On converse duality in
complex nonlinear programming

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In this note a converse duality theorem is proved for a class of
nonlinear programming problems over polyhedral cones in finite
dimensional complex space by a direct use of a Kuhn-Tucker type
necessary and sufficient condition for constrained optimization
in complex space.

0. Introduction

Abrams and Ben-Israel [2] and Abrams [1] have established a converse
duality theorem for nonlinear programming in complex space. For the proof
they have assumed the existence of an auxiliary analytic function. Craven
and Mond [5] have given a converse duality theorem for complex space
through a method of proof that depends on mapping a complex minimization
problem onto an equivalent minimization problem in real space. In [2],
Abrams and Ben-Israel have obtained a necessary condition of the Kuhn-
Tucker type for a class of nonlinear programming problems in complex space.
In the present paper, we have modified their result so as to be applicable
to the case where there may occur equality constraints in the constraint
set and then used this modified result to prove a converse duality theorem
for complex space.

1. Notations and preliminaries

Let C^n (R^n) denote the n-dimensional complex (real) space, with
hermitian (euclidean) norm. R_+ denotes the half line $[0, \infty)$. If A is

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a complex matrix or vector, then A^T, $\overline{A}$ and A^H denote its transpose, complex conjugate, and conjugate transpose. $S \subset C^m$ is a polyhedral cone if it is finite intersection of closed half-spaces in C^m, each containing 0 on its boundary. The polar S^* of S is defined by

$$S^* = \{y \in C^m : x \in S \Rightarrow \text{Re } y^H x \geq 0\} .$$

We shall make use of the following [1], [2], [5]: if S and T are polyhedral cones in C^m, then

$$(S \times T)^* = S^* \times T^* ,$$

$S \times T$ is a polyhedral cone.

If $S = C^m$, then $S^* = \{0\}$.

Define the linear manifold $Q \subset C^{2n} = C^n \times C^n$ by

$$Q = \{(w^1, w^2) \in C^{2n} : w^2 = \overline{w^1}\} .$$

The analytic function $f : Q \to C$ has a convex real part with respect to R_+ on Q if for all z^1, $z^2 \in Q$,

$$\text{Re}\left[f(z^1, \overline{z^1}) - f(z^2, \overline{z^2}) - \nabla_z^T f(z^2, \overline{z^2})(z^1 - z^2) - \nabla_{\overline{z}}^T f(z^2, \overline{z^2})(\overline{z^1} - \overline{z^2})\right] \geq 0 ,$$

where $\nabla_z f(z^2, \overline{z^2})$ and $\nabla_{\overline{z}} f(z^2, \overline{z^2})$ denote, respectively, the column vectors of partial derivatives

$$\frac{\partial f(z^2, \overline{z^2})}{\partial w_i^1} \quad \text{and} \quad \frac{\partial f(z^2, \overline{z^2})}{\partial w_i^2} .$$

The analytic function $g : Q \to C^m$ is concave with respect to S on Q if for all z^1, $z^2 \in Q$,

$$D_z g(z^2, \overline{z^2})(z^1 - z^2) + D_{\overline{z}} g(z^2, \overline{z^2})(\overline{z^1} - \overline{z^2}) + g(z^2, \overline{z^2}) - g(z^1, \overline{z^1}) \in S$$

where $D_z g(z^2, \overline{z^2})$ and $D_{\overline{z}} g(z^2, \overline{z^2})$ denote, respectively, the $m \times m$ matrices whose i, j-th elements are

$$\frac{\partial g_i\left(z^2, \overline{z^2}\right)}{\partial w_j^1} \quad \text{and} \quad \frac{\partial g_i\left(z^2, \overline{z^2}\right)}{\partial w_j^2} \; .$$

If $f : Q \to C$, $f^R : Q \to R$, and $f^I : Q \to R$ are analytic on Q , and $f(z, \overline{z}) = f^R(z, \overline{z}) + if^I(z, \overline{z})$ then

(A) $$\overline{\nabla_z f(z, \overline{z})} + \nabla_{\overline{z}} f(z, \overline{z}) = 2\nabla_{\overline{z}} f^R(z, \overline{z}) \; .$$

If $f : Q \to C$ is an analytic function, then (see [5], p. 619)

(B) $$\overline{\left(\frac{\partial f}{\partial w_i^1}\right)} = \frac{\partial \overline{f}}{\partial w_i^2} \quad \text{and} \quad \overline{\left(\frac{\partial f}{\partial w_i^2}\right)} = \frac{\partial \overline{f}}{\partial w_i^1} \; .$$

2.   Complex version of the Kuhn-Tucker conditions

We shall make use of the following results of Abrams and Ben-Israel [2].

