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ADDENDUM: REAL TOPOLOGICAL HOCHSCHILD HOMOLOGY OF SCHEMES

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For commutative rings A in which 2 is invertible, as well as for $A=\mathbb{Z}$, [3, Proposition 2.3.5] is true. However, for some rings A in which 2 is not invertible, the result is not correct as stated. The reason is the description of the ideal T in the proof of loc. cit. is not correct in this case, and this might imply that the map α in loc. cit. is not an isomorphism. In more detail, let A be a commutative ring. Due to [1, Corollary 5.2], $\underline{\pi}_0\mathrm{THR}(A)$ is isomorphic to the Mackey functor

$$\begin{tikzcd} A \arrow[r, shift left, "\mathrm{tran}"] \arrow[loop below, "\mathrm{id}"] & (A\otimes A)/T_A, \arrow[l, shift left, "\mathrm{res}"] \end{tikzcd}$$

where $\mathrm{res}(x\otimes y)=xy$ for $x,y\in A$, $\mathrm{tran}(a)=2a\otimes 1$ for $a\in A$, and T_A is the subgroup generated by $x\otimes a^2y-a^2x\otimes y$ and $x\otimes 2ay-2ax\otimes y$ for $a,x,y\in A$. Let $2A$ denote the ideal (2) in A. We have the monomorphism $2A\to(A\otimes A)/T_A$ given by $2a\mapsto 2a\otimes 1$ for $2a\in 2A$. Its cokernel is isomorphic to $(A/2\otimes A/2)/T_{A/2}$, where $A/2:=A/2A$. Observe that $T_{A/2}$ is the subgroup generated by $x\otimes a^2y-a^2x\otimes y$ for $a,x,y\in A/2$. Hence, we have the short exact sequence

$$0\to 2A\to(A\otimes A)/T_A\to A/2\otimes_{\varphi,A/2,\varphi}A/2\to 0,$$

where $\varphi\colon A/2 \to A/2$ denotes the Frobenius (i.e., the squaring map). In particular, [3, Proposition 2.3.5] holds if and only if φ is surjective.

The only statement in [3] where this proposition is used is the following one in the proof of [3, Proposition 3.2.2], for which we now provide an alternative proof.

Proposition 1. *Let $A \to B$ be an étale homomorphism of commutative rings. Then the induced morphism of Mackey functors*

$$\underline{\pi}_0\mathrm{THR}(\iota A)\square_{\iota A}\iota B \to \underline{\pi}_0\mathrm{THR}(\iota B)$$

is an isomorphism.

Proof. By [2, Lemma 5.1], the Mackey functor $\underline{\pi}_0\mathrm{THR}(\iota A)\square_{\iota A}\iota B$ is isomorphic to

$$\begin{tikzcd} B \arrow[loop below, "w"] \arrow[r, shift left, "\mathrm{tran}"] & (A\otimes A)/T_A \otimes_A B. \arrow[l, shift left, "\mathrm{res}"] \end{tikzcd}$$

Using the above computations, we have the following commutative diagram where the vertical maps are induced by multiplication:

$$\begin{tikzcd} 0 \arrow[r] & 2A\otimes_A B \arrow[r] \arrow[d, "\alpha"'] & (A\otimes A)/T_A\otimes_A B \arrow[r] \arrow[d, "\beta"] & A/2\otimes_{\varphi,A/2,\varphi} A/2\otimes_A B \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & 2B \arrow[r] & (B\otimes B)/T_B \arrow[r] & B/2\otimes_{\varphi,B/2,\varphi} B/2 \arrow[r] & 0. \end{tikzcd}$$

We only need to show that β is an isomorphism. Since B is flat over A, the rows are short exact sequences, and α is an isomorphism. The induced square of commutative rings

$$\begin{tikzcd} A/2 \arrow[r, "\varphi"] \arrow[d] & A/2 \arrow[d] \\ B/2 \arrow[r, "\varphi"] & B/2 \end{tikzcd}$$

is coCartesian by [4, Tag 0EBS] since $A/2 \to B/2$ is étale. It follows that γ is an isomorphism. Hence, β is an isomorphism by the five lemma. $\square$

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