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ESSENTIAL EXTENSIONS IN RADICAL THEORY FOR RINGS

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Abstract

The essential cover M_k of a class M is defined as the class of all essential extensions of rings belonging to M. M is called essentially closed if $M_k = M$. Every class M has a unique essential closure, i.e. a smallest essentially closed class containing M.

Let M be a hereditary class of (semi)prime rings. Then M is proved to be a (weakly) special class if and only if M is essentially closed. A main result is that M_k is the smallest (weakly) special class containing M. Further it is shown that the upper radical UM determined by M, is hereditary if and only if UM has the intersection property with respect to M_k.

Introduction

Armendariz (1968) has found that a radical σ is hereditary if and only if every essential extension of a σ-semisimple ring is σ-semisimple. He used this result to give a new and quite simple proof of a result of Hoffman and Leavitt (1968), namely that the lower radical determined by a homomorphically closed and hereditary class of rings is hereditary.

It is our intention to derive some more results in radical-theory for rings with the help of the notion of essential extension. In section 1 the *essential cover* M_k and the *essential closure* M^c of an arbitrary class M of rings are defined, and a characterization of essentially closed classes is given. Moreover it is established that a ring R has no proper essential extensions if and only if R has a unity element. In section 2 we show that the essential cover of a hereditary class of (semi)prime rings is essentially closed. In section 3 the results of section 2 are used for proving that for each hereditary class M of (semi)prime rings the essential cover M_k of M is the smallest (weakly) special class containing M. At the end of this section we find as a corollary a necessary and sufficient condition for the upper radical UM determined by a hereditary class M of (semi)prime rings to be supernilpotent.

The rings in this paper are assumed to be associative. They may fail to have unity elements.

1. Essentially closed classes of rings

Following Armendariz (1968) we say that an ideal A of a ring R is an *essential ideal* if and only if $B \cap A \neq 0$ for each nonzero ideal B of R, and when this is the case R is said to be an *essential extension* of A. Clearly each ring R is an essential extension of itself. If R is a prime ring, R is an essential extension of any of its nonzero ideals, whereas R is subdirectly irreducible if and only if it is an essential extension of a simple ring (note: all simple rings are prime).

Now let M be an arbitrary class of rings. The class M_k, consisting of all essential extensions of rings belonging to M, shall be called the *essential cover* of the class M. The class M will be called *essentially closed* if $M = M_k$, i.e. if every essential extension of a ring belonging to M, belongs to M. Each class M can be embedded in an essentially closed class, namely as follows: define $M^{(1)} = M$ and $M^{(i+1)} = M_k^{(i)}$ for each natural number i. Then the class $M^c = \bigcup_i M^{(i)}$ is essentially closed, it contains the class M, and moreover, it is contained in every other essentially closed class containing M. The latter statements are immediate consequences of the above definitions; we therefore omit their proofs. The class M^c shall be called the *essential closure* of the class M.

Recall that a class M is called *hereditary* if $A \lhd R \in M$ implies $A \in M$. For each class M there exists a smallest hereditary class containing M, namely the class HM consisting of all accessible subrings of rings belonging to M. The class of all essential ideals of rings belonging to M shall be denoted by EM, and the subclass of M consisting of all rings in M with a unity element shall be written as M_1. We have $M_1 \subset M \subset EM \subset HM$ for every class M. The class M will be said to have property (P) if the following holds:

(P) a. $M_1 = M$, i.e. *each ring in M has a unity element*;
b. *if $A \in EM$ and S with unity element is an essential extension of A then $S \in M$.*

LEMMA 1. *Every ring R has an essential extension R' with unity element, such that each ideal of R is an ideal of R' too.*

PROOF. Let R be an arbitrary ring, and let R_1 denote the well-known Dorroh-extension of R. Using Zorn's lemma we have the existence of an ideal A of R_1 which is maximal with respect to $A \cap R = 0$. The maximality of A with respect to $A \cap R = 0$ ensures that $R' = R_1/A$ is an essential extension of $(A + R)/A$. The latter ring being isomorphic to R this proves that R' is an essential extension of R with unity element. As is known, each ideal of R is an ideal of R_1 too, so the same will be true for R'.

