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AN ESSENTIAL RING WHICH IS NOT A v-MULTIPLICATION RING

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An integral domain D is called an *essential ring* if $D = \bigcap_\alpha V_\alpha$ where the V_α are valuation rings which are quotient rings of D. D is called a *v-multiplication ring* if the finite divisorial ideals of D form a group. Griffin [**2**, pp. 717-718] has observed that every v-multiplication ring is essential and that an essential ring having a defining family of valuation rings $\{V_\alpha\}$ which is of finite character (i.e. every nonzero element of D is a non-unit in at most finitely many V_α) is necessarily a v-multiplication ring; but he conjectures that, in general, there exists an essential ring which is not a v-multiplication ring. We give in §2 such an example. §1 is devoted to putting the definitions in a usable setting.

1. Preliminaries. Many of the definitions and results of this section can be found in one form or another in Jaffard [**5**] (see also [**6**] and [**2**]). However, we we shall work out the details and put together the pieces as needed in §2.

1.1 *Ordered sets and maps.* Let A denote a set with a (partial) ordering $\leq$. *We shall tacitly assume throughout this paper that all of our ordered sets are filtered below,* i.e. given $a_1, a_2 \in A$, there exists $a \in A$ such that $a \leq a_1$ and $a \leq a_2$. If $a_0, a_1, \ldots, a_n \in A$, we define the expression $a_0 \geq \inf_A\{a_1, \ldots, a_n\}$ as follows:

$$a_0 \geq \inf_A\{a_1, \ldots, a_n\} \text{ if and only if } a_0 \geq a \text{ for all } a \in A \text{ such that } a \leq a_1, \ldots, a_n.$$

If there exists $a_0 \in A$ such that $a_0 \geq \inf_A\{a_1, \ldots, a_n\}$ and $a_0 \leq a_1, \ldots, a_n$, then we call a_0 the infimum of $a_1, \ldots, a_n$ in A and we write $a_0 = \inf_A\{a_1, \ldots, a_n\}$. If every finite set of elements of A has an infimum in A, we say that A has infs. (A is semi-réticulé inférieurement in Jaffard's terminology [**5**, p. 2].) The finite v-ideal in A generated by $a_1, \ldots, a_n$, denoted $(a_1, \ldots, a_n)_v$, is defined as follows:

$$(a_1, \ldots, a_n)_v = \{a \in A | a \geq \inf_A\{a_1, \ldots, a_n\}\}.$$

If B is another ordered set, a map $\phi: A \rightarrow B$ will be called an order (respectively, equi-order) map if for all $a_1, a_2 \in A$, $\phi(a_1) \geq \phi(a_2)$ if (respectively, if and only if) $a_1 \geq a_2$. ϕ will be called a v-map if for all $a_0, a_1, \ldots, a_n \in A$,

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$a_0 \geq \inf_A\{a_1, \ldots, a_n\}$ implies $\phi(a_0) \geq \inf_B\{\phi(a_1), \ldots, \phi(a_n)\}$. Note that a v-map is an order map and that an equi-order map is injective. If B has infs, we use $(A, \phi, B)^\wedge$ to denote $\{b \in B | b = \inf_B\{\phi(a_1), \ldots, \phi(a_n)\}$ for some $a_1, \ldots, a_n \in A\}$; and we call this set the inf hull of $\phi(A)$ in B, or merely the inf hull of A in B when ϕ is equi-order. When the ϕ and B involved are clear, we shall merely write $A^\wedge$. We shall always regard $A^\wedge$ as an ordered set with respect to the order conferred on it by the order of B. $A^\wedge$ is then an ordered set with infs.

By an ordered semi-group we shall mean an ordered set together with a commutative associative operation $+$ which is compatible with the ordering and for which there exists an identity element 0; ordered groups are defined similarly. One now carries over the above concepts to define the corresponding notions of v-homomorphism, order homomorphism, etc. Note that a group with infs is a lattice group.

