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DOMAIN OF ATTRACTION OF THE QUASISTATIONARY DISTRIBUTION FOR BIRTH-AND-DEATH PROCESSES

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Abstract

We consider a birth–death process $\{X(t),\ t \geq 0\}$ on the positive integers for which the origin is an absorbing state with birth coefficients $\lambda_n,\ n \geq 0$, and death coefficients $\mu_n,\ n \geq 0$. If we define $A = \sum_{n=1}^{\infty} 1/\lambda_n \pi_n$ and $S = \sum_{n=1}^{\infty} (1/\lambda_n \pi_n) \sum_{i=n+1}^{\infty} \pi_i$, where $\{\pi_n,\ n \geq 1\}$ are the potential coefficients, it is a well-known fact (see van Doorn (1991)) that if $A = \infty$ and $S < \infty$, then $\lambda_C > 0$ and there is precisely one quasistationary distribution, namely, $\{a_j(\lambda_C)\}$, where λ_C is the decay parameter of $\{X(t),\ t \geq 0\}$ in $C = \{1, 2, \ldots\}$ and $a_j(x) \equiv \mu_1^{-1} \pi_j x Q_j(x),\ j = 1, 2, \ldots$. In this paper we prove that there is a unique quasistationary distribution that attracts all initial distributions supported in C, if and only if the birth–death process $\{X(t),\ t \geq 0\}$ satisfies both $A = \infty$ and $S < \infty$. That is, for any probability measure $M = \{m_i,\ i = 1, 2, \ldots\}$, we have $\lim_{t \to \infty} \mathbb{P}_M(X(t) = j \mid T > t) = a_j(\lambda_C),\ j = 1, 2, \ldots$, where $T = \inf\{t \geq 0 \colon X(t) = 0\}$ is the extinction time of $\{X(t),\ t \geq 0\}$ if and only if the birth–death process $\{X(t),\ t \geq 0\}$ satisfies both $A = \infty$ and $S < \infty$.

Keywords: Domain of attraction; quasistationary distribution; birth-and-death process; orthogonal polynomial; duality

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1. Introduction

Quasistationary distributions (QSDs) for continuous-time Markov chains have recently attracted much attention because of their theoretical and practical interest. A complete treatment of the QSD problem for a given family of processes should accomplish two things (see [9]):

(i) determination of all QSDs; and

(ii) solve the domain of attraction problem, namely, characterize all laws ν such that a given QSD M is a ν-limiting conditional distribution.

Although (i) has been addressed for several cases, details about (ii) are known only for finite Markov processes, and for subcritical Markov branching processes. Determination of all QSDs for birth–death processes has been studied by Cavender [2] and van Doorn [11]; a complete answer is as follows (see [11]).

1. If $A = \sum_{n=1}^{\infty} 1/\lambda_n \pi_n = \infty$ and $S = \sum_{n=1}^{\infty} (1/\lambda_n \pi_n) \sum_{i=n+1}^{\infty} \pi_i = \infty$, either $\lambda_C = 0$ and there is no QSD, or $\lambda_C > 0$ and there is a one-parameter family of QSDs, namely, $\{a_j(x),\ 0 < x \leq \lambda_C\}$, where $a_j(x) \equiv \mu_1^{-1} \pi_j x Q_j(x),\ j = 1, 2, \ldots$.

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2. If $A = \sum_{n=1}^{\infty} 1/\lambda_n \pi_n = \infty$ and $S = \sum_{n=1}^{\infty} (1/\lambda_n \pi_n) \sum_{i=n+1}^{\infty} \pi_i < \infty$, then $\lambda_C > 0$ and there is precisely one QSD, namely, $\{a_j(\lambda_C)\}$.

In the literature, there are two works directly related to problem (ii), i.e. the domain of attraction problem for birth–death processes. The first work is that of van Doorn [11], who solved the QSD problem when the initial distribution has finite support; in particular, when it concentrates all mass at a single state. The other related work is due to Zhang *et al.* [13], who proved that if $\sum_{i=1}^{\infty} m_i Q_i(\lambda_C) < \infty$ then any initial distribution $M = \{m_i,\ i = 1, 2, \dots\}$ is in the domain of attraction of $\{a_j(\lambda_C)\}$.

In this paper we prove that there is a unique QSD that attracts all initial distributions supported in $C = \{1, 2, \dots\}$ if and only if the birth–death process $\{X(t),\ t \geq 0\}$ satisfies both $A = \infty$ and $S < \infty$. That is, for any probability measure $M = \{m_i,\ i = 1, 2, \dots\}$, we have

$$\lim_{t\to\infty} \mathbb{P}_M(X(t) = j \mid T > t) = a_j(\lambda_C) = \mu_1^{-1} \pi_j \lambda_C Q_j(\lambda_C), \qquad j = 1, 2, \dots,$$

where $T = \inf\{t \geq 0 \colon X(t) = 0\}$ is the extinction time of $\{X(t),\ t \geq 0\}$ if and only if the birth–death process $\{X(t),\ t \geq 0\}$ satisfies both $A = \infty$ and $S < \infty$. As is well known, there are four classifications of birth–death processes, due to Feller [4]. In the case of a regular boundary and an exit boundary, there is no QSD. When ∞ is a natural boundary, there is a one-parameter family of QSDs if $\lambda_C > 0$; we will discuss the domain of attraction of QSDs for this classification in another paper. However, for an entrance boundary, which is equivalent to both $R = \infty$ and $S < \infty$, there exists a unique QSD. In this paper we give a necessary and sufficient condition for the existence of a unique QSD that attracts any initial distribution supported in $C = \{1, 2, \dots\}$.