THEOREM 2.1. *Let* (P) *denote the problem*

(P)   MINIMIZE   Re $f(z, \overline{z})$

                        SUBJECT TO $g(z, \overline{z}) \in S$

where S is a polyhedral cone in C^m , and $f : Q \to C$, $g : Q \to C^m$ are analytic in a neighborhood of a qualified point $\left(z^0, \overline{z^0}\right)$. A necessary condition for $\left(z^0, \overline{z^0}\right)$ to be optimal is that there exists a $u \in S^*$ such that

(2.1)   Re $u^H g\left(z^0, \overline{z^0}\right) = 0$

and

(2.2) $$\overline{\nabla_z f\left(z^0, \overline{z^0}\right)} + \nabla_{\overline{z}} f\left(z^0, \overline{z^0}\right) = D_z^H g\left(z^0, \overline{z^0}\right) u + D_{\overline{z}}^T g\left(z^0, \overline{z^0}\right) \overline{u} \; .$$

REMARKS 2.2. A point $\left(z^0, \overline{z^0}\right)$ qualified means it is a feasible point of (P) and the (Kuhn-Tucker) constraint qualification defined in [3] holds at this point.

If $f : Q \to C$ is analytic and has convex real part with respect to R_+ on Q and $g : Q \to C^m$ is analytic and concave with respect to S on

Q , then in Theorem 2.1 the necessary condition is also sufficient [1].

We want to modify Theorem 2.1 so as to include equality, as well as inequality, constraints.

COROLLARY 2.3. *If* (P) *itself is of the form*

$$\text{MINIMIZE} \quad \text{Re } f(z, \overline{z}, w, \overline{w})$$
$$\text{SUBJECT TO} \quad G(z, \overline{z}, w, \overline{w}) \in S ,$$

(2.3) $$H(z, \overline{z}, w, \overline{w}) = 0 , \quad w \in T ,$$

then a necessary condition for $\zeta = \left(z^0, \overline{z^0}, w^0, \overline{w^0}\right)$ *to be optimal is* $u \in S^*$,

(2.4) $\overline{\nabla_w f(\zeta)} + \nabla_{\overline{w}} f(\zeta) - D_w^H G(\zeta)u - D_{\overline{w}}^T G(\zeta)\overline{u} - D_w^H H(\zeta)v - D_{\overline{w}}^T H(\zeta)\overline{v} \in T^*$,

(2.5) $$\text{Re } u^H G(\zeta) = 0 ,$$

(2.6) $\text{Re}\left[\nabla_w^T f(\zeta) + \nabla_{\overline{w}}^H f(\zeta) - u^H D_w G(\zeta) - u^T \overline{D_{\overline{w}} G(\zeta)} - v^H D_w H(\zeta) - v^T \overline{D_{\overline{w}} H(\zeta)}\right] w^0 = 0$,

and

(2.7) $\overline{\nabla_z f(\zeta)} + \nabla_{\overline{z}} f(\zeta) = D_z^H G(\zeta)u + D_{\overline{z}}^T G(\zeta)\overline{u} + D_z^H H(\zeta)v + D_{\overline{z}}^T H(\zeta)\overline{v}$.

Proof. The constraint set (2.3) can be written as

$$\begin{Bmatrix} G(z, \overline{z}, w, \overline{w}) \\ H(z, \overline{z}, w, \overline{w}) \\ w \end{Bmatrix} \in S \times \{0\} \times T .$$

An easy application of Theorem 2.1 then gives the required result.

3. Converse duality

Assume $f(z, \overline{z})$ and $g(z, \overline{z})$ in (P) satisfy the hypothesis of Theorem 2.1. A dual of (P) in the sense of Wolfe [7] is

(D) MAXIMIZE $\text{Re } f(x, \overline{x}) - \text{Re } y^H g(x, \overline{x})$

SUBJECT TO $\psi(x, \overline{x}, y, \overline{y}) = \overline{\nabla_z f(x, \overline{x})} + \nabla_{\overline{z}} f(x, \overline{x}) - D_z^H g(x, \overline{x})y$
$$- D_{\overline{z}}^T g(x, \overline{x})\overline{y} = 0 , \quad y \in S^* .$$

THEOREM 3.1. *Let* $f : Q \to C$ *be analytic and have convex real part*

with respect to R_+ on Q, let S be a polyhedral cone in C^m, and let $g : Q \rightarrow C^m$ be analytic and concave with respect to S on Q. If $\left(z^0, \overline{z^0}, \overline{u^0}\right)$ is an optimal solution of (D) *such that the matrix*

(3.1)
$$\begin{bmatrix} D_{\underline{z}}^H \psi(x, \overline{x}, y, \overline{y}) & D_z^T \psi(x, \overline{x}, y, \overline{y}) \\ \\ D_{\underline{z}}^H \psi(x, \overline{x}, y, \overline{y}) & D_{\overline{z}}^T \psi(x, \overline{x}, y, \overline{y}) \end{bmatrix}$$

is nonsingular at $\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right)$, *then* $\left(z^0, \overline{z^0}\right)$ *is an optimal solution of* (P).