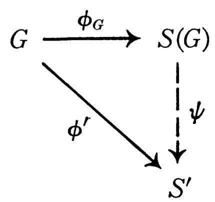
1.2 *The semi-group of finite v-ideals.* Let G denote an ordered (commutative) group with operation $+$. Then the set of all finite v-ideals of G can be given the structure of an ordered semi-group by defining for any finite subsets X, Y of G

$$X_v + Y_v = (X + Y)_v$$

and $X_v \leq Y_v$ if and only if $Y_v \subset X_v$ [**5**, p. 20]. We shall denote this ordered semi-group by $S(G)$. For any two elements $X_v, Y_v \in S(G)$, $\inf_{S(G)}\{X_v, Y_v\}$ exists and is just $(X \cup Y)_v$. Thus $S(G)$ is an ordered semi-group with infs. The *canonical map* $\phi_G: G \rightarrow S(G)$ defined by $\phi_G(x) = (x)_v$ is an (injective) equi-order v-homomorphism such that $G^\wedge = S(G)$.

The semi-group $S(G)$ has the following universal mapping property, which characterizes $S(G)$ up to a unique equi-order isomorphism.

1.3 Proposition. *Given an ordered semi-group S' with* infs *and a v-homomorphism $\phi': G \rightarrow S'$, then there exists a unique v-homomorphism $\psi: S(G) \rightarrow S'$ such that $\psi \circ \phi_G = \phi'$*



Moreover, the image of $S(G)$ under ψ is $(G, \phi', S')^\wedge$; and if ϕ' is equi-order, then ψ is equi-order (and a fortiori injective).

Proof. ψ is (necessarily) defined by writing any $s \in S(G)$ in the form $s = \inf_{S(G)}\{\phi_G(x_1), \ldots, \phi_G(x_n)\}$, $x_i \in G$, and then defining $\psi(s)$ to be $\inf_{S'}\{\phi'(x_1), \ldots, \phi'(x_n)\}$. Then ψ is well-defined: for suppose

$$s = \inf_{S(G)}\{\phi_G(x_1), \ldots, \phi_G(x_n)\} = \inf_{S(G)}\{\phi_G(y_1), \ldots, \phi_G(y_m)\}.$$

Then $y_i \geq \inf_G\{x_1, \ldots, x_n\}$; and hence since ϕ' is a v-homomorphism, $\phi'(y_i) \geq \inf_{S'}\{\phi'(x_1), \ldots, \phi'(x_n)\}$. Therefore

$$\inf_{S'}\{\phi'(y_1), \ldots, \phi'(y_m)\} \geq \inf_{S'}\{\phi'(x_1), \ldots, \phi'(x_n)\},$$

and the reverse inequality follows by symmetry.

One checks easily, using $\inf\{A + B\} = \inf A + \inf B$ and $\inf\{\inf A, \inf B\} = \inf\{A \cup B\}$, that ψ preserves sums and infs. It is clear from the definition of ψ that the image of $S(G)$ under ψ is $(G, \phi', S')^\wedge$. Finally, suppose ϕ' is equi-order, and $\psi((x_1, \ldots, x_n)_v) \geq \psi((y_1, \ldots, y_m)_v)$. By definition of ψ, then

$$\inf_{S'}\{\phi'(x_1), \ldots, \phi'(x_n)\} \geq \inf_{S'}\{\phi'(y_1), \ldots, \phi'(y_m)\};$$

and hence from the fact that ϕ' is equi-order, it follows that

$$x_i \geq \inf_G\{y_1, \ldots, y_m\}.$$

Thus

$$(x_1, \ldots, x_n)_v \geq (y_1, \ldots, y_m)_v.$$

A consequence of 1.3 is that $S(\ )$ is a functor from the category of ordered groups and v-homomorphisms into the category of ordered semi-groups with infs and v-homomorphisms.

