Note on Tortuous Curves.

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The condition that the principal normals of one curve may also be the principal normals of a second curve is, as found by Bertrand, that a linear relation with constant coefficients should exist between the curvature and torsion of each curve. In seeking for pairs of curves such that the tangents, principal normals or binormals of one may be the tangents, principal normals or binormals of the other, there are six cases to be considered. The curves of Bertrand are furnished by one case, and a second case, that of evolutes and involutes, is also discussed in the text-books. Of the remaining four only one gives results worthy of mention. Bertrand's problem suggests the inquiry into the nature of the pair of curves when the binormal of one is the principal normal of the other. A certain quadratic relation of a simple character found to exist between the curvature and torsion of the second curve led me to a paper in the *Comptes Rendus* of 1893, by Demoulin, in which the problem had been generalised. His method of solution is different, and no explicit results as to the nature of the curves are given in the paper. Since no indication of the discussion of the problem is given in the text-books I have seen, I venture to submit a note of some results.

Let (x, y, z) be a point on the curve;

$$a, \beta, \gamma; \quad l, m, n; \quad \lambda, \mu, \nu;$$

the direction cosines of the tangent, principal normal and binormal at (x, y, z); we shall take these lines to be so directed that by displacement they may be brought to coincide with the positive x, y and z axes respectively. Also let $1/\mathrm{R}$ and $1/\mathrm{T}$ be the curvature and torsion.

If (ξ, η, ζ) be the point on the second curve corresponding to (x, y, z) then

$$\xi = x + a\lambda,$$
$$\eta = y + a\mu,$$
$$\zeta = z + a\nu,$$

where a carrying its own sign is the distance between the points.

Taking the differentials and putting $\lambda d\xi + \mu d\eta + \nu d\zeta = 0$ we find that $da = 0$, that is, a is a constant.

From Frenet's formulæ

$$\frac{d\alpha}{ds} = \frac{l}{\mathrm{R}}, \quad \frac{dl}{ds} = -\frac{\alpha}{\mathrm{R}} - \frac{\lambda}{\mathrm{T}}, \quad \frac{d\lambda}{ds} = \frac{l}{\mathrm{T}},$$

we have

$$d\xi = \left(\alpha + \frac{al}{\mathrm{T}}\right)ds,$$
$$d\eta = \left(\beta + \frac{am}{\mathrm{T}}\right)ds,$$
$$d\zeta = \left(\gamma + \frac{an}{\mathrm{T}}\right)ds.$$

Denote corresponding quantities for the (ξ, η, ζ) curve by the same letters with suffixes.

Then

$$\left.\begin{aligned} \alpha_1 &= \kappa\left(\alpha + \frac{al}{\mathrm{T}}\right), \\ \beta_1 &= \kappa\left(\beta + \frac{am}{\mathrm{T}}\right), \\ \gamma_1 &= \kappa\left(\gamma + \frac{an}{\mathrm{T}}\right) \end{aligned}\right\} \qquad (1)$$

where $\kappa = \dfrac{ds}{ds_1}$.

By squaring and adding,

$$\kappa^2 = \frac{1}{1 + \dfrac{a^2}{\mathrm{T}^2}}.$$

$$\therefore \quad ds_1 = \sqrt{\left(1 + \frac{a^2}{\mathrm{T}^2}\right)}\, ds,$$

the positive sign of the root being taken so that s and s_1 increase together.

Also $\qquad \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = \kappa = \cos\theta$

where θ is the angle between the corresponding tangents.

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It is seen from the formula $\frac{d\mu}{ds}=\frac{m}{\mathrm{T}}$ that the torsion is positive when the positive binormal rotates round the tangent in the direction towards the centre of curvature. Hence the positive direction of θ being that from the tangent to the principal normal, we have

$$\cos\theta=\frac{1}{\sqrt{\left(1+\frac{a^2}{\mathrm{T}^2}\right)}},\quad \tan\theta=\frac{a}{\mathrm{T}}. \qquad (2)$$

Differentiation of the three formulæ (1) gives

$$\frac{l_1}{\mathrm{R}_1}ds_1=d\kappa\left(\alpha+\frac{al}{\mathrm{T}}\right)+\kappa\left(\frac{l}{\mathrm{R}}-\frac{a\alpha}{\mathrm{RT}}-\frac{a\lambda}{\mathrm{T}^2}-\frac{al}{\mathrm{T}^2}\frac{d\mathrm{T}}{ds}\right)ds, \qquad (3)$$

with two corresponding results.

Now let the binormal of the first curve be the principal normal of the second so that

$$\lambda=\epsilon l_1,\quad \mu=\epsilon m_1,\quad \nu=\epsilon n_1 \text{ where } \epsilon=\pm 1.$$

By multiplying the three equations (3) by λ, μ, ν and adding we have

$$\frac{\epsilon ds_1}{\mathrm{R}_1}=-\frac{a\kappa}{\mathrm{T}^2}ds,$$

$$\text{or}\quad \frac{\epsilon}{\mathrm{R}_1}=-\frac{a}{\mathrm{T}^2+a^2}. \qquad (4)$$

Since the curvature is positive in Frenet's formulæ, $\epsilon=-1$ if a is positive and $\epsilon=1$ if a is negative.