Our basic tools in this paper are Karlin and McGregor's [6] spectral representation for the transition probabilities of a birth–death process, and a duality concept for birth–death processes.

The remainder of the paper is organized as follows. After introducing the concepts and collecting some preliminary results in the next section, we will present results on some particular properties of birth–death processes in Section 3. Using some previous results, we obtain our main results in Section 4. Finally, we conclude in Section 5 with an example.

2. Preliminaries

In this paper we focus on a continuous-time Markov chain $\{X(t),\ t \geq 0\}$ on a state space $E \equiv \{0\} \cup C$, where $C \equiv \{1, 2, \dots\}$ is an irreducible transient class and 0 is an absorbing state. A continuous-time Markov chain $\{X(t),\ t \geq 0\}$ having state space E and q-matrix $Q \equiv (q_{ij},\ i, j \in E)$ given by

$$q_{ij} = \begin{cases} \lambda_i & \text{if } j = i+1,\ i \geq 0, \\ \mu_i & \text{if } j = i-1,\ i \geq 1, \\ -(\lambda_i + \mu_i) & \text{if } j = i,\ i \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

is called a birth-and-death process on E, with birth coefficients $\lambda_n \geq 0,\ n \geq 0$, and death coefficients $\mu_n \geq 0,\ n \geq 0$. Suppose that $\lambda_0 = \mu_0 = 0$, $\lambda_n > 0$, and $\mu_n > 0,\ n \geq 1$. Then Q will be conservative, 0 is an absorbing state, and $C = \{1, 2, \dots\}$ is irreducible for the minimal Q-function, F, and, hence, for any Q-function.

We will use Anderson's notion [1, pp. 103, 261]: define the potential coefficients $\pi = \{\pi_n,\ n \in C\}$ by $\pi_1 = 1$ and, for $n \geq 2$,

$$\pi_n = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_2\mu_3\cdots\mu_n}, \tag{2.2}$$

and let

$$A = \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n}, \qquad R = \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n}\sum_{i=1}^{n}\pi_i, \qquad S = \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n}\sum_{i=n+1}^{\infty}\pi_i.$$

Write $P_i(\cdot) = \mathbb{P}(\cdot \mid X(0) = i)$. We say that extinction occurs when the process reaches state 0. Furthermore, denote by

$$T = \inf\{t \geq 0\colon X(t) = 0\}$$

the hitting time of 0 or the extinction time of the process. Absorption at 0 is called certain if $P_i(T < +\infty) = 1$. Certain absorption is necessary for the existence of the QSD; therefore, before turning our attention to the QSD we have to assume that eventual absorption at state 0 is certain under the following condition (see [7]):

$$A = \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n} = \infty. \tag{2.3}$$

Imposing (2.3) implies that the condition

$$R = \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n}\sum_{i=1}^{n}\pi_i = \infty \tag{2.4}$$

is satisfied. Also, imposing (2.3) (and, hence, (2.4)) implies that the process $\{X(t),\ t \geq 0\}$ is nonexplosive (Q is regular) and, therefore, honest, in which case the transition probability function $P(t) = (P_{ij}(t),\ i \in C,\ j \in E)$, where

$$P_{ij}(t) = \mathbb{P}(X(t) = j \mid X(0) = i), \qquad i = 1, 2, \dots,\ j = 0, 1, \dots,\ t \geq 0,$$

is the unique solution of the system of Kolmogorov backward equations

$$P'(t) = QP(t), \qquad t \geq 0, \tag{2.5}$$

with initial conditions $P(0) = I$, where I is the identity matrix and

$$\sum_{j=0}^{\infty} P_{ij}(t) = 1, \qquad i = 1, \dots,\ t \geq 0. \tag{2.6}$$

We know that each Q-function, $P_{ij}(t)$, satisfies (2.5) since Q is conservative.

The state probabilities $P_j(t) \equiv \mathbb{P}(X(t) = j)$ are determined by the transition probabilities $P_{ij}(t)$ and the distribution of $X(0)$ through

$$P_j(t) = \sum_{i=0}^{\infty} m_i P_{ij}(t), \qquad j = 0, 1, 2, \dots,$$

where $m_i \equiv \mathbb{P}(X(0) = i) \geq 0$ and $\sum_{i=0}^{\infty} m_i = 1$ (so that $\sum_{j=0}^{\infty} P_j(t) = 1$).

It is well known (see, for example, [1, Theorem 5.1.9]) that, under our assumptions regarding the Markov chain, there exist a strictly positive constant c_{ij} (with $c_{ii} = 1$) and a parameter $\lambda_C \geq 0$ such that

$$P_{ij}(t) \leq c_{ij}\mathrm{e}^{-\lambda_C t}, \qquad i, j \in C,\ t \geq 0,$$

and

$$\lambda_C = -\lim_{t\to\infty} \frac{1}{t} \log P_{ij}(t), \qquad i, j \in C.$$

The parameter λ_C is known as the decay parameter of the Markov chain in C.