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THEOREM 1. *Let M be a hereditary class of rings. Then M is essentially closed if and only if the following two conditions are satisfied*:

(1) *M_1 has property* (P);

(2) *$M = EM_1$, i.e. M consists of all rings having an essential extension with unity element in M.*

PROOF. Supposing that the hereditary class M is essentially closed, let $R \in M_1$ be one essential extension of the ring A, and let S with unity element be another such essential extension. $M_1 \subset M$ implies $R \in M$. Since M is hereditary it follows that $A \in M$. Now S is an essential extension of $A \in M$ and M is essentially closed. Hence $S \in M$. Since S has a unity element we find $S \in M_1$, proving that the class M_1 has property (P). Since M is hereditary and $M_1 \subset M$ we have $EM_1 \subset M$. To get the inverse inclusion, let $R \in M$, and let R' denote an essential extension of R with unity element. By lemma 1 such an extension exists. Since M is essentially closed we have $R' \in M$. Hence $R' \in M_1$ by the definition of M_1. Since R is an essential ideal of R' it follows that $R \in EM_1$, proving the only if part of the theorem. To prove the converse part, let M be a hereditary class of rings such that M_1 has property (P) and $M = EM_1$, and let S be an essential extension of a ring R belonging to M. Then we have to prove that $S \in M$. To do so, let S' be an essential extension of S with unity element, such that every ideal of S is an ideal of S' too. Such an extension exists by lemma 1. Then R is an ideal of S', and since R is essential in S and S is essential in S' we even have that R is an essential ideal of S'. Now $R \in M = EM_1$ implies the existence of an essential extension $R' \in M_1$ of R. Since M_1 is supposed to satisfy condition (P) it follows that the ring S' belongs to the class M_1. Hence $S' \in M$, and since M is a hereditary class it follows from this that $S \in M$. This completes the proof.

COROLLARY 1. *A class of simple rings is essentially closed if and only if each ring in the class has a unity element.*

Theorem 1 makes clear that each ring in a hereditary essentially closed class M has an essential extension with unity element in the class M. In theorem 2 below we shall see that rings with unity element can be characterized as rings having no proper essential extensions. To get this result we need the following lemma.

LEMMA 2. *Let S be a ring with unity element* 1, *and let A be an ideal of a ring K. Then each ring epimorphism $\alpha: A \to S$ has an extension $\bar{\alpha}: K \to S$.*

PROOF. The method of the proof is due to Suliński (1958). Let S, A, K and $\alpha: A \to S$ be as given in the lemma.

Since α is an epimorphism there exists an element $e \in A$ such that $\alpha(e) = 1$. Since A is an ideal of K we have $exe \in A$ for each element x of K. Hence $\bar{\alpha}(x) = \alpha(exe)$ defines a map which is clearly a ring homomorphism extending α. This proves the lemma.

THEOREM 2. *For each ring K the following three statements are equivalent*:

(1) *K has no proper essential extensions*;

(2) *K is a direct summand of every ring containing K as an ideal*;

(3) *K has a unity element.*

PROOF. The proof is cyclic. By lemma 1 each ring K has an essential extension K' with unity element. If K has no proper essential extensions then K' and K must coincide, so K must have a unity element. This proves the implication (1) $\Rightarrow$ (3). That (3) $\Rightarrow$ (2) is well-known (see for example Szendrei (1953)). Finally, if K is a direct summand of any of its extensions let R be an essential extension of K. Then $R = K \oplus L$ for some ideal L of R. Since $K \cap L = 0$, and K is essential in R, it follows that $L = 0$, or $R = K$, proving that K has no proper essential extensions. Hence (2) implies (1), completing the proof.

2. The essential closure of a hereditary class of (semi)prime rings

In this section we show that the essential cover of a hereditary class of (semi)prime rings is essentially closed. As a consequence we have that the essential closure of such a class coincides with its essential cover.

LEMMA 3. *An essential extension of a (semi)prime ring is again a (semi)prime ring.*

The proof is easy and will be omitted.

