits a conformal involution. In [**2**], an infinite family of such surfaces is constructed, although the set of genera not accounted for is also infinite.

2. Coset diagrams for triangle groups. Consider the triangle group $\Delta = \Delta(p, q, r)$ having signature $(0; p, q, r)$ and presentation

$$\langle x_1, x_2, x_3 \mid x_1^p = x_2^q = x_3^r = x_1 x_2 x_3 = 1 \rangle \cong \langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle.$$

If Γ is a subgroup of finite index n in Δ then the action of the triangle group on the right cosets of Γ induces a homomorphism $\theta : \Delta \to S_n = \mathrm{Sym}\{1, 2, \ldots, n\}$. By theorem 1 of [**3**], Γ will be torsion-free when $\theta(x)$, $\theta(y)$ and $\theta(xy)$ are *regular* permutations—that is, they are composed entirely of p-cycles, q-cycles and r-cycles respectively. At this stage, we just insist that $\theta(y)$ is regular. We then depict this representation using a (slightly simplified)

Schreier coset diagram D: for each q-cycle in $\theta(y)$ draw a shaded q-gon, labelling its vertices in an anticlockwise direction with the points from the cycle; for each point $i \in \{1, 2, \ldots, n\}$ and its image j under $\theta(x)$, draw a directed arc running from i to j (except for those i fixed by $\theta(x)$ which are left unconnected by arcs).

On the other hand, suppose D is some diagram composed of n vertices grouped into shaded q-gons, and such that each vertex is incident with either no directed arcs or precisely two—one incoming and the other outgoing. The arcs and q-gons induce permutations $\alpha, \beta \in S_n$ according to the above scheme. These permutations are unique up to relabelling of the vertices and satisfy $\alpha^p = \beta^q = (\alpha\beta)^r = 1$, where p and r are the orders of the permutations α and $\alpha\beta$. Hence we define a homomorphism $\theta: \Delta = \Delta(p, q, r) \to S_n$ by $\theta(x) = \alpha$ and $\theta(y) = \beta$, and refer to D as a diagram *for* Δ. If $k \in \{1, 2, \ldots, n\}$, and G_k is the stabiliser in $G = \langle \theta(x), \theta(y) \rangle$ of k, then $\theta^{-1}(G_k)$ is a subgroup Γ_k of index n in Δ. For different vertex labellings and different k the subgroups Γ_k are conjugate in Δ, so without loss of generality we simply refer to the subgroup Γ arising from D. By construction, the permutation $\theta(y)$ is regular, and if $\theta(x)$ and $\theta(xy)$ are also regular, then Γ is torsion-free (again by [**3**]).

The relations $x^p = y^q = (xy)^r = 1$ induce a natural embedding of a diagram D into an orientable surface by defining the faces of the embedding to be the cycles of $\theta(x)$, $\theta(y)$ and $\theta(xy)$, ignoring repetitions. By Theorem 2 of [**4**], the genus of this embedding coincides with the genus of $S = \mathbb{H}^2/\Gamma$, where Γ is the group arising from D. We can thus compute the genus of this surface either by using the Riemann-Hurwitz formula $n\mu(\Delta) = \mu(\Gamma)$ (where n is the number of vertices of D and hence the index of Γ in Δ) or by taking an Euler count on the vertices, q-gonal sides, arcs and faces of D. Note that taking a face of the third kind—that is, one that is unshaded and not the interior of an x-cycle—and counting the number of q-gonal sides adjacent to it gives the length of a cycle of $\theta(xy)$. Hence, for the group Γ arising from D to be torsion-free, this count must be r for each such face of the embedding.

A family of examples, D'_m, $m \geq 4$, on $12m$ vertices and where the embedding is planar is illustrated in Figure 1—in fact the diagrams we shall use for our construction (where for transpositions in $\theta_m(x)$ we have replaced the two arcs running between a pair of vertices by a single undirected edge). Counting the number of triangular sides adjacent to the two unshaded faces we see that $\theta_m(xy)$ is composed of two cycles of length $6m$, $m \geq 4$. Clearly $\theta_m(x)$ and $\theta_m(y)$ have orders two and three respectively, so that the group Γ_m arising from D_m has index $12m$ in the triangle group $\Delta(2, 3, 6m)$, $m \geq 4$. Note that Γ_m is not torsion-free in this case, having signature $(0; 2^{4m})$.

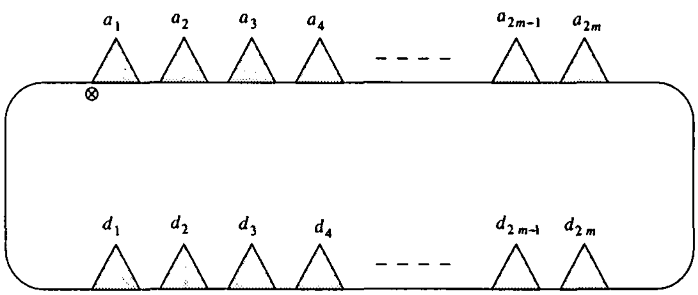

Figure 1.