The squaring and addition of (3) gives, after inserting the value of κ,

$$\frac{1}{\mathrm{R}_1{}^2}=\frac{a^2\mathrm{T}^2}{(\mathrm{T}^2+a^2)^3}\left(\frac{d\mathrm{T}}{ds}\right)^2-\frac{2a\mathrm{T}^2}{\mathrm{R}(\mathrm{T}^2+a^2)^2}\frac{d\mathrm{T}}{ds}+\frac{\mathrm{T}^2}{\mathrm{R}^2(\mathrm{T}^2+a^2)}+\frac{a^2}{(\mathrm{T}^2+a^2)^2}$$

$$=\frac{a^2}{(\mathrm{T}^2+a^2)^2}.$$

$$\therefore\quad a^2\mathrm{R}^2\left(\frac{d\mathrm{T}}{ds}\right)^2-2a\mathrm{R}(\mathrm{T}^2+a^2)\frac{d\mathrm{T}}{ds}+(\mathrm{T}^2+a^2)^2=0$$

$$\text{or}\quad a\mathrm{R}\frac{d\mathrm{T}}{ds}=\mathrm{T}^2+a^2.$$

$$\therefore\quad \tan^{-1}\left(\frac{\mathrm{T}}{a}\right)=\int\frac{ds}{\mathrm{R}}+\text{constant}.$$

Since a occurs in one of the intrinsic equations of the curve, there can only be *one* curve of the second kind associated with it.

If $T = T_0$ when $s = 0$ and $S = \int_0^s \frac{ds}{R}$,

then $$T = \frac{aT_0 + a^2\tan S}{a - T_0\tan S}. \qquad (5)$$

Let $s = f(R)$; then $S = \int_{R_0}^{R} \frac{f'(R)}{R} dR$,
where $R = R_0$ when $s = 0$.

A particular curve having this intrinsic equation free from the trigonometric function is got by putting

$$f'(R) = \frac{cR}{c^2 + R^2} \text{ or } s = \tfrac{1}{2}c\log\left(\frac{c^2 + R^2}{c^2 + R_0^2}\right);$$

$$\tan S \text{ is then } = \frac{c(R - R_0)}{c^2 + RR_0},$$

$$\text{and} \quad T = a\frac{R(R_0T_0 + ac) - c(aR_0 - cT_0)}{R(aR_0 - cT_0) + c(R_0T_0 + ac)}.$$

Let ψ be the angle between the tangents at $s = 0$ and $s = s$ when the surface formed by the tangents is developed on a plane;

then $$\int_0 \frac{ds}{R} = \psi \text{ and } T = a\tan\psi.$$

Since from (2) $$\tan\theta = \frac{a}{T},$$

$$\tan\theta = \cot\psi.$$

$$\therefore \quad \psi = (2n + 1)\frac{\pi}{2} - \theta.$$

It remains to determine T_1;

$$\lambda_1 = \beta_1 n_1 - \gamma_1 m_1 = \epsilon(\beta_1\nu - \gamma_1\mu).$$

$$\therefore \quad \frac{d\lambda_1}{ds_1} \cdot \frac{ds_1}{ds} = \frac{\epsilon}{T}(\beta_1 n - \gamma_1 m).$$

$$\therefore \quad \frac{d\lambda_1}{ds_1} = \frac{\kappa^2\epsilon}{T}(n\beta - m\gamma) \text{ from (1)}.$$

But $\frac{d\lambda_1}{ds_1} = \frac{l_1}{T_1}$, $n\beta - m\gamma = \lambda$ and $l_1 = \epsilon\lambda$.

Hence
$$\frac{1}{T_1}=\frac{T}{T^2+a^2} \quad - \quad - \quad - \quad - \quad (6)$$
and, from (4), $TT_1 = -a\epsilon R_1$.

Also
$$\frac{1}{T_1^2}+\frac{1}{R_1^2}=-\frac{\epsilon}{aR_1} \quad - \quad - \quad - \quad (7)$$
where ϵ and a are of opposite signs.

Referred to the tangent, principal normal and binormal at a point on a curve as axes, the equations of the axis of the osculating helix which has the same torsion as the curve at the point are

$$\frac{z}{x}=-\frac{T}{R},\ y=\frac{RT^2}{T^2+R^2}.$$

Here
$$\frac{R_1T_1^2}{R_1^2+T_1^2}=-a\epsilon.$$

Also
$$\frac{z}{x}=-\frac{T_1}{R_1}=\frac{a\epsilon}{T} \quad \text{from (6).}$$

But $\dfrac{z}{x}$ is the tangent of the angle this axis makes with the tangent to the curve, the angle being measured from the tangent to the binormal. Hence from the last two results and from (2), the tangent to the first curve is the axis of the helix which osculates the second curve and has the same torsion.