Karlin and McGregor [6] showed that the transition probabilities $P_{ij}(t)$, $i, j \in C$, can be represented by

$$P_{ij}(t) = \pi_j \int_0^\infty \mathrm{e}^{-xt} Q_i(x) Q_j(x)\, \mathrm{d}\psi(x), \tag{2.7}$$

which is the spectral representation referred to in the introduction. Here $\{Q_n(x),\ n \geq 1\}$ is a system of polynomials defined recursively by

$$\begin{aligned} \lambda_n Q_{n+1}(x) &= (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x), \qquad n = 2, 3, \dots, \\ \lambda_1 Q_2(x) &= \lambda_1 + \mu_1 - x, \qquad Q_1(x) = 1, \end{aligned} \tag{2.8}$$

and ψ is the unique (in our setting), positive measure on the nonnegative real axis of total mass 1 with respect to which $\{Q_n(x)\}$ constitutes an orthogonal polynomial sequence.

It is well known that $Q_n(x)$ has $n-1$ positive, simple zeros, $x_{n,i}$, $i = 1, 2, \dots, n-1$, which satisfy the 'interlacing' property

$$0 < x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \qquad i = 1, 2, \dots, n-1,\ n \geq 2, \tag{2.9}$$

from which it follows that the limits

$$\xi_i \equiv \lim_{n\to\infty} x_{n,i}, \qquad i \geq 1,$$

exist and satisfy $0 \leq \xi_i \leq \xi_{i+1} < \infty$. Actually, note that $\xi_1 = \lambda_C$.

It is easy to see from (2.9) that

$$x \leq \xi_1 \iff Q_n(x) > 0 \tag{2.10}$$

for all n. It also follows that $0 \leq \xi_i \leq \xi_{i+1} < \infty$, so that

$$\sigma \equiv \lim_{i\to\infty} \xi_i$$

exists, and $0 \leq \sigma \leq \infty$. Furthermore, we have

$$\xi_i = \xi_{i+1} \implies \sigma = \xi_i, \qquad i = 1, 2, \dots.$$

Now defining the (possibly finite) set $\Xi \equiv \{\xi_1, \xi_2, \dots\}$, we obtain from the following lemma (see [11]), which links the zeros of the polynomials $Q_n(x)$ to the support

$$S(\psi) \equiv \{x \mid \psi((x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}$$

of the measure ψ.

Lemma 2.1. *If $\sigma = \infty$ then $S(\psi) = \Xi$. If $\sigma < \infty$ then $S(\psi) \cap [0, \sigma] = \overline{\Xi}$ (a bar denotes closure) and σ is the smallest limit point of $S(\psi)$.*

Following Karlin and McGregor [7], we define the dual process to be a birth–death process on E with birth rates $\{\lambda_n^d,\ n \geq 0\}$ and death rates $\{\mu_n^d,\ n \geq 0\}$ given by

$$\lambda_n^d = \mu_n, \qquad \mu_1^d = 0, \qquad \mu_{n+1}^d = \lambda_n, \qquad n = 1, 2, \ldots .$$

Replacing λ_n by λ_n^d, μ_n by μ_n^d, and $Q_n(x)$ by Q_n^d in (2.8), we obtain a recurrence relation for the dual polynomials Q_n^d, $n = 1, 2, \ldots$, which constitutes an orthogonal polynomial sequence with respect to a (unique) positive measure ψ^d on the nonnegative real axis of total mass 1. This measure satisfies

$$\psi^d(\{0\}) = 0, \qquad \mathrm{d}\psi^d(x) = \mu_1 x^{-1}\,\mathrm{d}\psi(x), \quad x > 0, \tag{2.11}$$

and so, with

$$S(\psi) = S(\psi^d), \tag{2.12}$$

since, in view of [6],

$$\mu_1 \int_0^\infty x^{-1}\,\mathrm{d}\psi(x) = 1,$$

ψ cannot have positive mass at 0. It is not difficult to conclude from (2.12) that

$$\xi_i^d = \xi_i, \quad i = 1, 2, \ldots, \quad \text{and} \quad \sigma^d = \sigma.$$

We will use Chapter 5 of [1] to make the following classifications. Note that a set $\{x_j,\ j \in C\}$ of strictly positive numbers such that

$$\sum_{j \in C} q_{ij} x_j \leq -\mu x_i, \qquad i \in C, \tag{2.13}$$

is called a μ-subinvariant vector for Q on C. If the equality holds in (2.13) then $\{x_j,\ j \in C\}$ is called a μ-invariant vector for Q on C. A set $\{x_i,\ i \in C\}$ of strictly positive numbers such that

$$\sum_{j \in C} P_{ij}(t) x_j \leq \mathrm{e}^{-\mu t} x_i \tag{2.14}$$

for all $t \geq 0$ and all $i \in C$ is called a μ-subinvariant vector for $P_{ij}(t)$ on C. If the equality holds in (2.14) then $\{x_i,\ i \in C\}$ is called a μ-invariant vector for $P_{ij}(t)$ on C.

