J. Aust. Math. Soc. **116** (2024), 257–288
doi:10.1017/S1446788723000022

NONREALIZABILITY OF CERTAIN REPRESENTATIONS IN FUSION SYSTEMS

BOB OLIVER

(Received 30 September 2022; accepted 4 March 2023; first published online 11 April 2023)

Communicated by Michael Giudici

Abstract

For a finite abelian p-group A and a subgroup $\Gamma \leq \mathrm{Aut}(A)$, we say that the pair (Γ, A) is fusion realizable if there is a saturated fusion system $\mathcal{F}$ over a finite p-group $S \geq A$ such that $C_S(A) = A$, $\mathrm{Aut}_{\mathcal{F}}(A) = \Gamma$ as subgroups of $\mathrm{Aut}(A)$, and $A \ntrianglelefteq \mathcal{F}$. In this paper, we develop tools to show that certain representations are not fusion realizable in this sense. For example, we show, for $p = 2$ or 3 and Γ one of the Mathieu groups, that the only $\mathbb{F}_p\Gamma$-modules that are fusion realizable (up to extensions by trivial modules) are the Todd modules and in some cases their duals.

2020 *Mathematics subject classification*: primary 20D20; secondary 20C20, 20D05, 20E45.

Keywords and phrases: finite groups, Sylow subgroups, fusion, finite simple groups, modular representations.

Introduction

Fix a prime p. A saturated fusion system over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms between those subgroups that satisfy certain axioms formulated by Puig [Pu], motivated in part by the Sylow theorems for finite groups. See Definition 1.1 for more details.

Consider a pair (Γ, A), where A is a finite abelian p-group and $\Gamma \leq \mathrm{Aut}(A)$ is a group of automorphisms. We say that (Γ, A) is *fusion realizable* if there is a saturated fusion system $\mathcal{F}$ over some finite p-group $S \geq A$ such that $C_S(A) = A$, $A \ntrianglelefteq \mathcal{F}$, and $\mathrm{Aut}_{\mathcal{F}}(A) = \Gamma$ as groups of automorphisms of A. We also say that (Γ, A) is *realized by* $\mathcal{F}$ in this situation.

B. Oliver is partially supported by UMR 7539 of the CNRS. Part of this work was carried out at the Isaac Newton Institute for Mathematical Sciences during the GRA2 programme, supported by EPSRC Grant No. EP/K032208/1.

In an earlier paper [O2], we considered the special case where $p = 3$, $O^{3'}(\Gamma) \cong 2M_{12}$, M_{11}, or A_6, and A is an elementary abelian 3-group of rank 6, 5, or 4, respectively, and classified the saturated fusion systems that realize some pair (Γ, A) of this form. In this paper, we take the opposite approach, and develop tools that we use to show that 'most' $\mathbb{F}_p\Gamma$-modules are not fusion realizable, that is, cannot be realized by any saturated fusion system.

For example, in Definition 2.3 and Proposition 2.4, we define certain sets $\mathscr{R}_T(A)$, for A an abelian p-group and $T \le \mathrm{Aut}(A)$ a p-subgroup, with the property that $\mathscr{R}_T(A) \neq \emptyset$ if there is a fusion realizable pair (Γ, A) where $T \in \mathrm{Syl}_p(\Gamma)$. As one of the consequences of this proposition, we show (Corollary 2.9) that if A is elementary abelian and (Γ, A) is fusion realizable, then there are $m \ge 1$ and an elementary abelian p-subgroup $B \le \Gamma$ of rank m such that for each $g \in B^\#$, the action of g on A has at most m nontrivial Jordan blocks.

Theorems A and B as stated below are our main applications so far of these tools. For example, as one special case of Theorem A, we show that the Golay modules for M_{22} and M_{23} are not fusion realizable. In contrast, the Todd modules for M_{22} and M_{23} (dual to the Golay modules) are realized by the fusion systems of the Fischer groups Fi_{22} and Fi_{23}, and the Golay module for $\mathrm{Aut}(M_{22})$ (a case not covered by the statement of Theorem A) is realized by the fusion system of the Conway group Co_2.

THEOREM A (Theorem 3.3). *Fix a prime p, and let Γ be a finite group such that $\Gamma_0 = O^{p'}(\Gamma)$ is quasisimple and $\Gamma_0/Z(\Gamma_0)$ is one of Mathieu's five sporadic groups. Let A be an $\mathbb{F}_p\Gamma$-module such that (Γ, A) is fusion realizable, and set $A_0 = [\Gamma_0, A]/C_{[\Gamma_0,A]}(\Gamma_0)$. Then either*

- *$p = 2$, and A_0 is the Todd module for $\Gamma \cong M_{22}$, M_{23}, or M_{24} or the Golay module for $\Gamma \cong M_{24}$; or*
- *$p = 3$, $\Gamma \cong M_{11}$, $M_{11} \times C_2$, or $2M_{12}$, and A_0 is the Todd module or Golay module for Γ_0; or*
- *$p = 11$, $\Gamma_0 \cong 2M_{12}$ or $2M_{22}$, $\Gamma/Z(\Gamma_0) \cong \mathrm{Aut}(M_{12}) \times C_5$ or $\mathrm{Aut}(M_{22}) \times C_5$, and A_0 is a 10-dimensional simple $\mathbb{F}_{11}\Gamma$-module.*

When $p = 2$ or 3, the nonrealizability of (Γ, A) in Theorem A is shown in all cases by proving that the set $\mathscr{R}_T(A)$ mentioned above is empty for $T \in \mathrm{Syl}_p(\Gamma)$. For $p > 3$, it follows from results in [COS].

Theorem B is a restatement of a theorem of O'Nan [O'N, Lemma 1.10] in the context of fusion systems, included here to illustrate how these methods apply when A is not elementary abelian. Its proof is similar to O'Nan's, but is shortened by using results from Section 2.

THEOREM B (Theorem 4.3). *Assume, for some $n \ge 3$, that $A = \langle v_1, v_2, v_3 \rangle \cong C_{2^n} \times C_{2^n} \times C_{2^n}$, and that $S = A\langle s, t \rangle$ is an extension of A by D_8 with action as described in Table 4. Then A is normal in every saturated fusion system over S. Thus there is no $\Gamma \le \mathrm{Aut}(A)$ with $\mathrm{Aut}_S(A) \in Syl_2(\Gamma)$ such that (Γ, A) is fusion realizable.*

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

The paper is organized as follows. After summarizing in Section 1 the basic definitions and properties of fusion systems that are needed, we state and prove our main criteria for fusion realizability in Section 2. We then look at representations of Mathieu groups in Section 2 and prove Theorem A (Theorem 3.3), and study Alperin's 2-groups in Section 4 and prove Theorem B (Theorem 4.3). We finish with three appendices: Appendix A with some general results on representations, and Appendices B and C where we set up notation to work with the Golay modules for M_{22} and M_{23}, and the six-dimensional $\mathbb{F}_4 3M_{22}$-module, respectively.

Notation and terminology. Most of our notation for working with groups is fairly standard. When $P \le G$ and $x \in N_G(P)$, we let $c_x^P \in \mathrm{Aut}(P)$ denote conjugation by x on the left: $c_x^P(g) = {}^x g = xgx^{-1}$. Also, $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of a finite group G, and $G^\# = G \setminus \{1\}$. Other notation used here is as follows.

- E_{p^m} is always an elementary abelian p-group of rank m.
- $A \rtimes B$ and $A.B$, respectively, denote a semidirect product and an arbitrary extension of A by B.
- $2M_{12}$, nM_{22}, and $2A_4$ denote (nonsplit) central extensions of C_2 or C_n by the groups M_{12}, M_{22}, or A_4, respectively.

Also, composition of functions and homomorphisms is always written from right to left.

1. Background definitions and results

We recall here some of the basic definitions and properties of saturated fusion systems. Our main reference is [AKO], although most of the results are also shown in [Cr].

A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, such that for each $P, Q \le S$,

- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$; and
- every morphism in $\mathcal{F}$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

Here, $\mathrm{Hom}_S(P,Q) = \{c_g \in \mathrm{Hom}(P,Q) \mid g \in S,\ {}^gP \le Q\}$. We also write $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ for the set of $\mathcal{F}$-isomorphisms from P to Q, and $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$.

In order for fusion systems to be very useful, we need to assume they satisfy the following saturation properties, motivated by the Sylow theorems and first formulated by Puig [Pu].

DEFINITION 1.1. Let $\mathcal{F}$ be a fusion system over a finite p-group S.

(a) Two subgroups $P, Q \le S$ are *$\mathcal{F}$-conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P,Q) \neq \emptyset$, and two elements $x, y \in S$ are $\mathcal{F}$-conjugate if there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle)$ such that $\varphi(x) = y$. The $\mathcal{F}$-conjugacy classes of $P \le S$ and $x \in S$ are denoted $P^{\mathcal{F}}$ and $x^{\mathcal{F}}$, respectively.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

(b) A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ (*fully centralized* in $\mathcal{F}$) if $|N_S(P)| \geq |N_S(Q)|$ for each $Q \in P^{\mathcal{F}}$ ($|C_S(P)| \geq |C_S(Q)|$ for each $Q \in P^{\mathcal{F}}$).

(c) The fusion system $\mathcal{F}$ is *saturated* if it satisfies the following two conditions.

- (Sylow axiom) For each subgroup $P \leq S$ fully normalized in $\mathcal{F}$, P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
- (Extension axiom) For each isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ in $\mathcal{F}$ such that Q is fully centralized in $\mathcal{F}$, φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ where

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(Q)\}.$$

Definition 1.1 is the definition first given in [BLO], and is used here since it seems to be the easiest to apply for our purposes. It is slightly different from that given in [AKO, Definition I.2.2], but the two are equivalent by [AKO, Proposition I.2.5]. Its equivalence with Puig's original definition is shown in [AKO, Proposition I.9.3].

As one example, the fusion system of a finite group G with respect to a Sylow p-subgroup $S \leq G$ is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are those homomorphisms between subgroups that are induced by conjugation in G. It is clearly a fusion system and was shown by Puig to be saturated. (See [BLO, Proposition 1.3] for a proof of saturation in terms of Definition 1.1.)

We also need to work with certain classes of subgroups in a fusion system. Recall, for a pair of finite groups $H < G$, that H is *strongly p-embedded in G* if $p \mid |H|$, and $p \nmid |H \cap {}^gH|$ for $g \in G \setminus H$.

DEFINITION 1.2. Let $\mathcal{F}$ be a fusion system over a finite p-group S. For $P \leq S$,

- P is *$\mathcal{F}$-centric* if $C_S(Q) \leq Q$ for each $Q \in P^{\mathcal{F}}$;
- P is *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$ and the group $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ contains a strongly p-embedded subgroup;
- P is *weakly closed in $\mathcal{F}$* if $P^{\mathcal{F}} = \{P\}$;
- P is *strongly closed in $\mathcal{F}$* if for each $x \in P$, $x^{\mathcal{F}} \subseteq P$;
- P is *central* in $\mathcal{F}$ if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, for $Q, R \leq S$, extends to some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QP, RP)$ such that $\overline{\varphi}|_P = \mathrm{Id}_P$; and
- P is *normal in $\mathcal{F}$* ($P \trianglelefteq \mathcal{F}$) if each morphism in $\mathcal{F}$ extends to a morphism that sends P to itself.

We also let $\mathcal{F}^c$ and $\mathbf{E}_{\mathcal{F}}$, respectively, be the sets of subgroups of S that are $\mathcal{F}$-centric and $\mathcal{F}$-essential.

The following is one version of the Alperin–Goldschmidt fusion theorem for fusion systems.

THEOREM 1.3 [AKO, Theorem I.3.6]. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ for $R \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$.*

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

The next proposition is more technical.

PROPOSITION 1.4 [AKO, Lemma I.2.6(c)]. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then for each $P \le S$, and each $Q \in P^{\mathcal{F}}$ fully normalized in $\mathcal{F}$, there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $\psi(P) = Q$.*

Normal p-subgroups in a fusion system are strongly closed, but the converse does not always hold. The following is one situation where it does hold. For a much more detailed list of conditions under which strongly closed subgroups in a fusion system are normal, see [K1, Theorem B].

LEMMA 1.5 [AKO, Corollary I.4.7(a)]. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. If $A \trianglelefteq S$ is an abelian subgroup that is strongly closed in $\mathcal{F}$, then $A \trianglelefteq \mathcal{F}$.*

We next look at centralizers of p-subgroups in fusion systems. Normalizer subsystems are defined in a similar way (see [AKO, Section I.5]), but are not needed here.

DEFINITION 1.6. Let $\mathcal{F}$ be a fusion system over a finite p-group S. For each $Q \le S$, the *centralizer fusion subsystem* $C_{\mathcal{F}}(Q) \le \mathcal{F}$ is the fusion subsystem over $C_S(Q)$ defined by setting

$$\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, R) = \{\varphi|_P \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ),\ \varphi(P) \le R,\ \varphi|_Q = \mathrm{Id}_Q\}.$$

Note that a subgroup $Q \le S$ is central in $\mathcal{F}$ if and only if $C_{\mathcal{F}}(Q) = \mathcal{F}$.

THEOREM 1.7 [AKO, Theorem I.5.5]. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $Q \le S$. Then $C_{\mathcal{F}}(Q)$ is saturated if Q is fully centralized in $\mathcal{F}$.*

Weakly closed abelian subgroups play a central role in this paper, and the following lemma is of crucial importance when working with them.

LEMMA 1.8. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $A \le S$ is an abelian subgroup that is weakly closed in $\mathcal{F}$.*

(a) *If $R \le S$ is fully normalized and $\mathcal{F}$-conjugate to some $Q \le A$, then $R \le A$.*
(b) *For each $P, Q \le A$, each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ extends to some $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(A)$.*

PROOF. (a) Assume $Q \le A$ and $R \le S$ are $\mathcal{F}$-conjugate, and R is fully normalized in $\mathcal{F}$. By the extension axiom, each $\psi \in \mathrm{Iso}_{\mathcal{F}}(Q, R)$ extends to some $\overline{\psi} \in \mathrm{Hom}_{\mathcal{F}}(C_S(Q), S)$. Then $C_S(Q) \ge A$ since A is abelian, $\overline{\psi}(A) = A$ since A is weakly closed in $\mathcal{F}$, and so $R = \overline{\psi}(Q) \le A$.

(b) Assume $P, Q \le A$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, and choose $R \in Q^{\mathcal{F}}$ that is fully centralized in $\mathcal{F}$. Thus $R \le A$ by (a), and there is $\psi \in \mathrm{Iso}_{\mathcal{F}}(Q, R)$. By the extension axiom again, ψ extends to $\widehat{\psi} \in \mathrm{Hom}_{\mathcal{F}}(A, S)$ and $\psi\varphi$ extends to $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(A, S)$, and $\widehat{\psi}(A) = A = \widehat{\varphi}(A)$ since A is weakly closed. Then $\widehat{\psi}^{-1}\widehat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(A)$, and $(\widehat{\psi}^{-1}\widehat{\varphi})|_P = \psi^{-1}(\psi\varphi) = \varphi$. □

The proof of the next lemma gives another example of how the extension axiom can be used.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

LEMMA 1.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $A_0 \leq A_1 \leq S$ be a pair of abelian subgroups. If A_0 is fully centralized in $\mathcal{F}$ and A_1 is fully centralized in $C_{\mathcal{F}}(A_0)$, then A_1 is fully centralized in $\mathcal{F}$.*

PROOF. Choose $B_1 \in A_1^{\mathcal{F}}$ that is fully centralized in $\mathcal{F}$, fix $\chi \in \mathrm{Iso}_{\mathcal{F}}(A_1, B_1)$, and set $B_0 = \chi(A_0)$. By the extension axiom and since A_0 and B_1 are both fully centralized in $\mathcal{F}$, there are $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(A_1), C_S(B_1))$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(B_0), C_S(A_0))$ such that $\varphi|_{A_1} = \chi$ and $\psi|_{B_0} = (\chi|_{A_0})^{-1}$. Since $C_S(B_1) \leq C_S(B_0)$, the composite $\psi\varphi$ lies in $\mathrm{Hom}_{C_{\mathcal{F}}(A_0)}(C_S(A_1), C_S(A_0))$.

Since A_1 is fully centralized in $C_{\mathcal{F}}(A_0)$,

$$\psi\varphi(C_S(A_1)) = C_{C_S(A_0)}(\psi(B_1)) = C_S(\psi(B_1)) \geq \psi(C_S(B_1)),$$

and hence $\varphi(C_S(A_1)) \geq C_S(B_1)$. So A_1 is fully centralized in $\mathcal{F}$ since B_1 is. □

We need to work with quotient fusion systems in Section 4, but only quotients by subgroups normal in the fusion system.

