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A LAW OF THE ITERATED LOGARITHM FOR MARTINGALES

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Using a slight generalisation of Brown's inequality, we show that for martingales the existence of a weak nonuniform bound on the rate of convergence in the central limit theorem yields the usual upper bound part of the law of the iterated logarithm.

1. Introduction

For a given martingale it is frequently difficult or impossible to verify the assumptions of abstract martingale central limit theorems or laws of the iterated logarithms. On the other hand, it is possible to establish central limit theorems for certain martingales directly by using other methods as, for instance, generalised Fourier transforms. This possibility arises, for example, for some martingales that are needed to study Markov chains whose transition probabilities are associated with generalised convolution structures. Details on such martingales can be found in Zeuner [5] and Voit [4]. Thus it seems to be interesting to consider the problem of whether a central limit theorem implies a law of the iterated logarithm directly. For sums of independent random variables this problem was studied by Petrov in [3], Chapter X.3. It is the purpose of this paper to show by using elementary methods that for martingales the existence of a nonuniform bound on the rate of convergence in the central limit theorem implies the part of the law of the iterated logarithm which deals with the usual upper bound for the rate of convergence in the strong law of large numbers. It should be noted that the nonuniform bound needed below is very weak in view of the martingale central limit theorems presented in Chapter 3.6 of Hall and Heyde [1] or in Haeusler and Joos [2]. Moreover it is clear that further assumptions are necessary in order to establish a complete law of the iterated logarithm. Complete laws for martingales appearing in Zeuner [5] and Voit [4] will be presented in a forthcoming paper.

Theorem 2. *Let $(Z_n)_{n\in\mathbb{N}_0}$ be a real valued martingale with $Z_0 = 0$ and let $(s_n)_{n\in\mathbb{N}} \subset]0,\infty[$ be a nondecreasing sequence with $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} s_{n+1}/s_n = 1$ and $\liminf_{n\to\infty} n/s_n > 0$ such that $\left(Z_n/\sqrt{s_n}\right)_{n\in\mathbb{N}}$ converges in distribution to the normal*

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distribution $N(0,1)$. *Moreover, denoting the distribution functions of* $Z_n/\sqrt{s_n}$ *and* $N(0,1)$ *by* F_n *and* G *respectively, we assume that there exist constants* $\lambda > 1$, $\delta > 1$ *and* $C > 0$ *such that*

$$(1) \qquad |F_n(x) - G(x)| \leqslant C \cdot \frac{(\ln n)^{-\lambda}}{1 + |x|^{\delta}}$$

for all $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *Then*

$$(2) \qquad \mathbf{P}\left(\limsup_{n\to\infty} \frac{|Z_n|}{\sqrt{2s_n \cdot \ln\ln s_n}} \leqslant 1\right) = 1.$$

In particular, taking $s_n := \sigma^2 n$ and $\sigma^2 > 0$, we have the following

COROLLARY 3. *Let* $(Z_n)_{n\in\mathbb{N}_0}$ *be a martingale with* $Z_0 = 0$ *and let* $\sigma^2 > 0$ *be such that* $\left(Z_n/\sqrt{\sigma^2 n}\right)_{n\in\mathbb{N}}$ *converges in distribution to* $N(0,1)$. *Furthermore, defining* F_n *and* G *as above, we assume again that there exist constants* $\lambda > 1$, $\delta > 1$ *and* $C > 0$ *such that (1) is true for all* $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *Then*

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{|Z_n|}{\sqrt{2\sigma^2 n \cdot \ln\ln n}} \leqslant 1\right) = 1.$$

The proof of the theorem is based on the following lemma generalising Brown's inequality slightly (see Hall and Heyde [1], Theorem 2.4).