Proposition 2.1. *Let $\{x_j,\ j \in C\}$ be a set of strictly positive numbers. The following statements are equivalent.*

(a) *$\{x_j,\ j \in C\}$ is a μ-subinvariant vector for Q on C.*

(b) *$\{x_j,\ j \in C\}$ is a μ-subinvariant vector for $P_{ij}(t)$ on C.*

3. Related functions

A QSD on C is a proper probability distribution $\nu = \{\nu_j,\ j \in C\}$ such that, for all $t \geq 0$,

$$\nu_j = \mathbb{P}_\nu(X(t) = j \mid T > t), \qquad j \in C.$$

In other words, a QSD is an initial distribution on C so that the conditional probability of the process being in state j at time t, given that no absorption has taken place by that time, is independent of t for all j. We note that

$$\mathbb{P}_\nu(X(t) = j \mid T > t) = \frac{\mathbb{P}_\nu(X(t) = j)}{\mathbb{P}_\nu(T > t)},$$

while $\mathbb{P}_\nu(X(t) = j) \to 0$ as $t \to \infty$ for all $j \in C$ and any initial distribution ν. So ν can be a QSD only if $\mathbb{P}_\nu(T > t) \to 0$ as $t \to \infty$, that is, if absorption is certain, which is our assumption throughout this section.

We call $\nu = \{\nu_j\}$ the limiting condition distribution (LCD) on C if, for some initial distribution M on C, it satisfies

$$\nu_j = \lim_{t\to\infty} \mathbb{P}_M(X(t) = j \mid T > t), \qquad j \in C. \tag{3.1}$$

If we wish to describe the LCD corresponding to a particular initial distribution M, then we usually speak of the M-LCD (if it exists). If (3.1) holds, we also say that M is attracted to ν, or is in the domain of attraction of ν.

Obviously, every QSD is an LCD. Definition (3.1) only becomes interesting if it is satisfied with $M \neq \nu$. For the above concepts on LCD and QSD, we refer the reader to [5], [9], and [11].

The following series plays a crucial role in QSDs for birth–death processes:

$$S = \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i.$$

In this paper, under the conditions $A = \infty$ and $S < \infty$, we study the behavior of

$$\nu_j(t) \equiv \mathbb{P}_M(X(t) = j \mid T > t) = \frac{\sum_{i=1}^{\infty} m_i P_{ij}(t)}{\sum_{k=1}^{\infty} m_k (1 - P_{k0}(t))}, \qquad j \in C,$$

as $t \to \infty$, where $\{m_i\}_{i=1}^{\infty}$ is some initial distribution.

For notational convenience, we introduce the functions

$$a_j(x) \equiv \mu_1^{-1} \pi_j x Q_j(x), \qquad j = 1, 2, \ldots,$$

where the polynomials $Q_j(x)$ are recursively defined by (2.8) and the constants π_j are given in (2.2).

A useful lemma (see [11]) is the following.

Lemma 3.1. *If $A = \infty$ and $S < \infty$, then $\lambda_C > 0$, $\sigma = \infty$, and there is precisely one QSD, namely, $\{a_j(\lambda_C)\}$.*

Now we start with the limit of the function $\mathrm{e}^{\lambda_C t}(1 - P_{i0}(t))$, which is crucial to our main result.

Theorem 3.1. *If $A = \infty$ and $S < \infty$, then*

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_C t}(1 - P_{i0}(t)) = Q_i(\lambda_C)\psi^d(\{\lambda_C\})$$

and the limit is positive for all $i \in C$.

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Proof. Since the functions $P_j(t) = \sum_{i=1}^{\infty} m_i P_{ij}(t)$ satisfy the forward equations,

$$\begin{aligned} P_j'(t) &= \lambda_{j-1} P_{j-1}(t) - (\lambda_j + \mu_j) P_j(t) + \mu_{j+1} P_{j+1}(t), \qquad j = 1, 2, \dots, \\ P_0'(t) &= \mu_1 P_1(t). \end{aligned} \tag{3.2}$$

By (3.2) we have $P_{i0}(t) = \mu_1 \int_0^t P_{i1}(u)\,\mathrm{d}u$, so it follows from (2.7) that

$$\begin{aligned} P_{i0}(t) &= \mu_1 \int_0^t \left\{ \int_0^\infty \mathrm{e}^{-xu} Q_i(x)\,\mathrm{d}\psi(x) \right\} \mathrm{d}u \\ &= \mu_1 \int_0^\infty x^{-1} Q_i(x)\,\mathrm{d}\psi(x) - \mu_1 \int_0^\infty \mathrm{e}^{-xt} x^{-1} Q_i(x)\,\mathrm{d}\psi(x). \end{aligned}$$

Since $P_{i0}(t) \to 1$ as $t \to \infty$, the first term in this expression equals 1, so, with (2.11), we obtain

$$P_{i0}(t) = 1 - \int_0^\infty \mathrm{e}^{-xt} Q_i(x)\,\mathrm{d}\psi^d(x).$$

We now turn to the limiting behavior as $t \to \infty$ of the functions $\mathrm{e}^{\lambda_C t}(1 - P_{i0}(t))$:

$$\mathrm{e}^{\lambda_C t}(1 - P_{i0}(t)) = \int_{\lambda_C}^\infty \mathrm{e}^{-(x-\lambda_C)t} Q_i(x)\,\mathrm{d}\psi^d(x).$$