DEFINITION 1.10. Let $\mathcal{F}$ be a fusion system, and assume $Q \trianglelefteq S$ is normal in $\mathcal{F}$. Let $\mathcal{F}/Q$ be the fusion system over S/Q where for each $P, R \leq S$ containing Q, we set

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{F}/Q}(P/Q, R/Q)\\ &\quad = \{\varphi/Q \in \mathrm{Hom}(P/Q, R/Q) \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q),\ (\varphi/Q)(gQ) = \varphi(g)Q \text{ for all } g \in P\}.\end{aligned}$$

We refer to [Cr, Proposition II.5.11] for the proof that $\mathcal{F}/Q$ is saturated whenever $\mathcal{F}$ is. In fact, this definition and the saturation of $\mathcal{F}/Q$ hold whenever Q is weakly closed in $\mathcal{F}$. This is not surprising, since we are looking only at morphisms in $\mathcal{F}$ between subgroups containing Q, so that $\mathcal{F}/Q = N_{\mathcal{F}}(Q)/Q$.

2. Some criteria for realizing representations

In this section we state and prove our main technical results: the tools we later use to show that certain representations cannot be realized by any saturated fusion systems. Before doing that, we start by defining more formally what we mean by 'realizability'.

DEFINITION 2.1. Fix a prime p, a finite abelian p-group A, and a subgroup $\Gamma \leq \mathrm{Aut}(A)$. The pair (Γ, A) is *realized* by a saturated fusion system $\mathcal{F}$ over a finite p-group S if there is an abelian subgroup $B \leq S$ such that $C_S(B) = B$ and $B \ntrianglelefteq \mathcal{F}$, and such that $(\mathrm{Aut}_{\mathcal{F}}(B), B) \cong (\Gamma, A)$. The pair (Γ, A) is *fusion realizable* if it is realized by some saturated fusion system over a finite p-group.

If we drop the condition that $C_S(B) = B$, then it is easy to see that every pair (Γ, A) can be realized by a saturated fusion system. For example, if $m > 1$ is prime to p, then the fusion system $\mathcal{F}$ of $(A \rtimes \Gamma) \wr C_m$ contains a subgroup isomorphic to A with automizer isomorphic to Γ and which is not normal in $\mathcal{F}$. Hence, the importance of that condition in Definition 2.1, although it seems possible that we would get similar results if it were replaced by the condition that B be weakly closed.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

It is not yet clear to us whether the condition $B \ntrianglelefteq \mathcal{F}$ is the optimal one to use in Definition 2.1. It could be replaced by the slightly stronger condition that $\Omega_1(B) \ntrianglelefteq \mathcal{F}$, or by the even stronger condition that $O_p(\mathcal{F}) = 1$. In the cases dealt with in Theorems A and B, the result is the same independently of which definition we choose, but that probably does not hold in other situations.

When applying Definition 2.1, rather than assuming (Γ, A) and $(\mathrm{Aut}_{\mathcal{F}}(B), B)$ are abstractly isomorphic, it will in practice be more convenient to say that (Γ, A) is realized by a fusion system $\mathcal{F}$ over S if S contains A as a subgroup and $\mathrm{Aut}_{\mathcal{F}}(A) = \Gamma$.

We are now ready to start developing tools for showing that certain pairs (Γ, A) are not (weakly) fusion realizable. The starting point for all results in this section is the following proposition. It was inspired in part by [Gd, Corollary 4] and its proof, and also in part by arguments in [O'N, Section 1].

PROPOSITION 2.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $A \leq S$ be an abelian subgroup. Assume $A \ntrianglelefteq \mathcal{F}$, and consider the sets*

$$\begin{aligned}
\mathscr{U} &= \mathscr{U}_{\mathcal{F}}(A) = \{1 \neq U \leq N_S(A) \mid U \nleq A,\ \mathrm{Hom}_{\mathcal{F}}(U, A) \neq \emptyset\},\\
\mathscr{T} &= \mathscr{T}_{\mathcal{F}}(A) = \{t \in N_S(A) \setminus A \mid t^{\mathcal{F}} \cap A \neq \emptyset\} = \{t \in N_S(A) \setminus A \mid \langle t \rangle \in \mathscr{U}\},\\
\mathscr{W} &= \mathscr{W}_{\mathcal{F}}(A)\\
&= \{(t, U, A_*) \mid t \in \mathscr{T},\ U \in \mathscr{U},\ C_A(t) \geq A_* \in (U \cap A)^{\mathcal{F}}, |UA/A| = |C_{A/A_*}(t)|\}.
\end{aligned}$$

Then $\mathscr{U} \neq \emptyset$, $\mathscr{T} \neq \emptyset$, and $\mathscr{W} \neq \emptyset$, and the following assertions hold.

(a) *If A is not weakly closed in $\mathcal{F}$, there is $U \in A^{\mathcal{F}} \setminus \{A\}$ such that $[U, A] \leq U \cap A$, and such that $(t, U, U \cap A) \in \mathscr{W}$ for each $t \in U \setminus A$.*
(b) *If A is weakly closed in $\mathcal{F}$, then for each $t \in \mathscr{T}$, there are $U \in \mathscr{U}$ and $A_* \leq A$ such that $(t, U, A_*) \in \mathscr{W}$.*
(c) *If A is weakly closed in $\mathcal{F}$, then there is a subgroup $Z \leq A$, fully centralized in $\mathcal{F}$, such that $A \ntrianglelefteq C_{\mathcal{F}}(Z)$, and such that $U \cap A \leq Z$ for each $U \in \mathscr{U}_{C_{\mathcal{F}}(Z)}(A)$. In particular, $A_* = U \cap A$ for each $(t, U, A_*) \in \mathscr{W}_{C_{\mathcal{F}}(Z)}(A) \subseteq \mathscr{W}_{\mathcal{F}}(A)$.*

Thus in all cases, there are $t \in \mathscr{T}$ and $U \in \mathscr{U}$ such that $(t, U, U \cap A) \in \mathscr{W}$.

PROOF. By Lemma 1.5 and since $A \ntrianglelefteq \mathcal{F}$, A is not strongly closed. So $\mathscr{U} \neq \emptyset$ and $\mathscr{T} \neq \emptyset$ if $A \trianglelefteq S$, and we show when proving (a) that this also holds if $A \ntrianglelefteq S$. The last statement, and the claim that $\mathscr{W} \neq \emptyset$, follow from (a) when A is not weakly closed in $\mathcal{F}$, and from (b) and (c) otherwise.

(a) If A is not weakly closed in $\mathcal{F}$, then there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, S)$ such that $\varphi(A) \neq A$. So by Theorem 1.3 (Alperin's fusion theorem), there are $R \leq S$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $A \leq R$ and $\alpha(A) \neq A$. In the special case where $A \ntrianglelefteq S$, we take $R = N_S(A)$, and set $\alpha = c_x^R$ for some $x \in N_S(R) \setminus R$. So in all cases, we can arrange that $A \trianglelefteq R$ and hence $\alpha(A) \leq N_S(A)$.

Set $U = \alpha(A) \in \mathscr{U}$ and $A_* = U \cap A$. Then $[A, U] \leq A_*$ since A and U are both normal in R. So for each $t \in U \setminus A \subseteq \mathscr{T}$, we have $A_* \leq C_A(U) \leq C_A(t)$ and $|UA/A| = |U/A_*| = |A/A_*| = |C_{A/A_*}(t)|$, proving that $(t, U, A_*) \in \mathscr{W}$.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

(b) Assume A is weakly closed in $\mathcal{F}$ (in particular, $A \trianglelefteq S$). Fix $t \in \mathscr{T}$, and let $\mathscr{U}_t$ be the set of all $U \in \mathscr{U}$ such that $t \in U$. Choose $V \in \mathscr{U}_t$ such that $|V \cap A|$ is maximal among all $|U \cap A|$ for $U \in \mathscr{U}_t$. Set $A_* = V \cap A$ and $U_2^* = N_A(A_*\langle t\rangle)$. Then $A_*\langle t\rangle \cap A \leq V \cap A = A_*$, and so

$$U_2^*/A_* = \{x \in A \mid [x,t] \in A_*\}/A_* = C_{A/A_*}(t) \neq 1, \tag{2-1}$$

where $C_{A/A_*}(t) \neq 1$ since A/A_* and t both have p-power order.

Choose $W \in (A_*\langle t\rangle)^{\mathcal{F}}$ such that W is fully normalized in $\mathcal{F}$. Then $W \leq A$ by Lemma 1.8(a) and since A is weakly closed. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(A_*\langle t\rangle), S)$ be such that $\varphi(A_*\langle t\rangle) = W$ (see Proposition 1.4).

Set $U = \varphi(U_2^*)$ and $U_1^* = \varphi^{-1}(U \cap A)$. Then

$$\varphi(A_*) \leq \varphi(U_2^*) \cap A = U \cap A = \varphi(U_1^*),$$

so $A_* \leq U_1^* \leq U_2^* \leq A$. Also, $U_1^*\langle t\rangle \in \mathscr{U}_t$ since $\varphi(U_1^*\langle t\rangle) = (U \cap A)\langle \varphi(t)\rangle \leq A$, and hence

$$|U_1^*| \leq |U_1^*\langle t\rangle \cap A| \leq |V \cap A| = |A_*|$$

by the maximality assumption on V. Thus $U_1^* = A_* < U_2^*$ where the strict inclusion holds by (2-1), and $A_* = U_1^* \in (U \cap A)^{\mathcal{F}}$.

Now $U \cap A = \varphi(U_1^*) < \varphi(U_2^*) = U$, so $U \nleq A$. Since $U = \varphi(U_2^*)$, where $U_2^* \leq A$, this shows that $U \in \mathscr{U}$. Also, $U \cap A = \varphi(A_*)$, and so $UA/A \cong U/(U \cap A) \cong U_2^*/A_* = C_{A/A_*}(t)$. Thus $(t, U, A_*) \in \mathscr{W}$.

(c) Again assume A is weakly closed in $\mathcal{F}$, and let Z be maximal among all subgroups of A fully centralized in $\mathcal{F}$ such that $A \ntrianglelefteq C_{\mathcal{F}}(Z)$. Set $\mathcal{F}_0 = C_{\mathcal{F}}(Z)$ and $S_0 = C_S(Z)$ for short. Recall that $\mathcal{F}_0$ is saturated since Z is fully centralized in $\mathcal{F}$ (Theorem 1.7).

Fix $U \in \mathscr{U}_{\mathcal{F}_0}(A)$, choose a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(U, A)$, and set $A_* = U \cap A$. We must show that $A_* \leq Z$. Since $UZ \in \mathscr{U}_{\mathcal{F}_0}(A)$, we can assume $U \geq Z$.

Choose $B_* \in (A_*)^{\mathcal{F}_0}$ that is fully normalized in $\mathcal{F}_0$. Then $B_* \leq A$ by Lemma 1.8(a) and since A is weakly closed. By Proposition 1.4, there is $\chi \in \mathrm{Hom}_{\mathcal{F}_0}(N_{S_0}(A_*), S_0)$ such that $\chi(A_*) = B_*$. Then $\chi(A) = A$ since A is weakly closed, so $\chi\varphi(\chi|_U)^{-1} \in \mathrm{Hom}_{\mathcal{F}_0}(\chi(U), A)$ where $Z \leq \chi(U) \nleq A$ and $B_* = \chi(U \cap A) = \chi(U) \cap A$, and where $B_* \leq Z$ if and only if $A_* \leq Z$. Upon replacing U by $\chi(U)$ and φ by $\chi\varphi(\chi|_U)^{-1}$, we are now reduced to showing that $A_* \leq Z$ when $A_* = U \cap A$ is fully centralized in $\mathcal{F}_0$, and hence in $\mathcal{F}$ by Lemma 1.9.

By Lemma 1.8(b), there is an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}_0}(A)$ such that $\alpha|_{A_*} = \varphi|_{A_*}$, hence such that $\alpha^{-1}\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(A_*)}(U, A)$. Since $U \nleq A$, this implies that $A \ntrianglelefteq C_{\mathcal{F}}(A_*)$, and so $A_* = Z$ by the maximality assumption on Z.

In particular, for each $(t, U, A_*) \in \mathscr{W}_{\mathcal{F}_0}(A)$, since $U \cap A \leq Z$ and $A_* \in (U \cap A)^{\mathcal{F}_0}$, we have $U \cap A = A_* \leq Z$. □

We now reformulate the criteria in Proposition 2.2 in terms of A and $\mathrm{Aut}_{\mathcal{F}}(A)$ only, that is, in terms that do not involve the fusion system $\mathcal{F}$ or its Sylow group S.

DEFINITION 2.3. Fix a finite abelian p-group A and a p-subgroup $T \le \mathrm{Aut}(A)$. Set

$$\begin{aligned}\widehat{\mathscr{R}}^+_T(A) = \{(\tau, B, A_*) \mid \tau \in T^\#,\ B \le T,\ \langle\tau\rangle \text{ and } B \text{ isomorphic to subgroups of } A,\\ A_* \le C_A(\langle B, \tau\rangle),\ |B| \ge |C_{A/A_*}(\tau)|\},\end{aligned}$$

$$\widehat{\mathscr{R}}_T(A) = \{(\tau, B, A_*) \in \widehat{\mathscr{R}}^+_T(A) \mid |B| = |C_{A/A_*}(\tau)|\}.$$

Let $\mathscr{R}_T(A)$ be the largest subset $\mathcal{R} \subseteq \widehat{\mathscr{R}}_T(A)$ that satisfies the condition

$$\text{for each } (\tau, B, A_*) \in \mathcal{R} \text{ and each } \tau_1 \in B^\#, \text{ there is } (\tau_1, B_1, A_{*1}) \in \mathcal{R}. \qquad (*)$$

Similarly, let $\mathscr{R}^+_T(A)$ be the largest subset $\mathcal{R} \subseteq \widehat{\mathscr{R}}^+_T(A)$ that satisfies $(*)$.

If $\mathscr{R}_1$ and $\mathscr{R}_2$ are two subsets of $\widehat{\mathscr{R}}_T(A)$ or of $\widehat{\mathscr{R}}^+_T(A)$ that satisfy $(*)$, then their union also satisfies $(*)$. So there are unique largest subsets $\mathscr{R}_T(A) \subseteq \mathscr{R}^+_T(A)$ that satisfy the condition.

PROPOSITION 2.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $A \le S$ is an abelian subgroup such that $C_S(A) = A$ and $A \ntrianglelefteq \mathcal{F}$. Then $\mathscr{R}_{\mathrm{Aut}_S(A)}(A) \ne \emptyset$, and hence $\mathscr{R}^+_{\mathrm{Aut}_S(A)}(A) \ne \emptyset$. More precisely, the following assertions hold, where $T = \mathrm{Aut}_S(A)$.*

(a) *In all cases, if $(t, U, A_*) \in \mathscr{W}_{\mathcal{F}}(A)$ is such that $U \cap A = A_*$, then $(c^A_t, \mathrm{Aut}_U(A), A_*) \in \widehat{\mathscr{R}}_T(A)$.*
(b) *If A is not weakly closed in $\mathcal{F}$, then there is a subgroup $U \in A^{\mathcal{F}} \setminus \{A\}$ such that $(c^A_t, \mathrm{Aut}_U(A), A \cap U) \in \mathscr{R}_T(A)$ for each $t \in U \setminus A$.*
(c) *If A is weakly closed in $\mathcal{F}$, then there is a subgroup $Z \le A$ fully centralized in $\mathcal{F}$ such that $A \ntrianglelefteq C_{\mathcal{F}}(Z)$, and such that for each $t \in \mathscr{T}_{C_{\mathcal{F}}(Z)}(A)$, there is $U \in \mathscr{U}_{C_{\mathcal{F}}(Z)}(A)$ such that*

$$U \cap A \le Z \quad \textit{and} \quad (c^A_t, \mathrm{Aut}_U(A), U \cap A) \in \mathscr{R}_{C_T(Z)}(A) \subseteq \mathscr{R}_T(A).$$

PROOF. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S as above. Thus $A \le S$ is such that $C_S(A) = A$ and $A \ntrianglelefteq \mathcal{F}$. Once we have proven (a), (b), and (c), it will then follow immediately that $\mathscr{R}_T(A) \ne \emptyset$.

(a) Fix $(t, U, A_*) \in \mathscr{W}_{\mathcal{F}}(A)$ such that $A_* = U \cap A$, and set $\tau = c^A_t \in T$ and $B = \mathrm{Aut}_U(A) \le T$. Then $A_* = U \cap A \le C_A(B)$. Also, by definition of $\mathscr{W}_{\mathcal{F}}(A)$, we have $A_* \le C_A(t) = C_A(\tau)$ and $|UA/A| = |C_{A/A_*}(t)| = |C_{A/A_*}(\tau)|$.

By definition of $\mathscr{T}_{\mathcal{F}}(A)$ and $\mathscr{U}_{\mathcal{F}}(A)$, the subgroups $\langle\tau\rangle$ and B are both isomorphic to subgroups of A. So to prove that $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T(A)$, it remains only to show that $|UA/A| = |B|$. But $C_S(A) = A$ by assumption, so $|B| = |\mathrm{Aut}_U(A)| = |UA/A|$.