LEMMA 4. *Let* $(Z_n)_{n\in\mathbb{N}_0}$ *be a martingale with* $Z_0 = 0$. *Then for all* $s, t > 0$ *and* $m, n \in \mathbb{N}$ *with* $m < n$ *it follows that*

$$\mathbf{P}\left(\max_{m\leqslant i\leqslant n} |Z_i| > s + t\right) \leqslant \frac{1}{t}\int_{\{|Z_n|\geqslant s\}} (|Z_n| - s)d\mathbf{P} + \mathbf{P}(|Z_m| > s).$$

PROOF: Let $A := \{\max_{m\leqslant i\leqslant n} Z_i > s + t\}$ and let U be the number of upcrossings of the interval $[s, s+t]$ by $(Z_i)_{m\leqslant i\leqslant n}$. Then, using the upcrossing inequality (Hall and Heyde [1], Theorem 2.3), we have

$$(3) \qquad \begin{aligned} \mathbf{P}(A) &= \mathbf{P}(A \cap \{Z_m > s\}) + \mathbf{P}(A \cap \{Z_m \leqslant s\}) \\ &\leqslant \mathbf{P}(Z_m > s) + E(U) \leqslant \mathbf{P}(Z_m > s) + \frac{1}{t}E\Big((Z_n - s)^+\Big). \end{aligned}$$

Furthermore, by considering the number of upcrossings of $[s, s+t]$ by $(-Z_i)_{m\leqslant i\leqslant n}$, we derive

$$(4) \qquad \mathbf{P}\left(\min_{m\leqslant i\leqslant n} Z_i < -s - t\right) \leqslant \mathbf{P}(Z_m < -s) + \frac{1}{t}E\Big((-Z_n - s)^+\Big).$$

The desired inequality follows immediately from (3) and (4). □

PROOF OF THE THEOREM: Throughout this proof let $C_1, C_2, \ldots$ be suitable positive constants.

Take constants $b, c, d > 1$ with $c\sqrt{d} < b$. Put $x_n := \sqrt{2s_n \cdot \ln\ln s_n}$ for $s_n \geqslant 3$ and $x_n := 1$ otherwise. For $r \in \mathbb{N}$ we define $n_r := \min\{n \in \mathbb{N} : s_n \geqslant d^r\}$ and

$$A_r := \{\max_{n_r \leqslant i \leqslant n_{r+1}} |Z_i| > bx_{n_r}\}.$$

Using the properties of $(s_n)_{n\in\mathbb{N}}$, we have

$$1 \geqslant d^r/s_{n_r} \geqslant s_{n_r-1}/s_{n_r} \overset{r\to\infty}{\longrightarrow} 1.$$

Thus $\lim_{r\to\infty} d^r/s_{n_r} = 1$ and $\lim_{r\to\infty} s_{n_{r+1}}/s_{n_r} = d$. Hence, for r sufficiently large,

$$1 \leqslant \frac{\ln\ln s_{n_{r+1}}}{\ln\ln s_{n_r}} \leqslant \frac{\ln(\ln s_{n_r} + \ln 2d)}{\ln\ln s_{n_r}} \overset{r\to\infty}{\longrightarrow} 1.$$

In summary, it follows that $\lim_{r\to\infty} x_{n_{r+1}}/x_{n_r} = \sqrt{d}$. Therefore, using $c\sqrt{d} < b$ and our assumptions, we can choose an index $r_0 \in \mathbb{N}$ such that

(5)
$$c\big(x_{n_{r+1}} + \sqrt{s_{n_{r+1}}}\big) \leqslant bx_{n_r}$$

and

(6)
$$\ln n_{r+1} \geqslant \frac{1}{2}\ln s_{n_{r+1}} \geqslant \frac{(r+1)\ln d}{2} \geqslant 3$$

are true for all $r \geqslant r_0$. Using the Lemma and formula (5), we have

(7)
$$\begin{aligned}
\mathbf{P}(A_r) &\leqslant \mathbf{P}\left(\max_{n_r\leqslant i\leqslant n_{r+1}} |Z_i| > c\big(x_{n_{r+1}} + \sqrt{s_{n_{r+1}}}\big)\right)\\
&\leqslant \mathbf{P}\big(|Z_{n_r}| > c\cdot x_{n_{r+1}}\big) + \frac{1}{c\sqrt{s_{n_{r+1}}}} \cdot \int_{\{|Z_{n_{r+1}}|\geqslant cx_{n_{r+1}}\}} \big(|Z_{n_{r+1}}| - cx_{n_{r+1}}\big)d\mathbf{P}\\
&\leqslant \mathbf{P}(|Z_{n_r}| > c\cdot x_{n_r}) + \frac{C_1 x_{n_{r+1}}}{\sqrt{s_{n_{r+1}}}} \cdot \sum_{l=1}^{\infty} \mathbf{P}\big(|Z_{n_{r+1}}| \geqslant lcx_{n_{r+1}}\big)\\
&\leqslant \mathbf{P}(|Z_{n_r}| > c\cdot x_{n_r}) + C_2\sqrt{\ln\ln s_{n_{r+1}}} \cdot \sum_{l=1}^{\infty} \mathbf{P}\left(\frac{|Z_{n_{r+1}}|}{\sqrt{s_{n_{r+1}}}} \geqslant lc\sqrt{2\ln\ln s_{n_{r+1}}}\right)
\end{aligned}$$