1.4 LEMMA. *Let G and G' be ordered groups and let $\phi: G \rightarrow G'$ be an equi-order homomorphism. If $G' = G^\wedge$, then ϕ is a v-homomorphism.*

Proof. Let $x, x_1, \ldots, x_n$ be elements of G such that $x \geq \inf_G\{x_1, \ldots, x_n\}$, and let y' be an element of G' such that $y' \leq \phi(x_1), \ldots, \phi(x_n)$. Since G', G are groups and $G' = G^\wedge$, every element of G' is the supremum of finitely many elements of $\phi(G)$; so there exist $y_1, \ldots, y_m \in G$ such that

$$y' = \sup_{G'}\{\phi(y_1), \ldots, \phi(y_m)\}.$$

Then

$$\begin{aligned} &\phi(x_1), \ldots, \phi(x_n) \geq y' \geq \phi(y_1), \ldots, \phi(y_m) \\ &\Rightarrow x_1, \ldots, x_n \geq y_1, \ldots, y_m \Rightarrow x \geq y_1, \ldots, y_m \\ &\Rightarrow \phi(x) \geq \phi(y_1), \ldots, \phi(y_m) \Rightarrow \phi(x) \geq y'. \end{aligned}$$

1.5 PROPOSITION. *Let G be an ordered group. The following are equivalent:*

(i) *$S(G)$ is a group.*

(ii) *There exists a lattice group G' and an equi-order homomorphism $\phi': G \rightarrow G'$ such that $G^\wedge = G'$.*

(iii) *There exists a lattice group G' and an equi-order v-homomorphism $\phi': G \rightarrow G'$ such that $G^\wedge$ is a group.*

Moreover, when these equivalent conditions hold, then for any lattice group G' and any equi-order v-homomorphism $\phi': G \rightarrow G'$, the semi-group $G^\wedge$ is actually a group.

Proof. (i) $\Rightarrow$ (ii): Since $S(G)$ has infs, if it is a group, then it is a lattice group. We have already observed that the canonical map ϕ_G has the properties required in (ii).

(ii) $\Rightarrow$ (iii): ϕ' is a v-homomorphism by Lemma 1.4.

(iii) $\Rightarrow$ (i): Let $\psi: S(G) \rightarrow G'$ be the homomorphism given by 1.3. Then by 1.3, ψ is injective and has image $(G, \phi', G')\hat{} = G\hat{}$. Thus $S(G)$ is a group if $G\hat{}$ is. The last assertion follows similarly by 1.3.

1.6 *Groups of divisibility.* We shall now connect the above group theoretic considerations with integral domains. We use $(\ )^*$ to denote nonzero elements and $U(\ )$ to denote units. Let K be a field. To any domain D with quotient field K, we associate the group $\mathscr{G}(D) = K^*/U(D)$ with the order given by taking $D^*/U(D)$ to be the positive elements. (Thus, $\mathscr{G}(D)$ is the multiplicative group of nonzero principal fractional ideals of D with the integral ideals as positive elements.) That K is the quotient field of D reflects in $\mathscr{G}(D)$ being filtered. If $D_1 \subset D_2$ are two domains with quotient field K and $\phi_i: K^* \rightarrow \mathscr{G}(D_i)$ is the canonical map, then there exists a unique order homomorphism $\phi: \mathscr{G}(D_1) \rightarrow \mathscr{G}(D_2)$ such that $\phi \cdot \phi_1 = \phi_2$. $\mathscr{G}$ may thus be thought of as a functor from the category of domains with quotient field K and inclusion homomorphisms to the category of ordered groups and order homomorphisms. We want to observe next that if D' is a quotient ring of D with respect to a multiplicative system of D, then the homomorphism $\phi: \mathscr{G}(D) \rightarrow \mathscr{G}(D')$ is a v-homomorphism. This will follow from the next lemma and the observation that for D' a quotient ring of D if $a_1, \ldots, a_n \in K$ and $a' \in K$ are such that $a_1, \ldots, a_n \in a'D'$, then there exists $u \in U(D')$ such that $a = ua'$ and $a_1, \ldots, a_n \in aD$.