(b) If A is not weakly closed in $\mathcal{F}$, then by Proposition 2.2(a), there is $U \in A^{\mathcal{F}} \setminus \{A\}$ such that $[U, A] \le U \cap A$, and such that $(t, U, U \cap A) \in \mathscr{W}_{\mathcal{F}}(A)$ for each $t \in U \setminus A$. Thus $(c^A_t, \mathrm{Aut}_U(A), U \cap A) \in \widehat{\mathscr{R}}_{\mathcal{F}}(A)$ for each $t \in U \setminus A$ by (a).

Now set $\mathscr{R} = \{(\tau, \mathrm{Aut}_U(A), U \cap A) \mid \tau \in B^\#\} \subseteq \widehat{\mathscr{R}}_{\mathcal{F}}(A)$. Then $\mathscr{R}$ satisfies condition $(*)$ in Definition 2.3, so $\mathscr{R}_T(A) \supseteq \mathscr{R} \ne \emptyset$.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

(c) Assume A is weakly closed in $\mathcal{F}$, and let $Z \leq A$ be as in Proposition 2.2(c). Thus Z is fully centralized in $\mathcal{F}$, $A \ntrianglelefteq C_{\mathcal{F}}(Z)$, and $U \cap A \leq Z$ for each $U \in \mathscr{U}_{C_{\mathcal{F}}(Z)}(A)$.

Let $\mathscr{T} = \mathscr{T}_{C_{\mathcal{F}}(Z)}(A) \neq \emptyset$, $\mathscr{U} = \mathscr{U}_{C_{\mathcal{F}}(Z)}(A) \neq \emptyset$, and $\mathscr{W} = \mathscr{W}_{C_{\mathcal{F}}(Z)}(A) \neq \emptyset$ be as in Proposition 2.2, and set

$$\mathscr{R} = \{(c_t^A, \mathrm{Aut}_U(A), U \cap A) | t \in \mathscr{T},\ U \in \mathscr{U},\ (t, U, A_*) \in \mathscr{W}\},$$

where $A_* \in (U \cap A)^{C_{\mathcal{F}}(Z)}$ and hence $A_* = U \cap A$ since $U \cap A \leq Z$. By (a), $\mathscr{R} \subseteq \widehat{\mathscr{R}}_{C_T(Z)}(A)$. By Proposition 2.2(b),(c), for each $t \in \mathscr{T}$, there is $U \in \mathscr{U}$ such that $(t, U, U \cap A) \in \mathscr{W}$. So $\mathscr{R} \neq \emptyset$, and condition $(*)$ in Definition 2.3 holds for the pair $\mathscr{R}$. Thus $\mathscr{R} \subseteq \mathscr{R}_{C_T(Z)}(A) \subseteq \mathscr{R}_T(A)$. □

The next proposition is our main reason for defining $\mathscr{R}_T^+(A)$.

Proposition 2.5. *Fix a finite abelian p-group A and a p-subgroup $T \leq \mathrm{Aut}(A)$. Let $A_1 < A_2 \leq A$ be T-invariant subgroups such that T acts faithfully on A_2/A_1. If $\mathscr{R}_T^+(A) \neq \emptyset$, then $\mathscr{R}_T^+(A_2/A_1) \neq \emptyset$. More precisely,*

$$\mathscr{R}_T^+(A_2/A_1) \supseteq \{(\tau, B, (A_*A_1 \cap A_2)/A_1) \mid (\tau, B, A_*) \in \mathscr{R}_T^+(A)\}.$$

Proof. Assume $A_1 < A_2 \leq A$ are as above. If $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T^+(A)$, then

$$|C_{A_2/(A_*A_1 \cap A_2)}(\tau)| \leq |C_{A_2/(A_* \cap A_2)}(\tau)| = |C_{A_2A_*/A_*}(\tau)| \leq |C_{A/A_*}(\tau)| \leq |B|,$$

the first inequality by Lemma A.4 and the second by inclusion. So we have $(\tau, B, (A_*A_1 \cap A_2)/A_1) \in \widehat{\mathscr{R}}_T^+(A_2/A_1)$.

In particular, if $\mathscr{R}$ satisfies condition $(*)$ in Definition 2.3 for the pair (T, A), then $\mathscr{R}'$ satisfies $(*)$ for $(T, A_2/A_1)$, where

$$\mathscr{R}' = \{(\tau, B, (A_*A_1 \cap A_2)/A_1) \mid (\tau, B, A_*) \in \mathscr{R}\}.$$ □

It remains to find some strong necessary conditions on A and T for the set $\mathscr{R}_T(A)$ or $\mathscr{R}_T^+(A)$ to be nonempty.

Proposition 2.6. *Fix a finite abelian p-group A and a subgroup $T \leq \mathrm{Aut}(A)$. Then for each $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T(A)$,*

$$|B| = \frac{|A|}{|A_*[\tau, A]|} \quad \text{and} \quad \frac{|B|}{|C_A(\tau) \cap [\tau, A]|} = \frac{|C_A(\tau)[\tau, A]|}{|A_*[\tau, A]|}, \tag{2-2}$$

while for each $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T^+(A)$,

$$|B| \geq \frac{|A|}{|A_*[\tau, A]|} \quad \text{and} \quad \frac{|B|}{|C_A(\tau) \cap [\tau, A]|} \geq \frac{|C_A(\tau)[\tau, A]|}{|A_*[\tau, A]|} \geq 1. \tag{2-3}$$

In particular, for each $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T^+(A)$,

$$|B| \geq |C_A(\tau) \cap [\tau, A]|, \tag{2-4}$$

and $|B| \geq |[\tau, A]|$ if $p = 2$ and A is elementary abelian.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

PROOF. For each $\tau \in T^\#$, let $\varphi_\tau \in \mathrm{End}(A)$ be the map $\varphi_\tau(a) = [\tau, a]$. For each $A_* \le C_A(\tau)$, we have $C_A(\tau) = \mathrm{Ker}(\varphi_\tau)$ and $C_{A/A_*}(\tau) = \varphi_\tau^{-1}(A_*)/A_*$, and hence

$$|C_{A/A_*}(\tau)| = \frac{|C_A(\tau)| \cdot |A_* \cap [\tau, A]|}{|A_*|} = \frac{|C_A(\tau)| \cdot |[\tau, A]|}{|A_*[\tau, A]|} = \frac{|A|}{|A_*[\tau, A]|}$$
$$= \frac{|C_A(\tau)[\tau, A]| \cdot |C_A(\tau) \cap [\tau, A]|}{|A_*[\tau, A]|}. \tag{2-5}$$

Since $|B| \ge |C_{A/A_*}(\tau)|$ for each $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T^+(A)$ with equality if $(\tau, B, A_*) \in \widehat{\mathscr{R}}_T(A)$, points (2-2) and (2-3) follow immediately from (2-5) (and since $A_* \le C_A(\tau)$). Inequality (2-4) follows from (2-3), and the last statement holds since $[\tau, A] \le C_A(\tau)$ if $p = 2$ and A is elementary abelian. □

The following corollary describes one easy consequence of the above results.

COROLLARY 2.7. *Fix a finite abelian p-group A and a p-subgroup $T \le \mathrm{Aut}(A)$ such that $\mathscr{R}_T^+(A) \ne \emptyset$. Then there is $B_0 \le T$, isomorphic to a subgroup of A, such that $|B_0| \ge |C_A(\tau) \cap [\tau, A]|$ for each $\tau \in B_0^\#$.*

PROOF. Assume $\mathscr{R}_T^+(A) \ne \emptyset$. Choose $(\tau_0, B_0, A_{*0}) \in \mathscr{R}_T^+(A)$ such that $|C_A(\tau_0) \cap [\tau_0, A]|$ is the largest possible. By condition $(*)$ in Definition 2.3, for each $\tau \in B_0^\#$, there is $(\tau, B, A_*) \in \mathscr{R}_T^+(A)$, and hence

$$|C_A(\tau) \cap [\tau, A]| \le |C_A(\tau_0) \cap [\tau_0, A]| \le |B_0|,$$

where the second inequality holds by (2-4). □

We can think of the inequality $|B_0| \ge |C_A(\tau) \cap [\tau, A]|$ in Corollary 2.7 as a generalization of the condition $|Z(S) \cap [S, S]| = p$ in [O1, Lemma 2.3(b)]. More precisely, when A has index p in S and S is nonabelian, the corollary says that $|C_A(\tau) \cap [A, \tau]| = p$ for $\tau \in S \setminus A$, and hence that $|Z(S) \cap [S, S]| = p$.

We next look at the case where A is elementary abelian. For $\tau \in \mathrm{End}(A)$, we regard A as an $\mathbb{F}_p[X]$-module, and let the 'Jordan blocks' for τ be the factors under some decomposition of A as a product of indecomposable submodules. As usual, by 'nontrivial Jordan blocks' we really mean 'Jordan blocks with nontrivial action'.

The following notation will be used when reformulating Corollary 2.7 in terms of Jordan blocks.

NOTATION 2.8. Let A be an elementary abelian p-group, and let $\tau \in \mathrm{Aut}(A)$ be an automorphism of p-power order. Set $\mathscr{J}_A(\tau) = \mathrm{rk}(C_A(\tau) \cap [\tau, A])$, the number of nontrivial Jordan blocks for the action of τ on A.

In these terms, Corollary 2.7 takes the following form when A is elementary abelian.

COROLLARY 2.9. *Assume Γ is a finite group such that $\Gamma = O^{p'}(\Gamma)$, and let A be a finite faithful $\mathbb{F}_p\Gamma$-module. Assume there is a saturated fusion system $\mathcal{F}$ over a finite p-group S that realizes (Γ, A) as in Definition 2.1. Then there are $m \ge 1$ and an elementary abelian p-subgroup $B \le \Gamma$ of rank m such that $\mathscr{J}_A(\tau) \le m$ for each $\tau \in B^\#$.*

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

PROOF. Since $\mathcal{F}$ realizes (Γ, A), we can arrange that $A \le S$, $A \ntrianglelefteq \mathcal{F}$, and $\mathrm{Aut}_{\mathcal{F}}(A) = \Gamma$. Set $T = \mathrm{Aut}_S(A) \in \mathrm{Syl}_p(\Gamma)$. Then $\mathscr{R}_T^+(A) \ne \emptyset$ by Proposition 2.4. So by Corollary 2.7, there is an elementary abelian p-subgroup $B \le \Gamma$ such that $|B| \ge |C_A(\tau) \cap [\tau, A]|$ for all $\tau \in B^\#$. Thus $\mathrm{rk}(B) \ge \mathrm{rk}(C_A(\tau) \cap [A, \tau]) = \mathscr{J}_A(\tau)$ for each $\tau \in B^\#$. □

The special case of fusion realizability when $|T| = p$ has already been handled in the earlier papers [COS, O1]. We state the main conditions found in those papers.

LEMMA 2.10. *Fix a finite abelian p-group A and subgroups $\Gamma \le \mathrm{Aut}(A)$ and $T \in Syl_p(\Gamma)$, and assume that $|T| = p$ and $|[T, A]| > p$. If (Γ, A) is fusion realizable, then*

$$|C_A(T) \cap [T, A]| = p \quad \textit{and} \quad |N_\Gamma(T)/C_\Gamma(T)| = p - 1.$$

PROOF. The first equality is just a special case of Corollary 1.7.

To see the second equality, assume that (Γ, A) is realized by the fusion system $\mathcal{F}$ over $S \ge A$. In particular, we can assume that $\mathrm{Aut}_S(A) = T$, and so $|N_S(A)/A| = |T| = p$. Also, $|A/C_A(T)| = |[T, A]| > p$ by assumption, so A is the only abelian subgroup of index p in $N_S(A)$. Hence, $A \trianglelefteq S$, since otherwise $A \ne {}^xA \le N_S(A)$ for $x \in N_S(N_S(A)) \setminus N_S(A)$.

By Theorem 1.3 and since $A \ntrianglelefteq \mathcal{F}$ (recall that $\mathcal{F}$ realizes (Γ, A)), there must be some $\mathcal{F}$-essential subgroup $P \le S$ other than A, and by [COS, Lemma 2.2(a)], $P \in \mathcal{H} \cup \mathcal{B}$ where the classes $\mathcal{H}$ and $\mathcal{B}$ of subgroups of S are defined in [COS, Notation 2.1]. By [COS, Lemma 2.6(a)] (and in terms of Notation 2.4 in [COS]), we have $\mu(\mathrm{Aut}_{\mathcal{F}}{}^{(P)}(S)) = \Delta_t$ for $t = 0$ or -1, and from the definition of μ it then follows that $\mathrm{Aut}_\Gamma(T) = \mathrm{Aut}(T)$ and hence has order $p - 1$. □

3. Representations of Mathieu groups

We next look at representations of the Mathieu groups M_n and their central extensions. The main theorem is stated for an arbitrary prime p, but we focus attention mostly on the cases $p = 2, 3$, since the others follow from Lemma 2.10 and results in [COS].

We apply Corollary 2.9 in most cases, using Lemma A.1 and the character tables in [JLPW] to find lower bounds for $\mathscr{J}_A(x)$ when $|x| = 2$ or 3. The notation **2X** and **3X** refers to the classes as named in the Atlas [Atl] and in [JLPW]. In the following lemma, we restrict attention to M_{12} and M_{24} since they are the only Mathieu groups with more than one conjugacy class of elements of order 2 or 3.

LEMMA 3.1. *Assume $\Gamma \cong M_{12}$ or M_{24}. Then*

(a) *each element of order 2 in Γ is contained in some $H_1 \le \Gamma$ with $H_1 \cong D_{10}$; and*
(b) *each element of order 3 in Γ is contained in some $H_3 \le \Gamma$ with $H_3 \cong A_4$, and with elements of order 2 in class* **2A** *(if $\Gamma \cong M_{12}$) or* **2B** *(if $\Gamma \cong M_{24}$).*

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

TABLE 1. The number of orbits in the action on X by each element of order 2 or 3 in Γ. For example, a **2B**-element in M_{12} acts with four orbits of length 2 and four fixed points.

Γ	**2A**	**2B**	**3A**	**3B**
M_{12}	2^6	$2^4 \cdot 1^4$	$3^3 \cdot 1^3$	3^4
M_{24}	$2^8 \cdot 1^8$	2^{12}	$3^6 \cdot 1^6$	3^8

PROOF. Let $n = 12, 24$ be such that $\Gamma \cong M_n$, and let X be a 5-fold transitive Γ-set of order n. In each case, Γ has two classes of elements of order 2 and two classes of elements of order 3, and they are distinguished by whether they act on X freely or with fixed points as described in Table 1. The outer automorphism of M_{12} sends each of these classes to itself, and so the inclusion of $\mathrm{Aut}(M_{12})$ into M_{24} sends distinct classes to distinct classes. It thus suffices to prove the lemma when $\Gamma \cong M_{12}$.

(a) Fix an element $g \in \mathbf{2A}$. By [GL, page 41], $C_\Gamma(g) \cong C_2 \times \Sigma_5$, and the second factor must faithfully permute the six orbits under the action of g. Fix $N \leq C_\Gamma(g)$ of order 5, and let $h \in C_\Gamma(g) \setminus \langle g \rangle$ be such that $N\langle h \rangle \cong D_{10}$. Then $N\langle gh \rangle \cong D_{10}$, and we are done upon showing that h and gh lie in different classes.

Set $X_0 = C_X(N)$, a subset of order 2 whose elements are exchanged by g, and set $X_1 = X \setminus X_0$. Of the two elements h and gh, one fixes the two points in X_0 and the other exchanges them, and we can assume that h fixes them. Hence, $C_X(h) \neq \emptyset$, so $h \in \mathbf{2B}$. Also, $C_X(gh) \subseteq X_1$, and since gh freely permutes four of the five $\langle g \rangle$-orbits in X_1, we have $|C_X(gh)| \leq 2$. Since no involution in M_{12} acts with exactly two fixed points, this shows that $gh \in \mathbf{2A}$, finishing the proof of (a).

(b) Now fix an element $g \in \mathbf{3A}$. Then $C_\Gamma(g) \cong C_3 \times A_4$ by [GL, page 41]. Set $N = O_2(C_\Gamma(g)) \cong E_4$. The group $C_\Gamma(g)/\langle g \rangle \cong A_4$ acts faithfully on the set of four orbits of g, so the elements of order 2 in N all act freely on X and hence lie in class **2A**.

Fix $h \in C_\Gamma(g)$ such that $N\langle h \rangle \cong A_4$. Then $N\langle gh \rangle$ and $N\langle g^2h \rangle$ are also isomorphic to A_4. Also h freely permutes three of the four $\langle g \rangle$-orbits in X, and the fourth orbit is fixed by exactly one of the elements h, gh, or g^2h. So one of these three elements lies in class **3A**, and the other two in class **3B**. □

There are two special cases that we need to consider separately. The statement and proof of the following proposition are based on notation set up in Appendices B and C.