As $t \to \infty$, the right-hand expression, $\int_{\lambda_C}^\infty \mathrm{e}^{-(x-\lambda_C)t} Q_i(x)\,\mathrm{d}\psi^d(x)$, is readily seen to tend to 0 except at the single point λ_C, so we can write

$$\int_{\lambda_C}^\infty \mathrm{e}^{-(x-\lambda_C)t} Q_i(x)\,\mathrm{d}\psi^d(x) \to Q_i(\lambda_C)\psi^d(\{\lambda_C\}).$$

Next we show that $Q_i(\lambda_C)\psi^d(\{\lambda_C\}) > 0$. Lemma 3.1 tells us that if $A = \infty$ and $S < \infty$, then $\lambda_C > 0$ and $\sigma = \infty$. Then, from Lemma 2.1, $S(\psi) = \Xi = \{\xi_1, \xi_2, \dots\}$. Moreover, using (2.12), we easily have

$$S(\psi^d) = \{x \mid \psi^d((x - \varepsilon, x + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\} = \{\xi_1, \xi_2, \dots\}. \tag{3.3}$$

Recall that $\xi_1 = \lambda_C$. In view of (3.3) we have thus proved that $\psi^d(\{\lambda_C\}) > 0$.

Finally, by (2.10), $Q_i(\lambda_C) > 0$. Hence, $\lim_{t\to\infty} \mathrm{e}^{\lambda_C t}(1 - P_{i0}(t)) = Q_i(\lambda_C)\psi^d(\{\lambda_C\})$ and $Q_i(\lambda_C)\psi^d(\{\lambda_C\}) > 0$ for all $i \in C$. This completes the proof.

Lemma 3.2. *The transition probability $P_{ij}(t)$ is stochastically monotone if and only if $\sum_{j\ge k} P_{ij}(t)$ is a nondecreasing function of i for every fixed k and t.*

A stochastic matrix is monotone if its row vectors are stochastically increasing. It is easy to check that the birth–death process q-matrix is monotone.

Theorem 3.2. *If $A = \infty$ and $S < \infty$, then $Q_i(\lambda_C) \le Q_{i+1}(\lambda_C)$ and there exists a constant K such that $Q_i(\lambda_C) \le K$ for all $i \in C$, i.e. $Q_\infty(\lambda_C) \equiv \lim_{i\to\infty} Q_i(\lambda_C) < \infty$.*

Remark. Theorem 3.2 is very important and innovative. By using it we can deduce that the λ_C-invariant vector (see Theorem 3.4) $\{Q_i(\lambda_C),\ i \ge 1\}$ is bounded. Furthermore, it plays a critical role in the proof of our main result, Theorem 4.1 below.

Proof of Theorem 3.2. Since the birth–death process q-matrix is monotone and conservative and satisfies the regularity, then Theorem 3.2 of [3] tells us that the corresponding minimal Q-function is stochastically monotone. Let $k = 1$ in Lemma 3.2. Then $1 - P_{i0}(t) = \sum_{j\geq 1} P_{ij}(t)$ is a nondecreasing function of i for every fixed t. By Theorem 3.1, if $A = \infty$ and $S < \infty$, then, for all $i \in C$, we have

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_C t}(1 - P_{i0}(t)) = Q_i(\lambda_C)\psi^d(\{\lambda_C\}),$$

and $\psi^d(\{\lambda_C\}) > 0$. So, for every fixed t, $Q_i(\lambda_C)$ is also a nondecreasing function of i, that is, $Q_i(\lambda_C) \leq Q_{i+1}(\lambda_C)$.

On the other hand, defining $\gamma_{2i+2} \equiv \mu_i$, $\gamma_{2i+3} \equiv \lambda_i$, $i = 0, 1, \dots$, the polynomials $Q_i(x)$ are readily seen to be related to the polynomials $P_i(x)$ in [10] by

$$Q_{2i}(x) = \frac{P_i(-x)}{P_i(0)}, \qquad i = 0, 1, \dots.$$

Theorem 1 of [10] then states that, as $i \to \infty$, $\{P_i(-x)/P_i(0)\}_i$ converges uniformly on bounded sets to an entire function whose zeros are simple and are precisely the points ξ_i, $i \geq 1$, if and only if the series

$$\sum_{n=0}^{\infty} \pi_n \left\{\mu_1^{-1} + \sum_{i=0}^{n-1} (\lambda_i \pi_i)^{-1}\right\}$$

converges, which is easily seen to be equivalent to $S < \infty$. So $\{P_i(-x)/P_i(0)\}_i$ converges uniformly on bounded sets to an entire function whose zeros are simple and are precisely the points ξ_i, $i \geq 1$. Because the entire function is bounded on bounded sets, we can easily conclude that $\{Q_{2i}(\lambda_C),\ i \geq 1\}$ is also bounded.

Finally, by using $Q_i(\lambda_C) \leq Q_{i+1}(\lambda_C)$ and the monotone convergence theorem, it is clear that $Q_\infty(\lambda_C) \equiv \lim_{i\to\infty} Q_i(\lambda_C) < \infty$, i.e. there exists a constant K such that $Q_i(\lambda_C) \leq K$ for all $i \in C$. This completes the proof.