for all $r \geqslant r_0$. Furthermore, since by our assumptions

$$1 - F_n(x) \leqslant C_3 \cdot \frac{(\ln n)^{-\lambda}}{1+|x|^{\delta}} + 1 - G(x) \leqslant C_4\left(\cdot\frac{(\ln n)^{-\lambda}}{1+|x|^{\delta}} + \frac{1}{x}e^{-x^2/2}\right)$$

for all $n \in \mathbb{N}$ and $x \geqslant 1$, and since (6) implies that $\ln\ln n_{r+1} \geqslant 1$ for $r \geqslant r_0$, (5) and (6) ensure that

(8)
$$\begin{aligned}
&\sum_{r=r_0}^{\infty} \sqrt{\ln\ln s_{n_{r+1}}} \cdot \sum_{l=1}^{\infty} \mathbf{P}\left(Z_{n_{r+1}}/\sqrt{s_{n_{r+1}}} \geqslant lc\sqrt{2\ln\ln s_{n_{r+1}}}\right) \\
&\quad \leqslant C_4 \cdot \sum_{r=r_0}^{\infty}\sum_{l=1}^{\infty} \sqrt{\ln\ln s_{n_{r+1}}} \left(\frac{(\ln n_{r+1})^{-\lambda}}{l^{\delta}c^{\delta}\left(2\ln\ln s_{n_{r+1}}\right)^{\delta/2}} + \frac{e^{-c^2 l^2 \ln\ln s_{n_{r+1}}}}{cl\sqrt{2\ln\ln s_{n_{r+1}}}}\right) \\
&\quad \leqslant C_5 \cdot \sum_{r=r_0}^{\infty} (\ln n_{r+1})^{-\lambda} \cdot \sum_{l=1}^{\infty} l^{-\delta} + C_6 \cdot \sum_{r=r_0}^{\infty}\sum_{l=1}^{\infty} \left(\ln s_{n_{r+1}}\right)^{-l^2c^2} \\
&\quad \leqslant C_7 \cdot \sum_{r=r_0}^{\infty} \left(\frac{(r+1)\ln d}{2}\right)^{-\lambda} + C_8 \cdot \sum_{r=r_0}^{\infty}\sum_{l=1}^{\infty} ((r+1)\ln d)^{-lc^2} \\
&\quad \leqslant C_9 + C_{10} \cdot \sum_{r=r_0}^{\infty} \frac{((r+1)\ln d)^{-c^2}}{1-((r+1)\ln d)^{-c^2}} < \infty.
\end{aligned}$$

In a similar but much more obvious way we get

(9) $$\sum_{r=r_0}^{\infty} \mathbf{P}(Z_{n_r} > cx_{n_r}) < \infty.$$

Moreover, using symmetry arguments, we also have

(10)
$$\sum_{r=r_0}^{\infty} \left(\mathbf{P}(Z_{n_r} < -cx_{n_r}) + C_2\sqrt{\ln\ln s_{n_{r+1}}} \cdot \sum_{l=1}^{\infty} \mathbf{P}\left(\frac{Z_{n_{r+1}}}{\sqrt{s_{n_{r+1}}}} \leqslant -lc\sqrt{2\ln\ln s_{n_{r+1}}}\right)\right) < \infty.$$

In summary, (7), (8), (9) and (10) ensure that $\sum_{r=1}^{\infty} \mathbf{P}(A_r) < \infty$. Thus, by the Borel-Cantelli Lemma,

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{|Z_n|}{x_n} \leqslant b\right) = 1.$$

Since this equation is true for all $b = 1 + 1/k$ $(k \in \mathbb{N})$, and since a countable intersection of sets of probability one is a set of probability one, the proof is complete. □

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