PROPOSITION 3.2. *Assume $p = 2$.*

(a) *If $\Gamma \cong M_{22}$ or M_{23} and A is the Golay module (dual Todd module) for Γ, then $\mathscr{R}^+_T(A) = \emptyset$ for $T \in Syl_2(\Gamma)$.*

(b) *If $\Gamma \cong 3M_{22}$ and A is the six-dimensional simple $\mathbb{F}_4\Gamma$-module, then $\mathscr{R}^+_T(A) = \emptyset$ for $T \in Syl_2(\Gamma)$.*

PROOF. In the first part of the proof, we consider cases (a) and (b) together. Assume that the proposition is not true, and fix a triple $(\tau, B, A_*) \in \mathscr{R}^+_T(A)$. Thus $\tau \in T$ has order 2, $B \leq T$ is an elementary abelian 2-subgroup, and $A_* \leq C_A(\langle B, \tau \rangle)$ is such that

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

$|B| \geq |C_{A/A_*}(\tau)|$. By Proposition 2.6 and since $[\tau, A] \leq C_A(\tau)$, we have

$$|B| \geq |[\tau, A]| \cdot |C_A(\tau)/A_*[\tau, A]| \geq |[\tau, A]|. \tag{3-1}$$

Since $\Gamma \cong M_{22}$, M_{23}, or $3M_{22}$ has only one conjugacy class of involution, we have $|[\tau, A]| = 2^4$: by Lemma B.3 in case (a), and by Lemma C.5(b) in case (b). Thus $|B| \geq 2^4$, with equality since $\mathrm{rk}_2(\Gamma) = 4$ in all cases. So the inequalities in (3-1) are equalities, $C_A(\tau) = A_*[\tau, A]$, and hence

$$\mathrm{rk}(C_A(B)) \geq \mathrm{rk}(A_*) \geq \mathrm{rk}(C_A(\tau)/[\tau, A]) = \mathrm{rk}(A) - 2 \cdot \mathrm{rk}([\tau, A]) = \mathrm{rk}(A) - 8. \tag{3-2}$$

(a) Assume $T < \Gamma$ and A are as in Notations B.1 and B.2. Since H_1 and H_2 are the only subgroups of $T \in \mathrm{Syl}_2(\Gamma_0)$ isomorphic to E_{16} by Lemma B.3, B must be equal to one of them. Since $C_A(H_1)$ has rank 1 by Lemma B.3 again, and $\mathrm{rk}(C_A(B)) \geq \mathrm{rk}(A) - 8 \geq 2$ by (3-2), we have $B = H_2$.

By condition $(*)$ in Definition 2.3, each element of $B^\#$ can appear as the first component in an element of $\mathscr{R}_T^+(A)$. So we can assume that (τ, B, A_*) was chosen such that $\tau = \mathfrak{tr}_{h_1}$ (and still $B = H_2$). Hence, by Tables 6 and 7,

$$\mathfrak{gr}_{h_2} + C_{56} \in C_A(\mathfrak{tr}_{h_1}) = A_*[\mathfrak{tr}_{h_1}, A] \leq C_A(H_2)[\mathfrak{tr}_{h_1}, A] = \langle C_{12}, C_{13}, C_{14}, C_{15}, \mathfrak{gr}_{h_1}\rangle,$$

a contradiction. We conclude that $\mathscr{R}_T^+(A) = \emptyset$.

(b) Now assume $T < \Gamma$ and A are as in Notations C.2 and C.3. By Lemma C.5(a), P_1 and P_2 are the only subgroups of T isomorphic to E_{16}. Since $\mathrm{rk}(C_A(P_2)) = 2 \cdot \dim_{\mathbb{F}_4}(C_A(P_2)) = 2$ by Lemma C.5(b), while $\mathrm{rk}(C_A(B)) \geq 4$ by (3-2), we have $B = P_1$. By condition $(*)$ in Definition 2.3, we can assume that the triple (τ, P_1, A_*) was chosen so that $\tau = \mu_{10}$. But then

$$\langle e_1, e_2, e_3, e_4\rangle = C_A(\mu_{10}) = A_*[\mu_{10}, A] \leq C_A(P_1)[\mu_{10}, A] = \langle e_1, e_2, e_3\rangle$$

by Lemma C.5(b), a contradiction. □

We now apply Corollary 2.9 and Lemma A.1, together with Proposition 3.2, to determine the realizability of $\mathbb{F}_p\Gamma$-modules when $O^{p'}(\Gamma)$ is a central extension of a Mathieu group. The following is a restatement of Theorem A.

THEOREM 3.3. *Fix a prime p and a finite group Γ, and set $\Gamma_0 = O^{p'}(\Gamma)$. Assume that Γ_0 is quasisimple, and that $\Gamma_0/Z(\Gamma_0)$ is one of the Mathieu groups. Let A be an $\mathbb{F}_p\Gamma$-module such that (Γ, A) is fusion realizable, and set $A_0 = [\Gamma_0, A]/C_{[\Gamma_0,A]}(\Gamma_0)$. Then either*

(a) *$p = 2$, $\Gamma \cong M_{22}$ or M_{23}, and A_0 is the Todd module for Γ; or*
(b) *$p = 2$, $\Gamma \cong M_{24}$, and A_0 is the Todd module or Golay module for Γ; or*
(c) *$p = 3$, $\Gamma \cong M_{11}$, $M_{11} \times C_2$, or $2M_{12}$, and A_0 is the Todd module or Golay module for Γ_0; or*
(d) *$p = 11$, $\Gamma_0 \cong 2M_{12}$ or $2M_{22}$, $\Gamma/Z(\Gamma_0) \cong \mathrm{Aut}(M_{12}) \times C_5$ or $\mathrm{Aut}(M_{22}) \times C_5$, and A_0 is a 10-dimensional simple $\mathbb{F}_{11}\Gamma$-module.*

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

TABLE 2. In all cases, A_0 is an $\mathbb{F}_2\Gamma$-module such that $C_{A_0}(\Gamma)=0$ and $[\Gamma,A_0]=A_0$, and the characters are taken with respect to $\mathbb{F}_2$. The bounds for $\mathscr{J}_{A_0}(\tau)$ all follow from Lemmas 3.1(a) and A.1(a).

Γ_0	$\mathrm{rk}_2(\Gamma_0)$	$\dim(A_0)$	$\tau\in$	$\mathscr{J}_{A_0}(\tau)$
M_{11}	2	>1	**2A**	$\mathscr{J}_{A_0}(\mathbf{2A})\ge\frac{2}{5}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{5A}))\ge 4$
M_{12}	3	>1	**2A, 2B**	$\mathscr{J}_{A_0}(\mathbf{2X})\ge\frac{2}{5}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{5A}))\ge 4$
M_{22}	4	>10	**2A**	$\mathscr{J}_{A_0}(\mathbf{2A})\ge\frac{2}{5}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{5A}))\ge 8$
$3M_{22}$	4	>12	**2A**	$\mathscr{J}_{A_0}(\mathbf{2A})\ge\frac{2}{5}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{5A}))\ge 6$
M_{23}	4	>11	**2A**	$\mathscr{J}_{A_0}(\mathbf{2A})\ge\frac{2}{5}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{5A}))\ge 8$
M_{24}	6	>11	**2A, 2B**	$\mathscr{J}_{A_0}(\mathbf{2X})\ge\frac{2}{5}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{5A}))\ge 8$

TABLE 3. In all cases, A_0 is an $\mathbb{F}_3\Gamma$-module such that $C_{A_0}(\Gamma)=0$ and $[\Gamma,A_0]=A_0$, and the characters are taken with respect to $\mathbb{F}_3$. Thus when $\Gamma\cong 2M_{22}$, the character values for the simple 10-dimensional $\overline{\mathbb{F}}_3\Gamma$-module are doubled here since it can only be realized over $\mathbb{F}_9$. When $\Gamma\cong M_{11}$, the bounds for $\mathscr{J}_{A_0}(\tau)$ apply only when A_0 is not the 10-dimensional permutation module. The bounds for $\mathscr{J}_{A_0}(\tau)$ all follow from Lemmas 3.1 and A.1(c), except when $\Gamma\cong M_{11}$ or $2M_{12}$ where Lemma A.1(d) is used.

Γ	$\mathrm{rk}_3(\Gamma)$	$\dim(A_0)$	$\tau\in$	$\mathscr{J}_{A_0}(\tau)$
M_{11}	2	≥ 10 (*)	**3A**	$\mathscr{J}_{A_0}(\mathbf{3A})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{4A}))\ge\frac{5}{2}$
M_{12}	2	>1	**3A, 3B**	$\mathscr{J}_{A_0}(\mathbf{3X})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{2A}))\ge 3$
$2M_{12}$	2	>6	**3A, 3B**	$\mathscr{J}_{A_0}(\mathbf{3X})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{2A}))\ge\frac{5}{2}$
M_{22}	2	>1	**3A**	$\mathscr{J}_{A_0}(\mathbf{3A})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{2A}))\ge 4$
$2M_{22}$	2	>1	**3A**	$\mathscr{J}_{A_0}(\mathbf{3A})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{2A}))\ge 4$
M_{23}	2	>1	**3A**	$\mathscr{J}_{A_0}(\mathbf{3A})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{2A}))\ge 4$
M_{24}	2	>1	**3A, 3B**	$\mathscr{J}_{A_0}(\mathbf{3X})\ge\frac{1}{4}(\chi_{A_0}(1)-\chi_{A_0}(\mathbf{2B}))\ge 6$

PROOF. Let $n\in\{11,12,22,23,24\}$ be such that $\Gamma_0/Z(\Gamma_0)\cong M_n$. Fix $T\in\mathrm{Syl}_p(\Gamma)=\mathrm{Syl}_p(\Gamma_0)$. We frequently refer to Tables 2 and 3 for our lower bounds on $\mathscr{J}_A(\tau)$ for $|\tau|=p$, and they in turn are based on Lemmas 3.1 and A.1 and the character tables in the Atlas of Brauer characters [JLPW].

Case 1. If $p>3$, then $|T|=p$ in all cases. So by Lemma 2.10, we have $|N_\Gamma(T)/C_\Gamma(T)|=p-1$ and $|C_A(T)\cap[T,A]|=p$. In the terminology of [COS], this translates to saying that $\Gamma\in\mathscr{G}_p^\wedge$ and A is minimally active, and so the result follows from [COS, Proposition 7.1].

Case 2. Assume $p=2$. By Table 2, for $\tau\in\Gamma$ of order 2, we have $\mathscr{J}_{A_0}(\tau)>\mathrm{rk}_2(\Gamma)$ (and hence $\mathscr{R}_T^+(A_0)=\emptyset$) for each nontrivial simple $\mathbb{F}_2\Gamma_0$-module A_0, except when $\Gamma_0\cong M_{22}$, M_{23}, or M_{24} and A_0 is the Todd module or Golay module.

Thus if $Z(\Gamma_0)$ has odd order, then either $n\ge 22$ and A_0 is the Todd module or Golay module for Γ, or $\Gamma_0\cong 3M_{22}$ and A_0 is the six-dimensional $\mathbb{F}_4\Gamma_0$-module. In these cases, $\mathscr{R}_T^+(A_0)=\emptyset$ by Proposition 3.2, and so they are impossible by Propositions 2.4 and 2.5.

It remains to consider the cases where $Z(\Gamma)$ has even order. Assume first that $\Gamma_0\cong 2M_{12}$. Then $\mathrm{rk}_2(\Gamma)=4$, and $\mathscr{J}_{A_0}(\tau)\ge 4$ for each $\mathbb{F}_2[\Gamma/Z(\Gamma)]$-module A_0 with

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

nontrivial action by Table 2. By the last statement in Lemma A.2 (applied with A in the role of V), for each elementary abelian 2-subgroup $B \le G$ of rank 4, since $Z(\Gamma) \le B$, there is $\tau \in B$ of order 2 such that $\mathscr{J}_A(\tau) \ge 5$. So Corollary 2.9 again applies to show that (Γ, A) is not fusion realizable.

Now assume that $\Gamma_0/Z(\Gamma_0) \cong M_{22}$, and let $Z \le Z(\Gamma)$ be the Sylow 2-subgroup. Thus $|Z| = 2$ or 4, and $\mathrm{rk}_2(\Gamma_0) \le 5$. By Table 2 and since $\mathscr{J}_{A_0}(\tau) \le \mathrm{rk}_2(\Gamma_0)$, either $\Gamma_0/Z \cong M_{22}$ and A_0 is its Todd module or its dual, or $\Gamma_0/Z \cong 3M_{22}$ and A_0 is the six-dimensional $\mathbb{F}_4\Gamma/Z$-module. By Lemma A.2(b) and since Γ acts faithfully on A, there must be indecomposable extensions of A_0 by $\mathbb{F}_2$ and of $\mathbb{F}_2$ by A_0. Thus $H^1(\Gamma/Z; A_0) \ne 0$ and $H^1(\Gamma/Z; A_0^*) \ne 0$ (where A_0^* is the dual module), contradicting [MS, Lemma 6.1]. We conclude that no such faithful $\mathbb{F}_2\Gamma$-modules exist.

Case 3. Assume $p = 3$. We claim that $\mathscr{J}_{A_0}(\tau) > \mathrm{rk}_3(\Gamma_0)$ (and hence (Γ, A) is not fusion realizable) in all cases except when $\Gamma_0 \cong M_{11}$ or $2M_{12}$ and A_0 is the Todd module for Γ_0 or its dual. This follows from Table 3 except when $\Gamma_0 \cong M_{11}$, $\dim(A_0) = 10$, and $A_0 \oplus \mathbb{F}_3$ is the 11-dimensional permutation module. But in that case, $\mathscr{J}_{A_0}(\tau) = 3$ whenever $|\tau| = 3$ since τ acts on an 11-set with three free orbits.

Finally, if $\Gamma_0 \cong M_{11}$ or $2M_{12}$ and A is the Todd module or its dual, then A is absolutely irreducible by [O2, Lemmas 4.2 and 5.2], and hence $\Gamma \cong M_{11}$, $M_{11} \times C_2$, or $2M_{12}$. □

4. Alperin's 2-groups of normal rank 3

As an example of how the results in Section 2 can be applied when the abelian p-subgroup $A < S$ is not elementary abelian, we next look at some 2-groups first studied by Alperin [Alp] and O'Nan [O'N]. These are groups $A \trianglelefteq S$ where $A \cong C_{2^n} \times C_{2^n} \times C_{2^n}$ and $S/A \cong D_8$, with presentation given in Table 4. They are characterized by Alperin [Alp, Theorem 1] as the Sylow 2-subgroups of groups G with normal subgroup $E \cong E_8$, such that $O(G) = 1$, $\mathrm{Aut}_G(E) = \mathrm{Aut}(E)$, and all involutions in $C_G(E)$ lie in E. Our goal is to show how results from Section 2 can be applied to prove in the context of fusion systems a theorem of O'Nan's, by showing that A is normal in all saturated fusion systems over S [O'N, Lemma 1.10].

Before considering the groups $A \trianglelefteq S$ directly, we must first handle the following, simpler case (compare with [O'N, Lemma 1.7]).

Lemma 4.1. *Fix $n \ge 2$, and let $\widehat{S} = \langle v, w, \sigma\rangle$ be a group of order 2^{2n+2}, where $\widehat{A} = \langle v, w\rangle \cong C_{2^n} \times C_{2^n}$, and $\widehat{S} = \widehat{A} \rtimes \langle\sigma\rangle$ where $\sigma^4 = 1$, $v^\sigma = w$, and $w^\sigma = v^{-1}$. Then $\widehat{A}$ is normal in every saturated fusion system over $\widehat{S}$.*

Proof. Assume otherwise: assume $\mathcal{F}$ is a saturated fusion system over $\widehat{S}$ for which $\widehat{A} \ntrianglelefteq \mathcal{F}$. Thus some element $t \in \widehat{S} \setminus \widehat{A}$ is $\mathcal{F}$-conjugate to an element of $\widehat{A}$, and upon replacing t by t^2 if necessary, we can arrange that $t \in \sigma^2\widehat{A}$. Since $|C_{\widehat{A}}(\sigma)| = 2$ and $|C_{\widehat{A}}(\sigma^2)| = 4$, each abelian subgroup of $\widehat{S}$ not contained in $\widehat{A}$ has order at most 8, and hence $\widehat{A}$ is weakly closed in $\mathcal{F}$.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

TABLE 4. Let $S = A\langle s, t\rangle$, where $A = \langle v_1, v_2, v_3\rangle \cong C_{2^n} \times C_{2^n} \times C_{2^n}$, the elements s and t act on A as described in the table, and also $t^2 = 1$ and $s^4 \in \langle v_1v_3\rangle$. Set $T = \mathrm{Aut}_S(A) = \langle c_s, c_t\rangle \cong D_8$.

v	v^t	v^s	v^{s^2}	v^{st}
v_1	v_3^{-1}	v_2	v_3	v_2^{-1}
v_2	v_2^{-1}	v_3	$v_1v_2^{-1}v_3$	v_1^{-1}
v_3	v_1^{-1}	$v_1v_2^{-1}v_3$	v_1	$v_1^{-1}v_2v_3^{-1}$

By Proposition 2.2(b),(c) and since $\widehat{A}$ is weakly closed in $\mathcal{F}$, there is $U \leq \widehat{S}$ that is $\mathcal{F}$-conjugate to a subgroup of $\widehat{A}$ and such that $(t, U, U \cap \widehat{A}) \in \mathscr{W}_{\mathcal{F}}(\widehat{A})$. In particular, $|U\widehat{A}/\widehat{A}| = |C_{\widehat{A}/(U\cap\widehat{A})}(t)|$.