Remark. Conditions for a conservative q-matrix Q being regular are well known. Indeed, it is equivalent to saying that the minimal Q-function is honest and thus unique. In this paper we verify that Theorem 3.1 and Theorem 3.2 are valid when $A = \infty$ and $S < \infty$. In fact, we can prove that if $S = \infty$ and $\lambda_C > 0$, then $Q_\infty(\lambda_C) \equiv \lim_{i\to\infty} Q_i(\lambda_C) = \infty$.

If the initial distribution concentrates all mass at a single state, we consider

$$\nu_{ij}(t) \equiv \mathbb{P}_i(X(t) = j \mid T > t) = \frac{P_{ij}(t)}{1 - P_{i0}(t)}, \qquad i, j \in C,$$

as $t \to \infty$. Recall the following lemma (see Theorem 4.1 of [11]).

Lemma 3.3. *If* $\lambda_C > 0$ *then*

$$\lim_{t\to\infty} \mathbb{P}_i(X(t) = j \mid T > t) = a_j(\lambda_C) = \mu_1^{-1}\lambda_C \pi_j Q_j(\lambda_C), \qquad i, j \in C.$$

Lemma 3.3 includes the relationship between $\lim_{t\to\infty} \mathrm{e}^{\lambda_C t} P_{ij}(t)$ and $\lim_{t\to\infty} \mathrm{e}^{\lambda_C t}(1 - P_{i0}(t))$. Note that

$$\lim_{t\to\infty} \frac{\mathrm{e}^{\lambda_C t} P_{ij}(t)}{\mathrm{e}^{\lambda_C t}(1 - P_{i0}(t))} = \lim_{t\to\infty} \nu_{ij}(t) = a_j(\lambda_C), \qquad i, j \in C.$$

Hence, we can write

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_C t} P_{ij}(t) = a_j(\lambda_C) \lim_{t\to\infty} \mathrm{e}^{\lambda_C t}(1 - P_{i0}(t)), \qquad i, j \in C.$$

The limit $\lim_{t\to\infty} \mathrm{e}^{\lambda_C t} P_{ij}(t)$ is given in the following theorem.

Theorem 3.3. *If $A = \infty$ and $S < \infty$, then*

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_C t} P_{ij}(t) = \mu_1^{-1} \lambda_C \pi_j Q_j(\lambda_C) Q_i(\lambda_C) \psi^d(\{\lambda_C\}) \tag{3.4}$$

exists and the limit is positive for all $i, j \in C$.

Indeed, we can state the following result.

Corollary 3.1. *If $A = \infty$ and $S < \infty$, then $\lambda_C > 0$ and C is λ_C-positive.*

Proof. Recall from the proof of Theorem 3.1 that if $A = \infty$ and $S < \infty$, then $\psi^d(\{\lambda_C\}) > 0$. From (3.4), we have

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_C t} P_{ii}(t) = \mu_1^{-1} \lambda_C \pi_i Q_i(\lambda_C) Q_i(\lambda_C) \psi^d(\{\lambda_C\}) > 0$$

for all $i \in C$. This completes the proof.

We now prove that $\{Q_i(\lambda_C),\ i \ge 1\}$ is a λ_C-invariant vector for $P_{ij}(t)$ on C, for which we need the following important result from [8].

Lemma 3.4. ([8].) (a) *Let C be a communicating class with decay parameter $\lambda_C \ge 0$. Then there exist λ_C-subinvariant vectors for $P_{ij}(t)$ on C.*

(b) *Suppose that the communicating class C has decay parameter λ_C and is λ_C-recurrent. Then the λ_C-subinvariant vector $\{x_i,\ i \in C\}$ of (a) is unique up to constant multiples, and is in fact λ_C-invariant.*

Theorem 3.4. *If $A = \infty$ and $S < \infty$, then $\{Q_i(\lambda_C),\ i \ge 1\}$ is a λ_C-invariant vector for $P_{ij}(t)$ on C. That is,*

$$\sum_{j\in C} P_{ij}(t) Q_j(\lambda_C) = \mathrm{e}^{-\lambda_C t} Q_i(\lambda_C), \qquad i \in C.$$

Proof. Let us recall $Q_i(x)$, $i \ge 1$, defined in (2.8), in obvious vector notion: $-xQ(x) = \bar{Q}Q(x)$ with $Q_1(x) = 1$, where the matrix $\bar{Q}$ can be described as the original q-matrix Q with the first row and the first column removed. Of course, $\{Q_i(\lambda_C),\ i \ge 1\}$ is a λ_C-invariant vector for Q on C since $Q_i(\lambda_C) > 0$. By Proposition 2.1 we have

$$\sum_{j\in C} P_{ij}(t) Q_j(x) \le \mathrm{e}^{-xt} Q_i(x), \qquad 0 < x \le \lambda_C.$$

From Corollary 3.1, if $S < \infty$ then $\lambda_C > 0$ and C is λ_C-positive. This implies that C is λ_C-recurrent. Then we can apply Lemma 3.4(b) to show that $\{Q_i(\lambda_C),\ i \ge 1\}$ is λ_C-invariant, that is,

$$\sum_{j\in C} P_{ij}(t) Q_j(\lambda_C) = \mathrm{e}^{-\lambda_C t} Q_i(\lambda_C).$$

From Theorem 3.2, we can deduce that the λ_C-invariant vector $\{Q_i(\lambda_C),\ i \ge 1\}$ is bounded.