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An *automorphism* of a coset diagram D for some Δ is a permutation $\sigma \in S_n$ of its vertices such that $\sigma\theta(x) = \theta(x)\sigma$ and $\sigma\theta(y) = \theta(y)\sigma$. The automorphisms form a group under composition, $\mathrm{Aut}(D)$, which is then the centraliser in S_n of $G = \langle\theta(x), \theta(y)\rangle$. Observe that σ is uniquely determined by its effect on a vertex of D, so that in particular, if σ fixes a vertex it must be the identity permutation. By Theorem 2.5 of [**6**] and Theorem 1 of [**4**], $\mathrm{Aut}(D)$ is isomorphic to $N_\Delta(\Gamma)/\Gamma$, where $N_\Delta(\Gamma)$ is the normaliser in Δ of the group Γ arising from D. Hence, $\mathrm{Aut}(D)$ acts as a group of conformal automorphisms of the Riemann surface $S = \mathbb{H}^2/\Gamma$, but of course in general, it will be a subgroup of the full automorphism group $\mathrm{Aut}(S)$. We are interested in when the two coincide.

Suppose that Δ is *non-arithmetic* and *maximal*, in the sense that it is not properly contained in any triangle group. If Γ is a subgroup of finite index in Δ and $g \in N(\Gamma)$, the normaliser in $PSL(2,\mathbb{R})$ of Γ, then g is contained in the commensurator of Γ, $\mathrm{Comm}(\Gamma) = \{g \in PGL(2,\mathbb{R}) \mid \Gamma, g^{-1}\Gamma g \text{ commensurable}\}$—where two Fuchsian groups are commensurable if their intersection has finite index in both. Since $\mathrm{Comm}(\Gamma)$ is contained in $\mathrm{Comm}(\Delta)$, we have $g \in \mathrm{Comm}(\Delta)$. By a theorem of Margulis [**1**], and since Δ is non-arithmetic, $\mathrm{Comm}(\Delta)$ contains Δ as a subgroup of finite index. An elementary area calculation using the Riemann-Hurwitz formula gives that $\mathrm{Comm}(\Delta)$ is also a triangle group, and hence by maximality $\mathrm{Comm}(\Delta) = \Delta$. Thus, $g \in \Delta$ and so $N(\Gamma)/\Gamma = N_\Delta(\Gamma)/\Gamma$, that is, $\mathrm{Aut}(S) \cong \mathrm{Aut}(D)$.

Finally, we describe a method for combining diagrams to obtain new ones. Suppose D_1 and D_2 are diagrams for the same triangle group $\Delta(p,q,r)$, where the groups Γ_1 and Γ_2 arising from them have index n_1 and n_2 in Δ respectively. A t-handle $[u,v]_t$ is a pair of vertices u and v such that $\theta_i(x){:}u \to u$, $\theta_i(x){:}v \to v$ and $\theta_i(xy)^t{:}u \to v$, $i = 1$ or 2. Suppose that between them, the diagrams D_1 and D_2 contain the p t-handles $[u_1,v_1]_t$, $[u_2,v_2]_t, \ldots, [u_p,v_p]_t$. The *composition* of D_1 and D_2 is then the diagram D obtained by connecting D_1 and D_2 with the two p-cycles $(u_1,u_2,\ldots,u_p)$ and $(v_1,v_p,\ldots,v_2)$: that is, drawing directed arcs from vertex u_1 to u_2, u_2 to $u_3,\ldots,u_p$ to u_1, v_1 to v_p, v_p to $v_{p-1},\ldots$, and v_2 to v_1.

Clearly, the permutations α and β arising from D are elements of $S_{n_1+n_2}$, with β composed entirely of q-cycles and α having order p. A cycle of $\alpha\beta$ that does not pass through any of the vertices u_i and v_i is unaffected by composition (up to the obvious relabelling of the vertices of one of the diagrams). If $(u_i, w_{i,1}, w_{i,2}, \ldots, w_{i,t-1}, v_i, w_{i,t+1}, \ldots)$ and $(u_{i+1}, w_{i+1,1}, w_{i+1,2}, \ldots, w_{i+1,t-1}, v_{i+1}, w_{i+1,t+1}, \ldots)$ are the cycles in $\alpha\beta$ that pass through the handles $[u_i, v_i]$ and $[u_{i+1}, v_{i+1}]$ respectively *before* composition, then after, the first becomes $(u_i, w_{i+1,1}, w_{i+1,2}, \ldots, w_{i+1,t-1}, v_i, w_{i,t+1}, \ldots)$, again, up to relabelling. Thus, for all i, the lengths of these cycles is unchanged, and so $\alpha\beta$ has order r. We define θ in the usual way, giving that D is also a diagram for $\Delta(p,q,r)$ with the resulting group Γ having index $n_1 + n_2$ in Δ.