Since conjugation by t sends each element of $\widehat{A}$ to its inverse, $U \cap \widehat{A} \leq C_{\widehat{A}}(t) = \Omega_1(\widehat{A})$, and hence $C_{\widehat{A}/(U\cap\widehat{A})}(t) = \Omega_1(\widehat{A}/(U \cap \widehat{A}))$ has order 4. Thus $|U\widehat{A}/\widehat{A}| = 4$, and so there is $u \in U$ such that $u \in \sigma\widehat{A}$.

We claim that for each $U^* \in U^{\mathcal{F}}$, either $U^*\widehat{A} = \widehat{S}$ or $U^* \leq \widehat{A}$. Assume otherwise; then $U^*\widehat{A} = \widehat{A}\langle\sigma^2\rangle$. So $U^* \cap \widehat{A} \leq C_{\widehat{A}}(\sigma^2) = \Omega_1(\widehat{A})$, and U^* is elementary abelian since each element of $\sigma^2\widehat{A}$ has order 2. Since $U \cong U^*$ is not elementary abelian (recall that $|u| = 4$), this is impossible.

By Theorem 1.3 (Alperin's fusion theorem), there is a subgroup $R \leq \widehat{S}$, together with an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ and subgroups A_1 and $U_1 = \alpha(A_1)$, such that $A_1, U_1 \in U^{\mathcal{F}}$, $A_1 \leq \widehat{A}$, and $U_1 \nleq \widehat{A}$. We just saw that this implies $U_1\widehat{A} = \widehat{S}$. So $\widehat{A} \cap R$ contains a cyclic subgroup of order 4 and is normalized by σ. Hence, $R \geq \langle v^{2^{n-1}}, (vw)^{2^{n-2}}\rangle$, and so $[R, R] \geq \Omega_1(\widehat{A})$. Since α sends some element of $\Omega_1(\widehat{A})$ to an element in the coset $\sigma^2\widehat{A} \nsubseteq [R, R]$, this is impossible. □

Lemma 4.1 can also be proved using the transfer for $\mathcal{F}$ (see, for example, [AKO, Section I.8]) to show that no element x^2, for $x \in \sigma\widehat{A}$, can be in the focal subgroup of $\mathcal{F}$. Such an argument would be closer to that used by O'Nan in the proof of [O'N, Lemma 1.7], but we wanted to apply the tools used elsewhere in this paper.

We now return to the groups $A \trianglelefteq S$ defined by the presentation in Table 4. We first check that when $n \geq 2$, A is weakly closed in every saturated fusion system over S.

LEMMA 4.2 [O'N, Lemma 1.5]. *Let $S = A\langle s, t\rangle$ be an extension of the form described in Table 4, where $n \geq 2$. Then A is the only abelian subgroup of index 8 in S, and hence is weakly closed in every saturated fusion system over S.*

PROOF. This follows immediately from the centralizers listed in Table 5, since if $A_1 < S$ were abelian of index 8 and $A_1 \neq A$, then for $x \in A_1 \setminus A$ the subgroup $C_A(x) \geq A \cap A_1$ would have index at most 4 in A. □

The arguments used in the proof of the following theorem are essentially the same as O'Nan's (when proving Lemma 1.10 in [O'N]), but repackaged with the help of Proposition 2.4 and the properties of the sets $\mathscr{R}_T(A)$.

TABLE 5. Centralizers and commutators involving some of the abelian subgroups $H \le \langle s, t\rangle$. Here, $\varepsilon = 2^{n-1}$ and $\delta = 2^{n-2}$.

H	$\langle t\rangle$	$\langle s^2\rangle$	$\langle st\rangle$	$\langle s\rangle$	$\langle s^2, t\rangle$	$\langle s^2, st\rangle$
$C_A(H)$	$\langle v_1v_3^{-1}, v_2^{\varepsilon}\rangle$	$\langle v_1v_3, v_2^{\varepsilon}v_3^{\varepsilon}\rangle$	$\langle v_1v_2^{-1}, v_2^{\varepsilon}v_3^{\varepsilon}\rangle$	$\langle v_1v_3\rangle$	$\langle v_1^{\delta}v_2^{\varepsilon}v_3^{-\delta}\rangle$	$\langle v_1^{\varepsilon}v_3^{\varepsilon}, v_2^{\varepsilon}v_3^{\varepsilon}\rangle$
$[H,A]$	$\langle v_1v_3, v_2^2\rangle$	$\langle v_1v_3^{-1}, v_1^2v_2^{-2}\rangle$	$\langle v_1v_2, v_2^2v_3^{-2}\rangle$	$\langle v_1v_2^{-1}, v_2v_3^{-1}\rangle$		

THEOREM 4.3 [O'N, Lemma 1.10]. *Let $S = A\langle s, t\rangle$ be an extension of the form described in Table 4, where $n \ge 3$. Then A is normal in every saturated fusion system $\mathcal{F}$ over S.*

PROOF. Assume otherwise: assume $\mathcal{F}$ is such that $A \ntrianglelefteq \mathcal{F}$. By Proposition 2.4(c) and since A is weakly closed in $\mathcal{F}$ by Lemma 4.2, there is a subgroup $Z \le A$ fully centralized in $\mathcal{F}$ such that $A \ntrianglelefteq C_{\mathcal{F}}(Z)$, and such that for each $u \in \mathscr{T}_{C_{\mathcal{F}}(Z)}(A)$ there is $U \in \mathscr{U}_{C_{\mathcal{F}}(Z)}(A)$ such that $U \cap A \le Z$ and $(c_u^A, \mathrm{Aut}_U(A), U \cap A) \in \mathscr{R}_T(A)$. Set $\tau = c_u^A$; we can assume that $|\tau| = 2$. Set $B = \mathrm{Aut}_U(A)$ and $A_* = U \cap A$.

By Table 5, we have $|C_A(\tau) \cap [\tau, A]| = 4$. So $|B| \ge 4$ by inequality (2-4) in Proposition 2.6, with equality since $T \cong D_8$ has no abelian subgroups of order 8. Hence,

$$C_A(B)[\tau, A] \ge A_*[\tau, A] = C_A(\tau)[\tau, A], \tag{4-1}$$

where the equality follows from (2-2) in Proposition 2.6.

Since $|B| = 4$, we have $c_{s^2} \in B$. So we can choose $u \in s^2A$ with $u \in \mathscr{T}_{C_{\mathcal{F}}(Z)}(A)$ (thus $C_{\mathcal{F}}(Z)$-conjugate to an element of A), and hence $\tau = c_u^A = c_{s^2}$. By Table 5,

$$[\tau, A] = \langle v_1v_3^{-1}, v_1^2v_2^{-2}\rangle \quad \text{and} \quad C_A(\tau)[\tau, A] = \langle v_1v_3, v_1^2, v_2^2\rangle.$$

So by Table 5, inequality (4-1) fails when $B = \langle s^2, t\rangle$ or $\langle s^2, st\rangle$, and holds only when $B = \langle s\rangle$ and $A_* = C_A(s) = \langle v_1v_3\rangle$. Since $A_* \le Z \le C_A(B)$ by assumption, we have $Z = \langle v_1v_3\rangle$.

Set $\widehat{\mathcal{F}} = C_{\mathcal{F}}(Z)/Z$, $\widehat{A} = A/Z$, and $\widehat{S} = C_S(Z)/Z$ (see Definition 1.10). Then $A \ntrianglelefteq C_{\mathcal{F}}(Z)$ by assumption, hence is not strongly closed by Lemma 1.5, and so A/Z is not strongly closed in $C_{\mathcal{F}}(Z)/Z$. Thus $\widehat{A} \ntrianglelefteq \widehat{\mathcal{F}}$. Let $v, w, \sigma \in \widehat{S}$ be the classes (modulo Z) of $v_1, v_2, s \in S$. Then $\widehat{A} \trianglelefteq \widehat{S}$ are as in Lemma 4.1, so $\widehat{A} \trianglelefteq \widehat{\mathcal{F}}$ by that lemma, giving a contradiction. □

A. Some lemmas in representation theory

Recall Notation 2.8: when V is an elementary abelian p-group and $\tau \in \mathrm{Aut}(V)$ has order p, we set

$$\mathscr{J}_V(\tau) = \mathrm{rk}(C_V(\tau) \cap [\tau, V]),$$

the number of nontrivial Jordan blocks under the action of τ on V. We derive here some formulas that give lower bounds for these functions in terms of Brauer characters.

The first lemma gives, in certain cases, lower bounds for $\mathscr{J}_V(x)$ in terms of the modular character of V. When q is a prime and $q \nmid n$, we let $\mathrm{ord}_q(n)$ denote the order of n in the group $\mathbb{F}_q^\times$.

LEMMA A.1. *Fix a prime p, an elementary abelian p-group V, and an element $x \in \mathrm{Aut}(V)$ of order p. Let $\chi = \chi_V$ be the modular character of V as an $\mathbb{F}_p\mathrm{Aut}(V)$-module.*

(a) *Assume $p = 2$, and let q be an odd prime such that $\mathrm{ord}_q(2) = q - 1$. Let $a \in \mathrm{Aut}(V)$ be such that $|a| = q$ and $\langle a, x\rangle \cong D_{2q}$. Then*

$$\mathscr{J}_V(x) \geq \frac{q-1}{2q}(\chi_V(1) - \chi_V(a)).$$

(b) *Let q be a prime such that $p \mid (q-1)$, and let $a \in \mathrm{Aut}(V)$ be such that $|a| = q$ and $\langle a, x\rangle$ is nonabelian of order pq. Then*

$$\mathscr{J}_V(x) \geq \frac{1}{pq}\sum_{i=1}^{q-1}(\chi_V(1) - \chi_V(a^i)).$$

(c) *Assume $p = 3$, and let $a \in \mathrm{Aut}(V)$ be such that $\langle a, x\rangle \cong A_4$ and $|a| = 2$. Then*

$$\mathscr{J}_V(x) \geq \tfrac{1}{4}(\chi_V(1) - \chi_V(a)).$$

(d) *Assume $p = 3$, and let $a \in \mathrm{Aut}(V)$ be such that $\langle a, x\rangle \cong 2A_4$ and $|a| = 4$. Then*

$$\mathscr{J}_V(x) \geq \tfrac{1}{4}(\chi_V(1) - \chi_V(a)).$$

PROOF. **(b)** Since $\langle a, x\rangle$ is nonabelian of order pq, where $p \mid (q-1)$ and $|a| = q$, we have

$$\dim(V/C_V(a)) = \chi_V(1) - \frac{1}{q}\sum_{i=0}^{q-1}\chi_V(a^i) = \frac{1}{q}\sum_{i=1}^{q-1}(\chi_V(1) - \chi_V(a^i)).$$

The action of x on $\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} (V/C_V(a))$ freely permutes the eigenspaces for a, corresponding to the primitive qth roots of unity in $\overline{\mathbb{F}}_p$. So all Jordan blocks for this action have length p, and the same holds for Jordan blocks for the action of x on $V/C_V(a)$. So $\mathscr{J}_V(x) \geq \mathscr{J}_{V/C_V(a)}(x) = \frac{1}{p}\dim(V/C_V(a))$.

(a) Since $|a| = q$ and $\mathrm{ord}_q(2) = q - 1$, we have $\chi_V(a^i) = \chi_V(a)$ for all i prime to q. So this is a special case of (b).

(c) Let $b \in \langle a, x\rangle \cong A_4$ be such that $\langle a, b\rangle \cong E_4$. Since a, b, and ab are permuted cyclically by x, they all have the same character. Hence, each of the three nontrivial irreducible characters for $\langle a, b\rangle \cong E_4$ appears with multiplicity

$$n = \tfrac{1}{3}\dim(V/C_V(\langle a, b\rangle)) = \tfrac{1}{3}(\chi_V(1) - \tfrac{1}{4}(\chi_V(1) + 3\chi_V(a))) = \tfrac{1}{4}(\chi_V(1) - \chi_V(a)).$$

Since x permutes those three characters cyclically, we have $\mathscr{J}_V(x) \geq n$.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

(d) Set $H = \langle a, x\rangle \cong 2A_4$ where $|a| = 4$, and set $z = a^2 \in Z(H)$. Then $V = V_+ \oplus V_-$ as $\mathbb{F}_3H$-modules, where $V_\pm$ are the eigenspaces for the action of z, and it suffices to prove the claim when $V = V_+$ or $V = V_-$. The case $V = V_+$ was shown in (c).

Now assume $V = V_-$, and set $m = \dim(V) = \chi_V(1)$ and $H_0 = O_2(H) \cong Q_8$. Let W be the (unique) irreducible two-dimensional $\mathbb{F}_3H_0$-module. Then $V|_{H_0} \cong W^{m/2}$, and $\mathrm{Hom}_{\mathbb{F}_3H_0}(W, V) \cong \mathbb{F}_3^{m/2}$ since $\mathrm{End}_{\mathbb{F}_3H_0}(W) \cong \mathbb{F}_3$. So there are $\frac{1}{2}(3^{m/2} - 1)$ submodules of $V|_{H_0}$ isomorphic to W, they are permuted by $\langle x\rangle \cong C_3$, and hence there is at least one two-dimensional $\mathbb{F}_3H$-submodule $W_1 \le V$. By applying the same argument to V/W_1 and then iterating, we get a sequence of $\mathbb{F}_3H$-submodules $0 = W_0 < W_1 < \cdots < W_k = V$ such that $\dim(W_i/W_{i-1}) = 2$ for each $1 \le i \le k$. Then $\dim(C_{W_i/W_{i-1}}(x)) = 1$ for each i, so $\dim(C_V(x)) \le m/2$, and $\dim([x, V]) \ge m/2$. Each nontrivial Jordan block in V has dimension 2 or 3, and intersects with $[x, V]$ with dimension 1 or 2, respectively. Thus

$$\mathscr{J}_V(x) \ge \tfrac{1}{2}\dim([x, V]) \ge \tfrac{1}{4}m = \tfrac{1}{4}\chi_V(1) = \tfrac{1}{4}(\chi_V(1) - \chi_V(a)),$$

the last equality since $\chi_V(a) = 0$ (recall that $a^2 = z$ acts on V via $-\mathrm{Id}$). □

The next lemma is needed to handle $\mathbb{F}_p\Gamma$-modules in certain cases where $O_p(\Gamma) \ne 1$.

LEMMA A.2. *Fix a prime p, a finite group G such that $O^p(G) = G$, and a subgroup $1 \ne Z \le Z(G)$ of p-power order. Set $\overline{G} = G/Z$. Let V be a faithful indecomposable $\mathbb{F}_pG$-module. Then either*

(a) *among the composition factors of V, there are at least two simple $\mathbb{F}_pG$-modules with nontrivial action of G; or*

(b) *there are submodules $0 \ne V_0 < V_1 < V$ such that G acts trivially on V_0 and on V/V_1, the $\mathbb{F}_p\overline{G}$-module V_1/V_0 is simple, and V_1 and V/V_0 have trivial Z-action and are indecomposable $\mathbb{F}_p\overline{G}$-modules.*

Furthermore, in the situation of (b), for each $g \in G \setminus Z$, we have $\mathrm{rk}([h, V_1/V_0]) = \mathrm{rk}([h, V])$ *for at most one element $h \in gZ$. Thus if $p = 2$ and $|g| = 2$, there is $h \in gZ$ of order 2 such that $\mathscr{J}_V(h) > \mathscr{J}_{V_1/V_0}(h)$.*

PROOF. Assume (a) does not hold. Thus all but one of the composition factors in V have trivial G-action, and there are $\mathbb{F}_pG$-submodules $V_0 < V_1 \le V$ such that V_1/V_0 is simple (hence Z acts trivially) and all composition factors of V_0 and of V/V_1 are trivial. Since $G = O^p(G)$ is generated by p'-elements, it acts trivially on V_0 and on V/V_1.