4. The domain of attraction

A necessary and sufficient condition for the existence of a unique QSD that attracts any initial distribution supported in $C = \{1, 2, \ldots\}$ to hold is given in this section. Although there are several works that have studied this problem, such as [11]–[13], the result of this paper is of particular interest in the analysis of the domain of attraction of QSDs for birth–death processes because we use the definition of the QSD to prove the domain of attraction but do not impose any condition on the initial distribution. Before stating our main result, Theorem 4.1, we require the following lemma (see Theorem 1 of [13]) and proposition (see Proposition 5.2.9 of [1]).

Lemma 4.1. *For the birth-and-death process defined by (2.1), let $M = \{m_i,\ i \in C\}$ be a probability distribution and assume that $\sum_{i\in C} m_i Q_i(\lambda_C) < \infty$. Then*

$$\lim_{t\to\infty} \mathbb{P}_M(X(t) = j \mid T > t) = \begin{cases} \mu_1^{-1}\pi_j\lambda_C Q_j(\lambda_C) & \text{if } \lambda_C > 0, \\ 0 & \text{if } \lambda_C = 0, \end{cases}$$

for all $j \in C$.

Proposition 4.1. *Suppose that the communicating class C for $P_{ij}(t)$ has decay parameter λ_C. Let μ be such that $0 \le \mu \le \lambda_C$, and let $\{u_i,\ i \in C\}$ and $\{v_i,\ i \in C\}$ be two sets of numbers.*

(a) *Suppose that there is a μ-subinvariant measure $\{m_k,\ k \in C\}$ such that*

$$\sum_{k\in C} m_k |v_k| < \infty. \tag{4.1}$$

Then

$$\lim_{t\to\infty} \sum_{j\in C} P_{ij}(t)\mathrm{e}^{\mu t} v_j = \sum_{j\in C} \Big[\lim_{t\to\infty} P_{ij}(t)\mathrm{e}^{\mu t}\Big] v_j, \qquad i \in C.$$

(b) *Suppose that there is a μ-subinvariant vector $\{x_k,\ k \in C\}$ such that*

$$\sum_{k\in C} x_k |u_k| < \infty. \tag{4.2}$$

Then

$$\lim_{t\to\infty} \sum_{i\in C} u_i P_{ij}(t)\mathrm{e}^{\mu t} = \sum_{i\in C} u_i \Big[\lim_{t\to\infty} P_{ij}(t)\mathrm{e}^{\mu t}\Big], \qquad j \in C.$$

(c) *Suppose that there is a μ-subinvariant measure $\{m_k,\ k \in C\}$ and a μ-subinvariant vector $\{x_k,\ k \in C\}$ such that both conditions (4.1) and (4.2) are satisfied. Moreover, suppose that either*

$$\sup_{k\in C} \frac{|u_k|}{m_k} < +\infty \quad \text{or} \quad \sup_{k\in C} \frac{|v_k|}{x_k} < +\infty.$$

Then

$$\lim_{t\to\infty} \sum_{i\in C}\sum_{j\in C} u_i P_{ij}(t)\mathrm{e}^{\mu t} v_j = \sum_{i\in C}\sum_{j\in C} u_i \Big[\lim_{t\to\infty} P_{ij}(t)\mathrm{e}^{\mu t}\Big] v_j.$$

Theorem 4.1. *There exists a unique QSD that attracts all initial distributions supported in $C = \{1, 2, \ldots\}$ if and only if the birth–death process $\{X(t),\ t \ge 0\}$ satisfies both $A = \infty$ and $S < \infty$. That is, for any probability measure $M = \{m_i,\ i = 1, 2, \ldots\}$, we have*

$$\lim_{t\to\infty} \mathbb{P}_M(X(t) = j \mid T > t) = a_j(\lambda_C), \qquad j = 1, 2, \ldots,$$

where $a_j(\lambda_C) \equiv \mu_1^{-1}\lambda_C\pi_j Q_j(\lambda_C)$ if and only if the birth–death process $\{X(t),\ t \ge 0\}$ satisfies both $A = \infty$ and $S < \infty$.

Remark. Lemma 4.1, which is the main result of Zhang *et al.* [13], shows that if the initial distribution $M = \{m_i,\ i = 1, 2, \dots\}$ satisfies $\sum_{i=1}^{\infty} m_i Q_i(\lambda_C) < \infty$, then it is in the domain of attraction of $\{a_j(\lambda_C)\}$. By using this result, Theorem 4.1 follows from Theorem 3.2. But, their result is too dependent on Karlin and McGregor's spectral representation for the transition probabilities of a birth–death process. Hence, we would like to give another proof, which is applicable to the more general case.

Proof of Theorem 4.1. If there exists a QSD then the eventual absorption at 0 is certain, which is equivalent to $A = \infty$. If $S = \infty$, by Theorem 3.2 of [11], either $\lambda_C = 0$ and there is no QSD, or $\lambda_C > 0$ and there is a one-parameter family of QSDs; however, we have a unique QSD, and so $S = \infty$ is impossible.