THEOREM 3. *The essential cover of a hereditary class of (semi)prime rings is again a hereditary class of (semi)prime rings.*

PROOF. Let M be a hereditary class of (semi)prime rings. By lemma 3 we only need to prove that M_k is a hereditary class. To do so, let $R \in M_k$ and I an arbitrary ideal of R. By the definition of M_k, R has an essential ideal K in M. Since $I \cap K$ is an ideal of K, the hereditariness of the class M implies $I \cap K \in M$. Now $I \cap K$ is also an ideal of I and we claim that it is essential in I. For let P be an ideal of I such that $P \cap (I \cap K) = 0$. Then $P \cap K = 0$. Let P' be the ideal of R generated by P. By Andrunakievič's lemma (1958) we

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have $(P')^3 \subset P$. Hence $(P')^3 \cap K = 0$. Because of the essentiallity of K in R it follows that $(P')^3 = 0$, and since R is a semiprime ring this implies $P' = 0$, or equivalently $P = 0$. Therefore $I \cap K$ is an essential ideal of I. Together with the fact that $I \cap K \in M$ this implies $I \in M_k$ by the definition of M_k, proving that the class M_k is hereditary indeed.

LEMMA 4. *Let K be an essential ideal of the ring B, where B is an essential ideal of the ring R. Then K', the ideal of R generated by K, is essential in R.*

PROOF. Suppose the opposite. Then there exists a nonzero ideal A of R such that $K' \cap A = 0$. Since B is essential in R it follows that $A \cap B$ is a nonzero ideal of B. Since K is essential in B it follows that $K \cap A \cap B$ is nonzero. Consequently $K \cap A$ is nonzero, which contradicts $K' \cap A = 0$. This proves the lemma.

LEMMA 5. *Let the semiprime ring I be an essential ideal in the ring R. Then I^n is an essential ideal too, for each natural number n.*

PROOF. Let n be any natural number such that I^n is not essential in R. Then there exists a nonzero ideal A of R such that $A \cap I^n = 0$. Now $A \cap I$ is an ideal of I as well as of A, and since I is essential in R one has that $A \cap I \neq 0$. Since I is a semiprime ring I does not contain nonzero nilpotent ideals. Hence we have $(A \cap I)^n \neq 0$. Since $(A \cap I)^n \subset A \cap I^n$ this implies $A \cap I^n \neq 0$, a contradiction. Hence the lemma follows.

THEOREM 4. *The essential cover of a hereditary class of (semi)prime rings is essentially closed.*

PROOF. Let M be a hereditary class of (semi)prime rings, and let R be an essential extension of a ring B belonging to M_k, the essential cover of M. Then, by the definition of M_k, B contains an essential ideal K in M. Now lemma 4 applies: K', the ideal of R generated by K, is essential in R. Furthermore, by theorem 3, M_k is a hereditary class of semiprime rings. So K', being an ideal of the semiprime ring B, is semiprime. Therefore $(K')^3$ is an essential ideal of the ring R, by lemma 5. By Andrunakievič's lemma we have $(K')^3 \subset K$. Since K is in M, and M is hereditary, it follows that $(K')^3$ belongs to M. Thus the ring R has an essential ideal in M. Therefore R belongs to M_k, proving that M_k is essentially closed indeed.

COROLLARY 2. *Let M be a class of prime simple rings. Then M_k, the essential cover of M is essentially closed and is precisely the class of all subdirectly irreducible rings for which the rings in M act as hearts.*

3. Special and weakly special classes

Our next theorem will show that special and weakly special classes of rings can equally well be defined by using the concept of essential closedness. For an ideal A of a ring R let $A^* = \{x \in R \mid xA = Ax = 0\}$. We recall that a *special class* M is defined by Andrunakievič (1958) as a hereditary class of prime rings satisfying the condition:

(C1) If $A \in M$ is an ideal of a ring R, then $R/A^* \in M$,

and a *weakly special class* M by Rjabuhin (1965) as a hereditary class of semiprime rings satisfying the condition:

(C2) If $A \in M$ is an ideal of a ring R, and $A^* = 0$, then $R \in M$.

These concepts are important since it was shown that the upper radical UM determined by a (weakly) special class M is hereditary, and has the *intersection property relative to the class* M; that is the property that the UM-radical of an arbitrary ring R equals the intersection of all ideals I of R such that $R/I \in M$ (see Leavitt (1973)). Clearly if the radical σ has the intersection property relative to a class N then all rings in N are σ-semisimple.