Let $W \le V_1$ be the submodule generated by the $[g, V_1]$ for all p'-elements $g \in G$. For each such g, $[g, V_1] \cap V_0 \le [g, V_1] \cap C_{V_1}(g) = 0$ since g acts trivially on V_0, so projection onto V_1/V_0 sends $[g, V_1]$ injectively, and Z acts trivially on $[g, V_1]$ since it acts trivially on V_1/V_0. Thus $[Z, W] = 0$, and $V_1 = W + V_0$ since V_1/V_0 is simple and $W \nleq V_0$. So Z acts trivially on V_1.

By a similar argument, Z acts trivially on the dual $(V/V_0)^*$, and hence acts trivially on V/V_0. Since Z acts nontrivially on V, we have $V_1 < V$ and $V_0 \ne 0$.

Assume V_1 is not indecomposable. Thus $V_1 = W_0 \oplus W_1$, where W_0 and W_1 are nontrivial $\mathbb{F}_p\overline{G}$-submodules of V_1 and $W_0 \leq V_0$. The action of G on V/W_1 is trivial (an extension of W_0 by V/V_1), so $[G, V] \leq W_1$, and W_0 splits off as a direct summand of V, contradicting the assumption that V be indecomposable. Thus V_1 is indecomposable as an $\mathbb{F}_p\overline{G}$-module, and a similar argument involving the dual module V^* shows that V/V_0 is also indecomposable, finishing the proof of (b).

Now fix $g \in G \setminus Z$, and assume that $h_1, h_2 \in gZ$ are distinct elements such that $\mathrm{rk}([h_i, V]) = \mathrm{rk}([h_i, V_1/V_0])$ for $i = 1, 2$. Set $z = h_1^{-1}h_2 \in Z^\#$. Since G acts faithfully on V by assumption, there is some $a_0 \in V$ such that $[z, a_0] \neq 0$. By (b), we have $a_0 \notin V_1$ and $[z, a_0] \in V_0$.

Set $h = h_1$ for short, so that $h_2 = zh$. Then $[h, V_1/V_0] = [hz, V_1/V_0]$, so $\mathrm{rk}([h, V]) = \mathrm{rk}([h, V_1/V_0]) = \mathrm{rk}([hz, V])$, and hence $[h, V] = [h, V_1] = [hz, V]$ and $[h, V_1] \cap V_0 = 0$. In particular, $[h, a_0]$ and $[hz, a_0]$ are both in $[h, V_1]$. Also,

$$[hz, a_0] = z(h(a_0) - a_0) + (z(a_0) - a_0) = z([h, a_0]) + [z, a_0],$$

so $0 \neq [z, a_0] \in [h, V_1] \cap V_0$, a contradiction.

The last statement now follows since if $p = 2$ and $|h| = 2$, then $\mathscr{J}_V(h) = \mathrm{rk}([h, V])$ and $\mathscr{J}_{V_1/V_0}(h) = \mathrm{rk}([h, V_1/V_0])$. □

The following example shows one way to construct examples of modules of the type described in Lemma A.2(b).

EXAMPLE A.3. Fix a prime p, a finite group G such that $O^p(G) = G$, and a subgroup $1 \neq Z \leq Z(G)$ of p-power order. Choose $k \geq 1$ such that Z has exponent at most p^k. Let $H < G$ be such that no nontrivial normal subgroup of G is contained in H. Set $\widehat{V} = \mathbb{Z}/p^k(G/H)$: the free $\mathbb{Z}/p^k$-module with basis the set G/H of left cosets. Regard $\widehat{V}$ as a left $\mathbb{Z}/p^kG$-module, set $V_2 = C_Z(\widehat{V})$, and let $V \leq \widehat{V}$ be such that $V/V_2 = C_{\widehat{V}/V_2}(G)$. Set $V_0 = C_V(G) = C_{V_2}(G)$ and $V_1 = [G, V_2]V_0$. Then V is a $\mathbb{Z}/p^kG$-module on which G acts faithfully. Also, G acts trivially on V_0 and on V/V_1, and Z acts trivially on V_1 and on V/V_0.

If, furthermore, $V_1 < V_2$ (equivalently, if $p \mid |G/HZ|$), then there is a $\mathbb{Z}/p^kG$-submodule $V' < V$ such that $V' > V_1$, G acts faithfully on V', and $V'/V_1 \cong V/V_2$.

PROOF. Set

$$\sigma_G = \sum_{gH \in G/H} gH \in C_{\widehat{V}}(G) = V_0 \quad \text{and} \quad \sigma_Z = \sum_{z \in Z} zH \in C_{\widehat{V}}(Z) = V_2.$$

Note that $Z \cap H = 1$ since it is normal in G and contained in H.

Since no nontrivial normal subgroup of G is contained in H, the group G acts faithfully on $\widehat{V}$ and G/Z acts faithfully on V_2. So G acts faithfully on V if Z does.

Fix an element $1 \neq z \in Z$; we show that $[z, V] \neq 0$. Let $Z_0 < Z$ and $x \in Z \setminus Z_0$ be such that $Z = Z_0 \times \langle x \rangle$ and $z \notin Z_0$, and set $p^\ell = |x|$ (thus $\ell \leq k$). Choose $\lambda \in \mathbb{Z}/p^k$ of

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

order p^ℓ, let $g_1,\dots,g_m\in G$ be representatives for the left cosets of HZ in G, and set

$$v=\sum_{i=1}^{m}\sum_{t\in Z_0}\sum_{s=0}^{p^\ell-1}s\lambda\cdot(tx^sg_iH)\in\widehat{V}.$$

Let $z_0\in Z_0$ and $0<r<p^\ell$ be such that $z=z_0x^r$. Then

$$zv=\sum_{i=1}^{m}\sum_{t\in Z_0}\sum_{s=0}^{p^\ell-1}s\lambda\cdot(tz_0x^{s+r}g_iH)=v-r\lambda\cdot\sigma_G,$$

and $[z,v]\neq 0$ since $r\lambda\neq 0$.

For each $g\in G$, let $z_1,\dots,z_m\in Z_0$ and $r_1,\dots,r_m\in\mathbb{Z}$ be such that for each i, $gg_iH=z_jx^{r_j}g_jH$ for some j. Then

$$gv=\sum_{i=1}^{m}\sum_{t\in Z_0}\sum_{s=0}^{p^\ell-1}s\lambda\cdot(tx^sgg_iH)=\sum_{j=1}^{m}\sum_{t\in Z_0}\sum_{s=0}^{p^\ell-1}s\lambda\cdot(tz_jx^{s+r_j}g_jH)=v-\sum_{j=1}^{m}r_j\lambda\cdot g_j\sigma_Z,$$

and so $[g,v]\in C_{\widehat{V}}(Z)=V_2$. Thus $v\in V$, finishing the proof that Z acts faithfully on V.

Since $[Z,[G,V]]=1$ by definition and $[Z,G]=1$, we have $[G,[Z,V]]=1$ by the three-subgroup lemma (see [Go, Theorem 2.2.3]). Hence, $[Z,V]\leq V_0$, so Z acts trivially on V/V_0.

If $V_1<V_2$, then G acts trivially on V_2/V_1 and on V/V_2, and hence acts trivially on V/V_1 (recall that G is generated by p'-elements). So $V/V_1=(V_2/V_1)\times(V'/V_1)$ for some $\mathbb{Z}/p^kG$-submodule $V'<V$ containing V_1 with $V'/V_1\cong V/V_2$. Also, Z acts faithfully on V' since it acts faithfully on $V=V'+V_2$ and trivially on V_2, so G acts faithfully on V' since G/Z acts faithfully on $[G,V_2]\leq V_1=V'\cap V_2$. □

For example, when $p=2$, $G=2M_{12}$, $Z=Z(G)\cong C_2$, and $H\cong M_{11}$, then by Example A.3, there is a 12-dimensional faithful $\mathbb{F}_2G$-module V with submodules $V_0<V_1<V$, where $\dim(V_0)=1$, $\dim(V_1)=11$, Z acts trivially on V_1 and on V/V_0, and V_1 has index 2 in the 12-dimensional permutation module for $G/Z\cong M_{12}$.

There are much more general ways to construct faithful $\mathbb{Z}/p^kG$-modules V with $V_0<V_1<V$ as in Lemma A.2, starting with a given $\mathbb{Z}/p^k\overline{G}$-module V_1 ($\overline{G}=G/Z$). But the ones we have found all seem to require certain conditions on $H^2(\overline{G};V_1)$ to hold.

We end this appendix with the following, more technical lemma needed in Section 2.

LEMMA A.4. *Let A be a finite abelian group, and fix $\alpha\in\mathrm{Aut}(A)$. Let $A_0\leq A$ be such that $\alpha(A_0)=A_0$. Then $|C_{A/A_0}(\alpha)|\leq|C_A(\alpha)|$.*

PROOF. Set $G=\langle\alpha\rangle\leq\mathrm{Aut}(A)$. The short exact sequence $0\to A_0\longrightarrow A\longrightarrow A/A_0\to 0$ induces an exact sequence in cohomology

$$0\longrightarrow C_{A_0}(G)\longrightarrow C_A(G)\longrightarrow C_{A/A_0}(G)\longrightarrow H^1(G;A_0)\longrightarrow\dots,$$

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

and hence

$$|C_A(G)| \geq |C_{A/A_0}(G)| \cdot |C_{A_0}(G)|/|H^1(G;A_0)|.$$

Since $G = \langle\alpha\rangle$ and A_0 is finite, we have $|H^1(G;A_0)| = |H^2(G;A_0)|$ where $H^2(G;A_0)$ is a quotient group of $C_{A_0}(G)$ (see [W, Theorem 6.2.2]). So $|C_A(G)| \geq |C_{A/A_0}(G)|$. □

B. The Golay modules for M_{22} and M_{23}

We now apply results in Section 2 to prove that the Golay modules (that is, dual Todd modules) for M_{22} and M_{23} are not fusion realizable in the sense of Definition 2.1. We do this by showing that $\mathscr{R}_T^+(A) = \emptyset$ (see Definition 2.3) whenever $T \in \mathrm{Syl}_2(M_n)$ ($n = 22$ or 23) and A is the Golay module of M_n.

We first set up our notation for handling these groups and modules. The notation used here for doing this is based mostly on that used by Griess [Gr, Chs. 4–5].

For a finite set I and a field K, let K^I be the vector space of maps $I \longrightarrow K$, with canonical basis $\{e_i \mid i \in I\}$. Let

$$\mathrm{Perm}_I(K) \leq \mathrm{Mon}_I^*(K) \leq \mathrm{Aut}^*(K^I)$$

be the groups of permutation automorphisms, semilinear monomial automorphisms, and all semilinear automorphisms, respectively (that is, linear with respect to some field automorphism of K). Thus if $|I| = n$, then $\mathrm{Perm}_I(K) \cong \Sigma_n$ and $\mathrm{Mon}_I^*(K) \cong (K^\times)^n \rtimes (\Sigma_n \times \mathrm{Aut}(K))$. Let

$$\pi = \pi_{I,K}\colon\ \mathrm{Mon}_I^*(K) \longrightarrow \mathrm{Perm}_I(K)$$

be the canonical projection that sends a monomial automorphism to the corresponding permutation automorphism; thus $\mathrm{Ker}(\pi_{I,K})$ is the group of semilinear automorphisms that send each Ke_i to itself.

More concretely, set

$$I = \{1,2,3,4,5,6\} \quad \text{and} \quad \Omega = \mathbb{F}_4 \times I.$$

Thus $\mathbb{F}_2^\Omega$ and $\mathbb{F}_4^I$ are the vector spaces of functions $\Omega \longrightarrow \mathbb{F}_2$ and $I \longrightarrow \mathbb{F}_4$, respectively. We also identify $\mathbb{F}_4^I$ with the space of 6-tuples in $\mathbb{F}_4$. Fix $\omega \in \mathbb{F}_4 \setminus \mathbb{F}_2$, and let $(x \mapsto \overline{x})$ be the field automorphism of $\mathbb{F}_4$ of order 2. Thus $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$, and $\overline{x} = x^2$ for $x \in \mathbb{F}_4$.

Let $\mathscr{H} \subseteq \mathbb{F}_4^I$ be the hexacode subgroup:

$$\mathscr{H} = \langle(\omega,\overline{\omega},\omega,\overline{\omega},\omega,\overline{\omega}),(\overline{\omega},\omega,\overline{\omega},\omega,\omega,\overline{\omega}),(\overline{\omega},\omega,\omega,\overline{\omega},\overline{\omega},\omega),(\omega,\overline{\omega},\overline{\omega},\omega,\overline{\omega},\omega)\rangle_{\mathbb{F}_4}. \tag{B.1}$$

Thus $\mathscr{H}$ is a three-dimensional $\mathbb{F}_4$-linear subspace of $\mathbb{F}_4^I$. When making computations, we frequently refer to the following elements in $\mathscr{H}$:

$$h_1 = (1,1,1,1,0,0), \quad h_2 = (1,1,0,0,1,1), \quad h_3 = (\omega,\overline{\omega},1,0,1,0). \tag{B.2}$$

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

NOTATION B.1. Let the group $\Gamma \stackrel{\text{def}}{=} \mathbb{F}_4^I \rtimes \mathrm{Mon}_I^*(\mathbb{F}_4)$ act on $\Omega = \mathbb{F}_4 \times I$ in the usual way: $\mathbb{F}_4^I$ acts via translation, $(\mathbb{F}_4^\times)^I$ acts via multiplication in each coordinate, $\mathrm{Perm}_I(\mathbb{F}_4)$ permutes the coordinates, and $\phi \in \mathrm{Aut}(\mathbb{F}_4)$ sends (c, i) to $(\overline{c}, i)$. This in turn induces an action on $\mathbb{F}_2^\Omega$, where $g \in \Gamma$ sends an element $e_{(c,i)}$ to $e_{g(c,i)}$. Equivalently, for $\xi \in \mathbb{F}_2^\Omega$ and $(c, i) \in \Omega$, define $g(\xi)$ by $(g(\xi))(c, i) = \xi(g^{-1}(c, i))$.

As special cases, $\mathrm{tr}_\eta \in \mathrm{Aut}(\mathbb{F}_2^\Omega)$ will denote translation by $\eta \in \mathbb{F}_4^I$, and $\tau(\alpha) \in \mathrm{Aut}(\mathbb{F}_2^\Omega)$ will be the automorphism induced by $\alpha \in \mathrm{Mon}_I^*(\mathbb{F}_4)$. Thus

$$\mathrm{tr}_\eta(\xi)(c, i) = \xi(c - \eta(i), i) \quad \text{and} \quad \tau(\alpha)(\xi)(c, i) = \xi(\alpha^{-1}(c, i)).$$

Now set

$$\mathrm{Aut}^*(\mathscr{H}) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{Mon}_I^*(\mathbb{F}_4) | \alpha(\mathscr{H}) = \mathscr{H}\}.$$

By [Gr, Proposition 4.5.ii], $\mathrm{Aut}^*(\mathscr{H}) \cong 3\Sigma_6$. In other words, each permutation of I is the image of some automorphism of $\mathscr{H}$, unique up to multiplication by $u \cdot \mathrm{Id}$ for some $u \in \mathbb{F}_4^\times$. More explicitly, $\mathrm{Aut}^*(\mathscr{H})$ is generated by the subgroup

$$\mathrm{Aut}_0^*(\mathscr{H}) = \langle (1\,2)(3\,4), (1\,2)(5\,6), (1\,3\,5)(2\,4\,6), (1\,3)(2\,4), (1\,2)(3\,4)(5\,6)\phi \rangle \cong \Sigma_4 \times C_2,$$

where ϕ is the field automorphism $\phi(x_1, \ldots, x_6) = (\overline{x_1}, \ldots, \overline{x_6})$, together with the elements

$$\omega \cdot \mathrm{Id} \quad \text{and} \quad \alpha = (1\,2\,3) \cdot \mathrm{diag}(1, 1, 1, 1, \overline{\omega}, \omega).$$

We refer to [Gr, Definition 5.15] for a definition of the Golay code $\mathscr{G} \leq \mathbb{F}_2^\Omega$. Here, rather than repeat that definition, we give a set of generators. Define $\mathfrak{Gr}\colon \mathbb{F}_4^I \longrightarrow \mathbb{F}_2^\Omega$ by setting

$$\mathfrak{Gr}(\xi) = \sum\nolimits_{i \in I} e_{(\xi(i), i)}$$

(the 'graph' of ξ). Define elements in $\mathbb{F}_2^\Omega$:

$$C_i = \sum_{c \in \mathbb{F}_4} e_{(c,i)} \quad (\text{for } i \in I) \quad \text{and} \quad \mathfrak{gr}_h = \mathfrak{Gr}(h) + \mathfrak{Gr}(0) \quad (\text{for } h \in \mathbb{F}_4^I),$$

and also $C_{ij} = C_i + C_j$ for distinct $i, j \in I$ and $C_{1234} = C_{12} + C_{34}$. Then $C_i + \mathfrak{Gr}(0)$ and $\mathfrak{gr}_h$ are in $\mathscr{G}$ for all $i \in I$ and all $h \in \mathscr{H}$. From the 'standard basis' for $\mathscr{G}$ given in [Gr, 5.35], we see that

$$\mathscr{G} = \langle C_i + \mathfrak{Gr}(h) \mid i \in I,\ h \in \mathscr{H} \rangle = \langle C_i + \mathfrak{Gr}(0), \mathfrak{gr}_h \mid i \in I,\ h \in \mathscr{H} \rangle.$$

This is a 12-dimensional subspace of $\mathbb{F}_2^\Omega$, with basis consisting of the six elements $C_i + \mathfrak{Gr}(0)$ for $i \in I$, together with six elements $\mathfrak{gr}_h$ for h in any given $\mathbb{F}_2$-basis of $\mathscr{H}$. By [Gr, Theorem 5.8], the weight of each element in $\mathscr{G}$ is 0, 8, 12, 16, or 24.