Conversely, recall from Lemma 3.1 that if $A = \infty$ and $S < \infty$, then there is precisely one QSD. Let $M = \{m_i\}$ be any initial distribution on C. On the one hand, using Theorem 3.2, if $S < \infty$ then there exists a constant K such that

$$\sum_{k=1}^{\infty} m_k Q_k(\lambda_C) \le K \sum_{k=1}^{\infty} m_k = K < \infty.$$

By Theorem 3.4, $\{Q_k(\lambda_C),\ k \ge 1\}$ is a λ_C-invariant vector; therefore, we can let $x_k = Q_k(\lambda_C)$ and $u_k = m_k$ in (4.2). Then by Proposition 4.1(b) and Theorem 3.3 we have

$$\begin{aligned} \lim_{t\to\infty} \sum_{i\in C} m_i P_{ij}(t) \mathrm{e}^{\lambda_C t} &= \sum_{i\in C} m_i \Big[\lim_{t\to\infty} P_{ij}(t)\mathrm{e}^{\lambda_C t}\Big] \\ &= \sum_{i\in C} m_i \mu_1^{-1} \lambda_C \pi_j Q_j(\lambda_C) Q_i(\lambda_C) \psi^d(\{\lambda_C\}). \end{aligned} \tag{4.3}$$

On the other hand, taking $v_i \equiv 1$ and using the fact that $Q_i(\lambda_C) \le Q_{i+1}(\lambda_C)$, we obtain $\sup_{k\in C} 1/Q_k(\lambda_C) = 1/Q_1(\lambda_C) = 1 < +\infty$. By Proposition 4.1(c) and Theorem 3.1, we have

$$\lim_{t\to\infty} \sum_{i\in C} \sum_{j\in C} m_i P_{ij}(t) \mathrm{e}^{\lambda_C t} = \sum_{i\in C} \sum_{j\in C} m_i \Big[\lim_{t\to\infty} P_{ij}(t) \mathrm{e}^{\lambda_C t}\Big]. \tag{4.4}$$

Recall that $P_{ij}(t)$ is honest from (2.6) and use (4.4) to obtain

$$\begin{aligned} \lim_{t\to\infty} \sum_{i\in C} m_i (1 - P_{i0}(t)) \mathrm{e}^{\lambda_C t} &= \sum_{i\in C} \sum_{j\in C} m_i \Big[\lim_{t\to\infty} P_{ij}(t) \mathrm{e}^{\lambda_C t}\Big] \\ &= \sum_{i\in C} \sum_{j\in C} m_i \mu_1^{-1} \lambda_C \pi_j Q_j(\lambda_C) Q_i(\lambda_C) \psi^d(\{\lambda_C\}) \\ &= \sum_{i\in C} m_i Q_i(\lambda_C) \psi^d(\{\lambda_C\}). \end{aligned} \tag{4.5}$$

In the above system of equations, we also used the fact that $\sum_{j\in C} \mu_1^{-1} \lambda_C \pi_j Q_j(\lambda_C) = 1$. As a consequence, by (4.3) and (4.5), for $j = 1, 2, \dots$, we have

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}_M(X(t) = j \mid T > t) &= \lim_{t\to\infty} \frac{\sum_{i=1}^{\infty} m_i P_{ij}(t)}{\sum_{k=1}^{\infty} m_k (1 - P_{k0}(t))} \\ &= \lim_{t\to\infty} \frac{\sum_{i=1}^{\infty} m_i P_{ij}(t) \mathrm{e}^{\lambda_C t}}{\sum_{k=1}^{\infty} m_k (1 - P_{k0}(t)) \mathrm{e}^{\lambda_C t}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sum_{i\in C} m_i \mu_1^{-1} \lambda_C \pi_j Q_j(\lambda_C) Q_i(\lambda_C) \psi^d(\{\lambda_C\})}{\sum_{i\in C} m_i Q_i(\lambda_C) \psi^d(\{\lambda_C\})} \\
&= \mu_1^{-1} \lambda_C \pi_j Q_j(\lambda_C) \\
&= a_j(\lambda_C). \qquad (4.6)
\end{aligned}
$$

Equation (4.6) not only implies that $\{a_j(\lambda_C)\}$ attracts all initial distributions M supported in $C = \{1, 2, \ldots\}$, it also implies the uniqueness of the QSD. In fact, take another QSD $\nu = \{\nu_j,\ j \in C\}$. Then $\nu_j = \mathbb{P}_\nu(X(t) = j \mid T > t)$ by definition and it is equal to $\{a_j(\lambda_C)\}$ by (4.6). Thus, if $A = \infty$ and $S < \infty$, then there exists a unique QSD that attracts all initial distributions supported in $C = \{1, 2, \ldots\}$.

5. An example

Set

$$\lambda_i = i^2, \qquad \mu_i = 2i^2, \qquad i \geq 0,$$

in (2.1). Then $\pi_1 = 1$,

$$
\begin{aligned}
\pi_i &= \frac{1}{i^2 2^{i-1}}, \\
A &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} = \sum_{n=1}^{\infty} 2^{n-1} = \infty, \\
S &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i = \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \sum_{n=m}^{\infty} \frac{1}{(n+1)^2} - \frac{1}{2} < \infty.
\end{aligned}
$$

It follows from Lemma 3.1 that $\lambda_C > 0$ and there is precisely one QSD, namely, $\{a_j(\lambda_C)\}$. Moreover, by Theorem 4.1, any initial distribution is in the domain of attraction of $\{a_j(\lambda_C)\}$ for this process. However, it remains a difficult problem to give an explicit λ_C.

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