LEMMA 6. *Let M be a class of (semi)prime rings. Then the following statements are equivalent*:

(1) *M satisfies* (C1);
(2) *M satisfies* (C2);
(3) *M is essentially closed.*

PROOF. Let M be a class of semiprime rings, and let $A \in M$ be an ideal in the ring R. Since A is a semiprime ring it is clear that A^* is maximal in R relative to having zero intersection with A. Consequently, R/A^* contains $(A^* + A)/A^*$ as an essential ideal. The latter ring being isomorphic to A, this together with $A \in M$ implies $R/A^* \in M_k$. Hence, if M is essentially closed, then $M_k = M$ yields $R/A^* \in M$, proving that (3) implies (1). The implication (1) $\Rightarrow$ (2) is trivial. Finally suppose that M satisfies (C2), and let $A \in M$ be an essential ideal in R. Since A^* is an ideal of R having zero intersection with A it follows that $A^* = 0$. Therefore (C2) applies, which yields $R \in M$, proving that M is essentially closed. This completes the proof.

As an immediate consequence of lemma 6 we have

THEOREM 5. *A hereditary class of (semi)prime rings is (weakly) special if and only if it is essentially closed.*

Now we are able to show that each hereditary class of (semi)prime rings generates a (weakly) special class.

THEOREM 6. *The essential cover M_k of a hereditary class of (semi)prime rings is a (weakly) special class. In fact M_k is the smallest (weakly) special class containing M.*

PROOF. Let M be a hereditary class of (semi)prime rings. By theorem 3 M_k is again a hereditary class of (semi)prime rings, and by theorem 4 M_k is essentially closed. Then theorem 5 yields that M_k is a (weakly) special class. This proves the first part of the theorem. To prove the second part, let N be a (weakly) special class containing M. By theorem 5 the class N is essentially closed. Hence $M \subset N$ implies $M_k \subset N$, completing the proof.

Regarding the upper radical classes UM and UM_k determined by a hereditary class M of (semi)prime rings and its essential cover respectively, we observe that, since $M \subset M_k$, the inclusion $UM_k \subset UM$ will hold. One may ask for conditions to ensure equality. In this connection we state

THEOREM 7. *Let M be a hereditary class of (semi)prime rings, and let M_k be its essential cover. Then the following statements are equivalent:*

(1) *UM is a hereditary radical*;
(2) $M_k \subset SUM$;
(3) $UM = UM_k$;
(4) *UM has the intersection property with respect to M_k*;
(5) $UM \cap M_k = 0$.

PROOF. The proof is cyclic.

(1) $\Rightarrow$ (2): Let UM be hereditary and $R \in M_k$. By the definition of M_k the ring R has an essential ideal I in M. Since UM is hereditary we have $UM(I) = I \cap UM(R)$. Now $I \in M$ implies $UM(I) = 0$. Hence $I \cap UM(R) = 0$. Since I is essential in R it follows that $UM(R) = 0$, or equivalently $R \in SUM$. This proves that $M_k \subset SUM$.

(2) $\Rightarrow$ (3): $M_k \subset SUM$ implies $SUM_k \subset SUM$, or equivalently $UM \subset UM_k$. But $M \subset M_k$, so $UM_k \subset UM$, and thus $UM = UM_k$.

(3) $\Rightarrow$ (4): Since M_k is a (weakly) special class (by theorem 6), it follows from Andrunakievič (1958) or Rjabuhin (1965) that $UM = UM_k$ has the intersection property relative to M_k.

(4) $\Rightarrow$ (5): Let UM have the intersection property relative to M_k. Then any $R \in M_k$ is UM-semisimple and so $UM \cap M_k = 0$.

(5) $\Rightarrow$ (1): Let A be an ideal of a ring R with $A \notin UM$. Then there is some ideal I of A such that $0 \neq A/I \in M$, and since all rings in M are semiprime, $I \triangleleft R$. But (by Andrunakievič (1958) or Rjabuhin (1965)) UM_k is hereditary and since $M \subset M_k$ there must be some nonzero image $(R/I)/(J/I) \cong R/J \in M_k$. Also $UM \cap M_k = 0$, so $R/J \notin UM$ and thus $R \notin UM$. Therefore UM is hereditary.

Corollary 3. *Let M be a hereditary class of (semi)prime rings. Then UM is a hypernilpotent radical if and only if each essential extension of a ring belonging to M is UM-semisimple.*

Recalling that a radical is called hypernilpotent if it is hereditary and it contains all nilpotent rings, this is only a restatement of the equivalence of the conditions (1) and (2) in theorem 7, because the fact that the rings in M are semiprime ensures that each nilpotent ring is UM-radical.

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