Define M_{24} to be the group of permutations of Ω that preserve $\mathscr{G}$, and set $\boldsymbol{Gol_{24}} = \mathscr{G}/\langle e_\Omega \rangle$, its Golay module. Also, define

$$\Delta_1 = \{(0, 6)\} \quad \text{and} \quad \Delta_2 = \{(0, 6), (1, 6)\},$$

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

TABLE 6. Commutators $[g, x] = g(x) - x$ for $g \in T$ and $x \in \boldsymbol{Gol_{23}}$. The first six elements in the top row form a basis for $C_{\boldsymbol{Gol_{22}}}(\mathrm{tr}_{h_1})$, and together with the seventh they form a basis for $C_{\boldsymbol{Gol_{23}}}(\mathrm{tr}_{h_1})$.

x	C_{1234}	C_{12}	C_{13}	C_{15}	$\mathfrak{gr}_{h_1}$	$\mathfrak{gr}_{h_2} + C_{56}$	$\mathfrak{gr}_{h_3+\omega h_2} + C_{56}$
$[\mathrm{tr}_{h_1}, x]$	0	0	0	0	0	0	0
$[\mathrm{tr}_{\omega h_1}, x]$	0	0	0	0	C_{1234}	C_{12}	C_{23}
$[\mathrm{tr}_{h_3}, x]$	0	0	0	0	C_{12}	C_{12}	C_{25}
$[\mathrm{tr}_{\omega h_3}, x]$	0	0	0	0	C_{13}	C_{15}	C_{35}
$[\tau_{12}\tau_{34}, x]$	0	0	C_{1234}	C_{12}	0	0	$\mathfrak{gr}_{h_1}$
$[\tau_{13}\tau_{24}, x]$	0	C_{1234}	0	C_{13}	0	$\mathfrak{gr}_{h_1}$	$\mathfrak{gr}_{h_1}$
$[\tau_{12}\phi, x]$	0	0	C_{12}	C_{12}	0	0	$\mathfrak{gr}_{h_2} + C_{56}$

and for $i = 1, 2$ set

$$M_{\mathbf{24}-i} = C_{M_{\mathbf{24}}}(\Delta_i) \quad \text{and} \quad \boldsymbol{Gol}_{\mathbf{24}-i} = \{\xi \in \mathscr{G} \mid \mathrm{supp}(\xi) \cap \Delta_i = \emptyset\}.$$

Thus $\dim(\boldsymbol{Gol_{24}}) = \dim(\boldsymbol{Gol_{23}}) = 11$, while $\dim(\boldsymbol{Gol_{22}}) = 10$.

Define permutations $\tau_{ij}, \mathrm{tr}_h \in \Sigma_\Omega$ for $i \neq j$ in I and $h \in \mathbb{F}_4^I$ by letting τ_{ij} exchange the ith and jth columns and letting tr_h be translation by h. More precisely,

$$\tau_{ij}(c, k) = (c, \sigma(k)) \text{ where } \sigma = (i\,j) \in \Sigma_6 \quad \text{and} \quad \mathrm{tr}_h(c, i) = (c + h(i), i).$$

Then $\mathrm{tr}_h \in M_{\mathbf{24}}$ for all $h \in \mathscr{H}$. By the above description of $\mathrm{Aut}_0^*(\mathscr{H}) \leq \mathrm{Aut}^*(\mathscr{H})$, the elements $\tau_{12}\tau_{34}$, $\tau_{12}\tau_{56}$, and $\tau_{13}\tau_{24}$ all lie in $M_{\mathbf{24}}$.

NOTATION B.2. Fix $n = 22$ or 23. Set $\Gamma = M_n$, and define subgroups

$$\begin{aligned} T &= \langle \mathrm{tr}_{h_1}, \mathrm{tr}_{\omega h_1}, \mathrm{tr}_{h_3}, \mathrm{tr}_{\omega h_3}, \tau_{12}\tau_{34}, \tau_{13}\tau_{24}, \tau_{12}\phi \rangle \in \mathrm{Syl}_2(\Gamma), \\ H_1 &= \langle \mathrm{tr}_{h_1}, \mathrm{tr}_{\omega h_1}, \tau_{12}\tau_{34}, \tau_{13}\tau_{24} \rangle, \\ H_2 &= \langle \mathrm{tr}_{h_1}, \mathrm{tr}_{\omega h_1}, \mathrm{tr}_{h_3}, \mathrm{tr}_{\omega h_3} \rangle. \end{aligned}$$

In the next lemma, we list the basic properties of these subgroups that are needed.

LEMMA B.3. *Assume Notation B.2, with $n = 22$ or 23. Then H_1 and H_2 are the only subgroups of T isomorphic to E_{16}. If we set $A = \boldsymbol{Gol}_n$, then*

$$\begin{aligned} [\mathrm{tr}_{h_1}, A] &= \langle C_{12}, C_{13}, C_{14}, \mathfrak{gr}_{h_1} \rangle \cong E_{16}, \\ C_A(H_1) &= C_A(T) = \langle C_{1234} \rangle, \\ C_A(H_2) &= \langle C_{12}, C_{13}, C_{14}, C_{15} \rangle \cong E_{16}. \end{aligned}$$

PROOF. The first statement is well known and easily checked. Note, for example, that $T/H_1 \cong D_8$, and that $C_{H_1}(x)$ has rank 2 for $x \in T \setminus H_1$. So if $E_{16} \cong H \leq T$ and $H \neq H_1$, then $HH_1 = H_1\langle \mathrm{tr}_{h_3}, \mathrm{tr}_{\omega h_3} \rangle$ or $H_1\langle \mathrm{tr}_{h_3}, \tau_{12}\phi \rangle$, and from this one easily reduces to the case $H = H_2$. (Note that all elements of order 2 in H_1H_2 lie in $H_1 \cup H_2$.)

The statements about commutators and centralizers follow from Tables 6 and 7. □

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

TABLE 7. The classes of these four elements x form a basis for $\boldsymbol{Gol}_n/C_{\boldsymbol{Gol}_n}(\mathrm{tr}_{h_1})$.

x	$\mathfrak{gr}_{\omega h_1}$	$\mathfrak{gr}_{h_3}$	$\mathfrak{gr}_{\omega h_3}$	$\mathfrak{Gr}(\omega h_2) + C_1$
$[\mathrm{tr}_{h_1}, x]$	C_{1234}	C_{12}	C_{13}	$\mathfrak{gr}_{h_1} + C_{12}$

C. The six-dimensional module for $3M_{22}$

We again fix an element $\omega \in \mathbb{F}_4 \setminus \mathbb{F}_2$, and let $(a \mapsto \overline{a})$ denote the field automorphism of $\mathbb{F}_4$. Thus $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$. We also use the bar over matrices to denote the field automorphism applied to the entries, that is, $\overline{(a_{ij})} = (\overline{a_{ij}})$. Let $\mathrm{Tr}\colon \mathbb{F}_4 \longrightarrow \mathbb{F}_2$ be the trace: $\mathrm{Tr}(a) = a + \overline{a}$.

Set $V = \mathbb{F}_4^3$ and $A = \mathbb{F}_4^6$, where elements of V are written as column matrices $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$ for $a, b, c \in \mathbb{F}_4$, and elements of A are written as column matrices $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ for $u, v \in V$. Let $\langle -, - \rangle$ be the hermitian form on A defined by

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \mathrm{Tr}(u^t\overline{y} + v^t\overline{x}).$$

The description here of the action of $\Gamma = 3M_{22}$ on A is based on that in [Ben, Ch. 2] and in [Atl, page 39], originally due to Benson and others. An element denoted $\boxed{\begin{smallmatrix} r & s & t \\ x & y & z \end{smallmatrix}}$ in [Ben] or `(rx sy tz)` in [Atl] is written here $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ where $u = \left(\begin{smallmatrix} r \\ s \\ t \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} r+x \\ s+y \\ t+z \end{smallmatrix}\right)$.

For $i, k = 1, 2, 3$ and $j = 1, 2$, define

$$b_{ijk} = \begin{cases} \omega^j & \text{if } i = k \\ 1 & \text{if } i \neq k \end{cases} \quad \text{and} \quad b_{ij} = \begin{pmatrix} b_{ij1} \\ b_{ij2} \\ b_{ij3} \end{pmatrix} \in V,$$

and set $\mathscr{B} = \{\langle b_{ij} \rangle \mid i = 1, 2, 3,\ j = 1, 2\}$. The following lemma is easily checked.

LEMMA C.1. *Consider the hermitian form $\mathfrak{h}\colon V \times V \longrightarrow \mathbb{F}_4$ defined by $\mathfrak{h}(v, w) = \overline{v}^t w$. Define elements $u_1, \ldots, u_6 \in V$ by setting*

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 1 \\ \omega \\ \overline{\omega} \end{pmatrix}, \quad u_6 = \begin{pmatrix} 1 \\ \overline{\omega} \\ \omega \end{pmatrix},$$

and set $\mathscr{U} = \{\langle u_i \rangle \mid 1 \leq i \leq 6\}$. Then the members of $\mathscr{U}$ are the only one-dimensional subspaces of V not orthogonal to any member of $\mathscr{B}$, and the members of $\mathscr{B}$ are the only one-dimensional subspaces of V not orthogonal to any member of $\mathscr{U}$. Hence, for $D \in GL_3(4)$, the action of D on V permutes the members of $\mathscr{U}$ if and only if the action of $\overline{D}^t$ on V permutes the members of $\mathscr{B}$.

Define matrices

$$M_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{01} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad M_{02} = \begin{pmatrix} 0 & \omega & \overline{\omega} \\ \overline{\omega} & 0 & \omega \\ \omega & \overline{\omega} & 0 \end{pmatrix},$$

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

and set $M_{00} = 0$, $M_{03} = M_{01} + M_{02}$, $M_{30} = M_{10} + M_{20}$, and $M_{ij} = M_{i0} + M_{0j}$ for $i, j = 1, 2, 3$. In other words, if we set $\mathbf{3} = \{0, 1, 2, 3\}$ and regard it as an elementary abelian 2-group via bitwise sum, then $((i, j) \mapsto M_{ij})$ is a homomorphism from $\underline{\mathbf{3}}\times\underline{\mathbf{3}}$ to $M_3(\mathbb{F}_4)$.

Finally, set

$$N_{ij} = I + M_{ij} \quad ((i, j) \in \underline{\mathbf{3}}\times\underline{\mathbf{3}}).$$

Note that

$$N_{i0} = \overline{u_i}u_i^t \quad \text{and} \quad N_{0i} = \overline{u_{i+3}}u_{i+3}^t \quad \text{for all } i = 1, 2, 3. \tag{C.1}$$

NOTATION C.2. Define maximal isotropic subspaces $X_{ij} \le A$ (for $i, j = 0, 1, 2, 3$) and $Y_{ij} \le A$ (for $i = 1, 2, 3$ and $j = 1, 2$) as follows:

$$X_{ij} = \left\{ \begin{pmatrix} N_{ij}v \\ v \end{pmatrix} \,\middle|\, v \in V \right\} \quad \text{and} \quad Y_{ij} = \left\{ \begin{pmatrix} u \\ b_{ij}\overline{b_{ij}}^t u \end{pmatrix} \,\middle|\, u \in V \right\}.$$

Set $\mathscr{X} = \{X_{ij} \mid i, j = 0, 1, 2, 3\}$ and $\mathscr{Y} = \{Y_{ij} \mid i = 1, 2, 3,\ j = 1, 2\}$. Let $\Gamma \le \mathrm{Aut}(A)$ be the group of unitary automorphisms of A that permute the members of $\mathscr{X} \cup \mathscr{Y}$.

The members of $\mathscr{X} \cup \mathscr{Y}$ are all totally isotropic since the matrices N_{ij} and $b_{ij}\overline{b_{ij}}^t$ are hermitian for all i, j. Following [Atl, Ben], we arrange them diagrammatically as follows:

$$\begin{array}{|cc|cc|cc|} \hline & & X_{00} & X_{01} & X_{02} & X_{03} \\ Y_{12} & Y_{11} & X_{10} & X_{11} & X_{12} & X_{13} \\ \hline Y_{22} & Y_{21} & X_{20} & X_{21} & X_{22} & X_{23} \\ Y_{32} & Y_{31} & X_{30} & X_{31} & X_{32} & X_{33} \\ \hline \end{array} \tag{C.2}$$

NOTATION C.3. For $M \in M_3(\mathbb{F}_4)$ and $D \in GL_3(\mathbb{F}_4)$, define $\varphi_M, \psi_D \in \mathrm{Aut}(A)$ by setting

$$\varphi_M\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \psi_D\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} D & 0 \\ 0 & \overline{D}^{-t} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $(-)^{-t}$ means transpose inverse. Set

$$D_0 = \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1&0&0 \\ 1&0&1 \\ 1&1&0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 0&0&1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1&0&0 \\ 0&\omega&0 \\ 0&0&\overline{\omega} \end{pmatrix};$$

and set

$$\mu_{ij} = \varphi_{M_{ij}} \quad \text{and} \quad \delta_i = \psi_{D_i} \quad \text{for } i, j = 0, 1, 2, 3.$$

Also, define the following subgroups of $\mathrm{Aut}_{\mathbb{F}_4}(A)$ (in fact, of Γ):

$$\begin{aligned} H &= N_\Gamma(\mathscr{Y}) = N_\Gamma(\mathscr{X}), & P_1 &= \{\mu_{ij} \mid i, j = 0, 1, 2, 3\}, \\ H_0 &= C_\Gamma(\mathscr{Y}), & P_2 &= \langle \mu_{10}, \mu_{01}, \delta_0, \delta_1 \rangle, \\ \Gamma_0 &= C_\Gamma(\mathscr{X} \cup \mathscr{Y}), & T &= P_1P_2\langle \delta_2 \rangle = P_1\langle \delta_0, \delta_1, \delta_2 \rangle. \end{aligned}$$

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

Note that φ_M is unitary whenever $\overline{M}^t = M$, and ψ_D is unitary for all $D \in GL_3(4)$. In particular, the μ_{ij} and the δ_i are all unitary.

Most of the information about Γ and its action on A in the following lemma is well known and implicit in Ch. 2 of [Ben], but we try here to make more explicit some of the details in the proofs.

LEMMA C.4. *Set* $A_0 = \{\binom{w}{0} \mid w \in V\}$. *Set* $\Delta = \langle D_0, D_1, D_2, D_3\rangle \le GL_3(4)$, *and set* $\psi_\Delta = \langle \delta_0, \delta_1, \delta_2, \delta_3\rangle = \{\psi_D \mid D \in \Delta\} \le \mathrm{Aut}(A)$. *Then*

(a) $\Gamma \cong 3M_{22}$ *and* $T \in Syl_2(\Gamma)$;
(b) $\Delta \cong \psi_\Delta \cong 3A_6$;
(c) $H_0 = P_1 \times \Gamma_0$ *where* $P_1 = \{\varphi \in \Gamma \mid \varphi|_{A_0} = \mathrm{Id}\} \cong E_{16}$ *and* $\Gamma_0 = \langle \omega \cdot \mathrm{Id}_A\rangle$; *and*
(d) $H = \{\varphi \in \Gamma \mid \varphi(A_0) = A_0\} = P_1\psi_\Delta$.

PROOF. For each $i = 1, 2, 3$ and $j = 1, 2$,

$$Y_{ij} \cap A_0 = \{\tbinom{u}{0} \mid u \in V,\ \overline{b_{ij}}^t u = 0\} = \{\tbinom{u}{0} \mid u \in b_{ij}^\perp\} \tag{C.3}$$

in the notation of Lemma C.1. Thus $\dim_{\mathbb{F}_4}(Y \cap A_0) = 2$ for $Y \in \mathscr{Y}$, and distinct members of $\mathscr{Y}$ have distinct intersections with A_0. So for each pair $Y \ne Y'$ in $\mathscr{Y}$, we have $Y \cap Y' \le A_0$ where $\dim(Y \cap Y') = 1$, and the set of all such intersections generates A_0.