Proof. We shall suppose that $\left(z^0, \overline{z^0}, \overline{u^0}\right)$ is a qualified point relative to the feasible region of (D). So $\left(z^0, \overline{z^0}, \overline{u^0}\right)$ is a feasible point of (D) and we have

(3.2) $\nabla_z f\left(z^0, \overline{z^0}\right) + \nabla_{\overline{z}} f\left(z^0, \overline{z^0}\right) - D_z^H g\left(z^0, \overline{z^0}\right) u^0 - D_{\overline{z}}^T g\left(z^0, \overline{z^0}\right) \overline{u^0} = 0$

and

(3.3) $u^0 \in S^*$.

Utilising Corollary 2.3, we get

(3.4) $g\left(z^0, \overline{z^0}\right) + D_z g\left(z^0, \overline{z^0}\right) v + D_{\overline{z}} g\left(z^0, \overline{z^0}\right) \overline{v} \in S$,

(3.5)   Re $u^{0^H} \left\{ g\left(z^0, \overline{z^0}\right) + D_z g\left(z^0, \overline{z^0}\right) v + D_{\overline{z}} g\left(z^0, \overline{z^0}\right) \overline{v} \right\} = 0$,

and

(3.6) $D_z^H \psi\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right) v + D_{\overline{z}}^T \psi\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right) \overline{v} = 0$.

Combining (3.6) with its complex conjugate, we obtain

(3.7)
$$\begin{bmatrix} D_{\underline{z}}^H \psi\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right) & D_z^T \psi\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right) \\ \\ D_{\underline{z}}^H \psi\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right) & D_{\overline{z}}^T \psi\left(z^0, \overline{z^0}, u^0, \overline{u^0}\right) \end{bmatrix} \begin{bmatrix} v \\ \overline{v} \end{bmatrix} = 0$$.

The second part of the statement of Theorem 3.1 assures us that the

solution $(v, \bar{v})$ of the homogeneous system (3.7) is unique, and therefore

(3.8) $v = 0$.

Using (3.8) we find that (3.4) and (3.5) become

(3.9) $g\left(z^0, \overline{z^0}\right) \in S$

and

(3.10) $\mathrm{Re}\; u^{0^H} g\left(z^0, \overline{z^0}\right) = 0$.

Equation (3.9) implies that $\left(z^0, \overline{z^0}\right)$ is feasible for (P). Since $\mathrm{Re}\, f$ is convex with respect to R_+ on Q and g is concave with respect to S on Q , the conditions (3.2), (3.3), and (3.10) are sufficient for optimality, and thus $\left(z^0, \overline{z^0}\right)$ is an optimal solution for (P).

REMARKS 3.2. Let $\varphi = f^R(z, \bar{z})$ and let partial derivatives be denoted by subscripts. Using identities (A) and (B) , the matrix (3.1) becomes

$$2\begin{bmatrix} \varphi_{zz} & \varphi_{\overline{zz}} \\ \varphi_{\overline{z}z} & \varphi_{\overline{zz}} \end{bmatrix} - \begin{bmatrix} g_{zz} & g_{\overline{zz}} \\ g_{\overline{z}z} & g_{\overline{zz}} \end{bmatrix}\bar{y} - \begin{bmatrix} \bar{g}_{zz} & \bar{g}_{\overline{zz}} \\ \bar{g}_{\overline{z}z} & \bar{g}_{\overline{zz}} \end{bmatrix}y \ ,$$

which is similar to the matrix obtained in [5] (Theorem 1, p. 622).

Abrams in [1] (Theorem 4, p. 627) has assumed the existence of an analytic function $\alpha : C^m \to C^n$ with certain properties. It can be checked that for the existence of such a function, non-singularity of the matrix (3.1) is a sufficient condition (see [4], Theorem 9, p. 39).

One can prove this theorem by using Fritz John type necessary conditions [6], which requires no constraint qualifications, thus being able to eliminate the assumption that the optimal solution of the dual is a qualified point.

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