3. The construction. Consider the diagrams D_m for $\Delta(2,3,6m)$, $m \geq 4$, illustrated in Figure 1. By [**5**], the triangle group $\Delta(2,3,r)$ is arithmetic exactly when $r = 7, 8, 9, 10, 11, 12, 14$ and 18, and a simple area calculation gives that $\Delta(2,3,r)$ is maximal for all r. We observed in the previous section that $\theta_m(xy)$ is composed of two cycles of length $6m$ while $\theta_m(y)$ is obviously composed of $4m$ 3-cycles. Thus, both $\theta_m(y)$ and $\theta_m(xy)$ are regular permutations.

The vertices $a_1, a_2, \ldots, a_{2m}$ and $d_1, d_2, \ldots, d_{2m}$ can be grouped into the 2-handles

$[a_1, a_2]_2$, $[a_3, a_4]_2, \ldots, [a_{2m-1}, a_{2m}]_2$, $[d_1, d_2]_2$, $[d_3, d_4]_2, \ldots, [d_{2m-1}, d_{2m}]_2$. Since $\theta_m(x)$ has order two in this case, a single composition requires just two of these 2-handles. Perform $2m$ such compositions on D_m by joining the 2-handle $[a_i, a_{i+1}]_2$ to the 2-handle $[d_{i+2}, d_{i+3}]_2$, where subscripts are taken modulo $2m$. Alternatively, we achieve the same result by considering the handles $[a_1, a_{2m-2}]_{4m-6}$, $[a_2, a_{2m-3}]_{4m-10}, \ldots, [a_{m-1}, a_m]_2$ and joining them to the handles $[d_3, d_{2m}]_{4m-6}$, $[d_4, d_{2m-1}]_{4m-10}, \ldots, [d_{m+1}, d_{m+2}]_2$, respectively; (this observation will become useful shortly). Call the resulting diagrams D'_m. Notice that after these compositions, the resulting $\theta'(x)$ is composed entirely of transpositions, and so is a regular permutation. Thus, the groups Γ'_m, $m \geq 4$, arising from the D'_m are torsion-free in $\Delta(2, 3, 6m)$. These diagrams have $12m$ vertices, $12m$ triangular sides, $6m$ edges, $4m$ shaded faces and 2 unshaded faces. An Euler count thus gives $2 - 2g = 12m - 12m - 6m + 4m + 2 = -2m + 2 \Rightarrow g = m$, $m \geq 4$.

Now, chasing a path around the diagram through the handles $[a_1, a_{2m-2}]_{4m-6}$ and $[d_3, d_{2m}]_{4m-6}$, we see that the element $\theta'((xy)^{4m-6}(xy^2)^{4m-6})$ fixes the vertices a_{2m-2} and d_{2m}. Moreover, a careful but routine examination of the cycles in $\theta'(xy)$ and $\theta'(xy^2)$ reveals that these are the only two vertices fixed by this element (essentially since $4m - 6$ is the largest t for which D_m contains a t-handle, the handles $[a_1, a_{2m-2}]_{4m-6}$ and $[d_3, d_{2m}]_{4m-6}$ being the only two such examples.) Thus, since any $\sigma \in \mathrm{Aut}(D'_m)$ centralises $\theta'((xy)^{4m-6}(xy^2)^{4m-6})$, it must either stabilise both a_{2m-2} and d_{2m}, or transpose them. Notice that the vertex in D'_m marked $\otimes$ and the vertex d_1 are the images of a_{2m-2} and d_{2m} respectively under $\theta'(y^2xy^2xyx)$. Hence if (a_{2m-2}, d_{2m}) is a transposition in σ, then so is $(\otimes, d_1)$. But (a_{2m-2}, d_{2m}) is a transposition in $\theta'(x)$ while $(\otimes, d_1)$ is not, and so σ does not commute with $\theta'(x)$, giving that σ must fix both a_{2m-2} and d_{2m}, and is the identity. Thus, $\mathrm{Aut}(D'_m) \cong \mathrm{Aut}(\mathbb{H}^2/\Gamma'_m)$ is trivial.

Finally (and somewhat unfortunately) the case $g = 3$ is handled separately using the diagram D in Figure 2. Chasing the action of $\theta(xy)$, it is easy to see that D is a diagram for $\Delta(2, 6, 9)$ where $\theta(y)$ and $\theta(xy)$ are regular permutations. Also, $\Delta(2, 6, 9)$ is non-arithmetic ([5]) and maximal. Composing the 2-handles $[a_1, a_2]_2$ and $[a_3, a_4]_2$ as well as the pair $[a_5, a_6]_2$ and $[a_7, a_8]_2$ yields D', in which the resulting $\theta'(x)$ is also regular. An Euler count gives that $\mathbb{H}^2/\Gamma'$ has genus 3, and a simple calculation using the group theory package MAGMA evaluates the order of the centraliser in S_{18} of $\langle \theta'(x), \theta'(y) \rangle$ (being trivial).

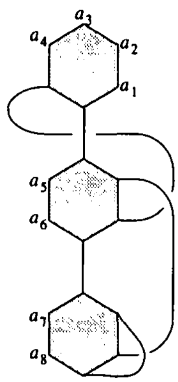

Figure 2.

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