Thus each $\varphi \in H$ sends A_0 to itself. If $\varphi \in H_0$, then φ sends each of the one-dimensional subspaces $Y \cap Y'$ to itself (for $Y \ne Y'$ in $\mathscr{Y}$), and hence $\varphi|_{A_0} \in \langle \omega \cdot \mathrm{Id}_{A_0}\rangle$.

By definition, $X \cap A_0 = 0$ for each $X \in \mathscr{X}$. So if $\varphi \in \Gamma$ is such that $\varphi(A_0) = A_0$, then φ permutes the members of $\mathscr{X}$ and those of $\mathscr{Y}$, and hence lies in H. If $\varphi|_{A_0} \in \langle \omega \cdot \mathrm{Id}_{A_0}\rangle$, then since the intersections $Y \cap A_0$ for $Y \in \mathscr{Y}$ are all distinct, φ sends each member of $\mathscr{Y}$ to itself and hence lies in H_0. To summarize, we have now shown that

$$H = \{\varphi \in \Gamma \mid \varphi(A_0) = A_0\} \quad \text{and} \quad H_0 = \{\varphi \in \Gamma \mid \varphi|_{A_0} \in \langle \omega \cdot \mathrm{Id}_{A_0}\rangle\}. \tag{C.4}$$

(b) Each of the matrices D_i for $i = 0, 1, 2, 3$ permutes the members of the set $\mathscr{U} = \{u_i \mid 1 \le i \le 6\}$, and does so via the permutations

$$D_0 : (2\,3)(5\,6), \quad D_1 : (1\,4)(2\,3), \quad D_2 : (1\,2)(3\,4), \quad D_3 : (4\,5\,6). \tag{C.5}$$

These generate the group of all even permutations of the set $\mathscr{U}$. In particular, there is a matrix $D_4 \in \Delta$ that induces the permutation $(1\,2\,3)$, and by considering its action on the u_i for $1 \le i \le 4$, we see that $D_4 = \left(\begin{smallmatrix} 0 & 0 & r \\ r & 0 & 0 \\ 0 & r & 0 \end{smallmatrix}\right)$ for some $r \in \mathbb{F}_4^\times$.

We claim that

$$\Delta = \{D \in GL_3(4) \mid D(\mathscr{U}) = \mathscr{U}\}. \tag{C.6}$$

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

To see this, assume $D \in GL_3(4)$ permutes the $\langle u_i \rangle$. Since all even permutations of $\mathscr{U}$ are realized by elements in Δ, there is $D' \equiv D \pmod{\Delta}$ that sends each of the subspaces $\langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle, \langle u_4 \rangle$ to itself. But then D' must have the form $s \cdot I$ for $s \in \mathbb{F}_4^\times$. Since

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \begin{pmatrix} 0 & 0 & r \\ r & 0 & 0 \\ 0 & r & 0 \end{pmatrix} \right] \in [\Delta, \Delta],$$

this proves (C.6), and also shows that $\Delta \cong 3A_6$.

The isomorphism $\psi_\Delta \cong \Delta$ follows directly from the definitions.

(c) We first check, for each $i, j = 0, 1, 2, 3$, $k = 1, 2, 3$, and $\ell = 1, 2$, that $\mu_{ij}(Y_{k\ell}) = Y_{k\ell}$. This means showing, for $u \in V$, that

$$b_{k\ell} \overline{b_{k\ell}}^t u = b_{k\ell} \overline{b_{k\ell}}^t (u + M_{ij} b_{k\ell} \overline{b_{k\ell}}^t u),$$

that is, that $\overline{b_{k\ell}}^t M_{ij} b_{k\ell} = 0$. It suffices to do this when $ij = 0$ and $(i, j) \neq (0, 0)$. In all such cases, by (C.1), there is $\langle c_{ij} \rangle \in \mathscr{U}$ such that $c_{ij} \overline{c_{ij}}^t = I + M_{ij}$. So it suffices to show that

$$(\overline{b_{k\ell}}^t c_{ij}) \cdot \overline{(\overline{b_{k\ell}}^t c_{ij})} = \overline{b_{k\ell}}^t b_{k\ell} = 1,$$

or equivalently that $b_{k\ell} \not\perp c_{ij}$, which follows from Lemma C.1.

For the same automorphism μ_{ij} with matrix $\left(\begin{smallmatrix} I & M_{ij} \\ 0 & I \end{smallmatrix}\right)$, an element $\left(\begin{smallmatrix} N_{k\ell} u \\ u \end{smallmatrix}\right) \in X_{k\ell}$ is sent to $\left(\begin{smallmatrix} N_{k\ell} u + M_{ij} u \\ u \end{smallmatrix}\right)$. Since $N_{k\ell} + M_{ij} = N_{k+i, \ell+j}$ where sums of indices are taken bitwise, this shows that $\mu_{ij}(X_{k\ell}) = X_{k+i, \ell+j}$. So μ_{ij} permutes the members of $\mathscr{X}$, finishing the proof that $\mu_{ij} \in H_0 \leq \Gamma$.

Conversely, for each $\varphi \in \Gamma$ such that $\varphi|_{A_0} = \mathrm{Id}$, φ induces the identity on A/A_0 since it is unitary and A_0 is a maximal isotropic subgroup, so φ has matrix $\left(\begin{smallmatrix} I & M \\ 0 & I \end{smallmatrix}\right)$ for some $M \in M_3(\mathbb{F}_4)$. Thus $\varphi = \varphi_M$ (see Notation C.3). Let (i, j) be such that $\varphi(X_{00}) = X_{ij}$; then $N_{00} + M = I + M = N_{ij}$, so $M = M_{ij}$, and $\varphi = \mu_{ij} \in P_1$. We now conclude that

$$P_1 = \{\varphi \in \Gamma \mid \varphi|_{A_0} = \mathrm{Id}\}.$$

By (C.4), $\varphi \in H_0$ implies that $\varphi|_{A_0} \in \langle \omega \cdot \mathrm{Id}_{A_0} \rangle$, and hence that $\varphi \in P_1 \times \langle \omega \cdot \mathrm{Id}_A \rangle$. Thus $H_0 \leq P_1 \times \langle \omega \cdot \mathrm{Id}_A \rangle$, and we have already proved the opposite inclusion. Also, $\langle \omega \cdot \mathrm{Id}_A \rangle \leq \Gamma_0 \leq H_0$, and $\Gamma_0 \cap P_1 = 1$ since P_1 acts faithfully on $\mathscr{X}$. So $\Gamma_0 = \langle \omega \cdot \mathrm{Id}_A \rangle$.

(d) Fix $D \in \Delta$; we show that $\psi_D \in H$. Let $\rho_D : M_3(\mathbb{F}_4) \longrightarrow M_3(\mathbb{F}_4)$ be the homomorphism $\rho_D(M) = DM\overline{D}^t$. Since D permutes the members of $\mathscr{U}$ by (C.5), ρ_D permutes the set

$$\{u\overline{u}^t \mid \langle u \rangle \in \mathscr{U}\} = \{N_{10}, N_{20}, N_{30}, N_{01}, N_{02}, N_{03}\}$$

(see (C.1)). This, together with the relations $N_{ij} + N_{k\ell} + N_{mn} = N_{i+k+m, j+\ell+n}$ (where indices are added bitwise), shows that ρ_D permutes the set of all N_{ij} for $i, j = 0, 1, 2, 3$. (Note, for example, that $N_{00} = N_{10} + N_{20} + N_{30}$.) If i, j, k, ℓ are such that $\rho_D(N_{ij}) = N_{k\ell}$, then

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

$$\psi_D(X_{ij}) = \left\{ \begin{pmatrix} DN_{ij}u \\ \overline{D}^{-t} u \end{pmatrix} \,\middle|\, u \in V \right\} = \left\{ \begin{pmatrix} DN_{ij}\overline{D}^t v \\ v \end{pmatrix} \,\middle|\, v \in V \right\} = X_{k\ell},$$

and thus ψ_D permutes the members of $\mathscr{X}$.

By Lemma C.1 and since D permutes the members of $\mathscr{U}$, the matrix $\overline{D}^t$ permutes the members of $\mathscr{B}$. So for each i,j there are k,ℓ such that $\overline{D}^t b_{k\ell} \in \langle b_{ij} \rangle$, and hence

$$\begin{aligned}\psi_D(Y_{ij}) &= \left\{ \begin{pmatrix} Du \\ \overline{D}^{-t} b_{ij}\overline{b_{ij}}^t u \end{pmatrix} \,\middle|\, u \in V \right\} = \left\{ \begin{pmatrix} v \\ \overline{D}^{-t} b_{ij}\overline{b_{ij}}^t D^{-1} v \end{pmatrix} \,\middle|\, v \in V \right\} \\ &= \left\{ \begin{pmatrix} v \\ b_{k\ell}\overline{b_{k\ell}}^t v \end{pmatrix} \,\middle|\, v \in V \right\} = Y_{k\ell}.\end{aligned}$$

Thus ψ_D also permutes the members of $\mathscr{Y}$, and it follows that $\psi_D \in H$.

Conversely, for each $\eta \in H$, $\eta(A_0) = A_0$ by (C.4), and $\eta|_{A_0}$ permutes the subspaces $Y \cap A_0$ for $Y \in \mathscr{Y}$. So η has matrix of the form $\left(\begin{smallmatrix} D & X \\ 0 & \overline{D}^t \end{smallmatrix}\right)$, where D permutes the subspaces $b^\perp \le V$ for all $\langle b \rangle \in \mathscr{B}$ by (C.3), and hence permutes the members of $\mathscr{U}$ by Lemma C.1. So by (C.6), there is $\delta \in \psi_\Delta$ such that $\eta|_{A_0} = \delta|_{A_0}$. Then $\delta^{-1}\eta \in P_1$ by (c), and $\eta \in P_1\psi_\Delta$. This finishes the proof that $H = P_1\psi_\Delta$.

(a) Set $\Gamma^* = \Gamma/\Gamma_0$, regarded as a group of permutations of the set $\mathscr{X} \cup \mathscr{Y}$. By [Ben, Theorem 2.3], Γ^* acts 3-transitively on the set $\mathscr{X} \cup \mathscr{Y}$. It is well known (see, for example, [Po, page 235]) that the only finite groups that act 2-transitively on a set of order 22 are M_{22}, A_{22}, and their automorphism groups. So once we have shown that $T \in \mathrm{Syl}_2(\Gamma)$ and $|T| = 2^7$, it will then follow that $\Gamma^* \cong M_{22}$, and that Γ is a central extension of $\Gamma_0 \cong C_3$ by Γ^*.

Recall that $T = P_1\langle \delta_0, \delta_1, \delta_2 \rangle$, where by (C.5) the action of the δ_i on $\mathscr{Y}$ generates a subgroup of Σ_6 isomorphic to D_8. Hence, $T/P_1 \cong D_8$, and $|T| = 2^7$. Alternatively, one can describe T by looking at the subgroup of $\mathrm{Aut}(A_0)$ generated by restrictions of its elements.

Under the action of Γ^*, the stabilizer of a subspace $X \in \mathscr{X} \cup \mathscr{Y}$ acts $\mathbb{F}_4$-linearly on X. If $\varphi \in \Gamma$ is such that $\varphi|_X = \mathrm{Id}_X$, then φ sends each member of $\mathscr{X} \cup \mathscr{Y}$ to itself since their intersections with X are distinct, and hence $\varphi \in \Gamma_0$. The point stabilizer for the action of Γ^* on $\mathscr{X} \cup \mathscr{Y}$ is thus isomorphic to a subgroup of $PGL_3(4)$, and hence the order of Γ^* divides $22 \cdot |PGL_3(4)| = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 3 \cdot |M_{22}|$. So $T \in \mathrm{Syl}_2(\Gamma)$ and $\Gamma^* \cong M_{22}$. Finally, Γ is the nonsplit central extension of $\Gamma_0 \cong C_3$ by Γ^* since it contains $\psi_D \cong 3A_6$ by (b,d). □

Thus $P_1 = O_2(H)$, where $H \cong E_{16} \rtimes 3A_6$ is a hexad subgroup of $\Gamma \cong 3M_{22}$. One can also show that $P_2 = O_2(K)$ where $K = N_\Gamma(\{Y_{11}, Y_{12}\})$ is a duad subgroup of Γ. Equivalently, $K \cong C_3 \times (E_{16} \rtimes \Sigma_5)$ is the group of elements of Γ that permute the five 2×2 blocks in diagram (C.2), that is, send the four members of each such block to those in another block.

The next lemma collects some technical properties of the action of Γ on A.

https://doi.org/10.1017/S1446788723000022 Published online by Cambridge University Press

LEMMA C.5. *Let* $\{e_1, e_2, \ldots, e_6\}$ *be the canonical basis for* $A = \mathbb{F}_4^6$. *Then for* $P_1, P_2, T \leq \Gamma$ *and* $\mu_{10} \in P_1 \cap P_2$ *as defined in Notation C.3,*

(a) P_1 *and* P_2 *are the only subgroups of* T *isomorphic to* E_{16}*; and*
(b) $C_A(\mu_{10}) = \langle e_1, e_2, e_3, e_4 \rangle$, $[\mu_{10}, A] = \langle e_2, e_3 \rangle$, $C_A(P_1) = \langle e_1, e_2, e_3 \rangle$, $C_A(P_2) = \langle e_2 + e_3 \rangle$.

PROOF. For point (a), see Lemma B.3. Point (b) follows easily from the above descriptions of the actions. □

Acknowledgments

The author would like to thank the Newton Institute in Cambridge for its hospitality while he was finishing the write-up of this paper. He would also like to thank the referee for carefully reading the paper and making several suggestions for improvements.

References

[AKO] M. Aschbacher, R. Kessar and B. Oliver, *Fusion Systems in Algebra and Topology* (Cambridge University Press, Cambridge, 2011).

[Alp] J. Alperin, 'Sylow 2-subgroups of 2-rank three', in: *Finite Groups' 72: Proceedings of the Gainesville Conference on Finite Groups*, North-Holland Mathematics Studies, 7 (eds. T. Gagen, M. P. Hale Jr and E. E. Shult) (North-Holland, Amsterdam, 1973), 3–5.

[Atl] J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson, *ATLAS of Finite Groups* (Clarendon Press, New York–Evanston–London, 1985).

[Ben] D. Benson, 'The simple group J_4', Thesis, University of Cambridge, 1981.

[BLO] C. Broto, R. Levi and B. Oliver, 'The homotopy theory of fusion systems', *J. Amer. Math. Soc.* **16** (2003), 779–856.

[COS] D. Craven, B. Oliver and J. Semeraro, 'Reduced fusion systems over p-groups with abelian subgroup of index p: II', *Adv. Math.* **322** (2017), 201–268.

[Cr] D. Craven, *The Theory of Fusion Systems* (Cambridge University Press, Cambridge, 2011).

[Gd] D. Goldschmidt, '2-Fusion in finite groups', *Ann. of Math. (2)* **99** (1974), 70–117.

[GL] D. Gorenstein and R. Lyons, *The Local Structure of Finite Groups of Characteristic 2 Type*, Memoirs of the American Mathematical Society, 276 (American Mathematical Society, Providence, RI, 1983).

[Go] D. Gorenstein, *Finite Groups* (Harper & Row, 1968).

[Gr] R. Griess, *Twelve Sporadic Groups* (Springer-Verlag, Berlin, 1998).

[JLPW] C. Jansen, K. Lux, R. Parker and R. Wilson, *An Atlas of Brauer Characters* (Oxford University Press, Oxford, 1995).

[Kı] Y. Kızmaz, 'A generalization of Alperin's fusion theorem and its applications', Preprint, 2022, arXiv:2204.11303.

[MS] U. Meierfrankenfeld and B. Stellmacher, 'The general FF-module theorem', *J. Algebra* **351** (2012), 1–63.

[O1] B. Oliver, 'Simple fusion systems over p-groups with abelian subgroup of index p: I', *J. Algebra* **398** (2014), 527–541.

[O2] B. Oliver, 'Fusion systems realizing certain Todd modules', *J. Group Theory*, to appear, arXiv:2204.07750.

[O'N] M. O'Nan, 'Some evidence for the existence of a new simple group', *Proc. Lond. Math. Soc. (3)* **32** (1976), 421–479.

[Po] B. Pogorelov, ‘Primitive permutation groups of low degree I’, *Algebra Logic* **19** (1980), 230–254.
[Pu] L. Puig, ‘Frobenius categories’, *J. Algebra* **303** (2006), 309–357.
[W] C. Weibel, *An Introduction to Homological Algebra* (Cambridge University Press, Cambridge, 1994).

BOB OLIVER, Université Sorbonne Paris Nord, LAGA, UMR 7539 du CNRS,
99, Av. J.-B. Clément, Villetaneuse 93430, France
e-mail: bobol@math.univ-paris13.fr