1.7 LEMMA. *Let A and A' be ordered sets and $\phi: A \rightarrow A'$ an order map such that for any $a_1, \ldots, a_n \in A$ and $a' \in A'$, $\phi(a_1), \ldots, \phi(a_n) \geq a'$ implies there exists $a \in A$ such that $\phi(a) = a'$ and $a_1, \ldots, a_n \geq a$. Then ϕ is a v-map.*

Proof. Let $a_0, a_1, \ldots, a_n \in A$ be such that $a_0 \geq \inf_A\{a_1, \ldots, a_n\}$ and suppose $a' \in A'$ is such that $a' \leq \phi(a_1), \ldots, \phi(a_n)$. By hypothesis there exists $a \in A$ such that $\phi(a) = a'$ and $a_1, \ldots, a_n \geq a$. Then $a_0 \geq a$, and hence $\phi(a_0) \geq \phi(a) = a'$. Thus $\phi(a_0) \geq \inf_{A'}\{\phi(a_1), \ldots, \phi(a_n)\}$.

If $S(\mathscr{G}(D))$ is a group, the domain D is called a *v-multiplication ring* (or a *pseudo-Prufer domain* by Bourbaki [**1**,(b), p. 96, Exercise 19]). Moreover, if $D = \bigcap_\alpha V_\alpha$ where the V_α are valuation rings which are quotient rings of D, then D is called an *essential ring*. Griffin has conjectured in [**2**, p. 717] that there exists an essential ring $D = \bigcap_\alpha V_\alpha$ which is not a v-multiplication ring (the question also appears in Griffin's paper [**3**, p. 25], where the answer is needed to complete a diagram of domains). The v-homomorphisms $\mathscr{G}(D) \rightarrow \mathscr{G}(V_\alpha)$ induce an equi-order v-homomorphism $\mathscr{G}(D) \rightarrow \Pi\mathscr{G}(V_\alpha)$, when $\Pi\mathscr{G}(V_\alpha)$ is given the coordinatewise order. To show then that $S(\mathscr{G}(D))$ is not a group, it suffices by 1.5 to prove that the inf hull $\mathscr{G}(D)\hat{}$ of $\mathscr{G}(D)$ in $\Pi\mathscr{G}(V_\alpha)$ is not a group. This is the approach that will be used in §2.

The following application of the above is perhaps worth noting.

1.8 PROPOSITION. *Let D' be a quotient ring with respect to a multiplicative system of the domain D. Then D is a v-multiplication ring implies D' is a v-multiplication ring.*

Proof. By 1.7, the homomorphism $\mathscr{G}(D) \to \mathscr{G}(D')$ is a v-homomorphism, and hence the composite homomorphism $\mathscr{G}(D) \to \mathscr{G}(D') \to S(\mathscr{G}(D'))$ is also a v-homomorphism. Now apply 1.3 to conclude that $S(\mathscr{G}(D'))$ is a homomorphic image of the group $S(\mathscr{G}(D))$ and hence is itself a group.

2. The example. Let k be a field, and let $y, z, x_1, x_2, \ldots$ be indeterminates. Let R denote the 2-dimensional regular local ring $k(x_1, x_2, \ldots)[y, z]_{(y,z)}$, and for each positive integer i let V_i denote the valuation ring containing the field $k(\{x_j\}_{j\neq i})$ obtained by giving x_i, y, and z the value 1 and then taking infimums, i.e. the value of any polynomial in $k[x_1, x_2, \ldots, y, z]$ is the infimum of the values of the monomials occurring in that polynomial [**1**-(a), p. 160]. Let $D = R \cap \{V_i | i = 1, 2, \ldots\}$.

CLAIM. *D is an essential ring which is not a v-multiplication ring.*

Proof. Note that $k[x_1, x_2, \ldots, y, z] \subset D$, so D has quotient field $k(x_1, x_2, \ldots, y, z)$. Since R is a Krull domain, R is an essential ring. Thus, to show D is an essential ring, it will suffice to show that R and each of the V_i's are quotient rings of D. Since $k[x_1, x_2, \ldots, y, z] \subset D$ and R is a quotient ring of $k[x_1, x_2, \ldots, y, z]$, it is clear that R is a quotient ring of D. To see that V_i is a quotient ring of D, we observe that if $R' = R \cap \{V_j | j \neq i\}$, then $1/x_i \in R'$ but $1/x_i \notin V_i$. Thus, $D = R' \cap V_i$ with $D < R'$. Since V_i is a discrete rank one valuation ring, V_i must be a quotient ring of D by [**4**, Lemma 1.3].

It remains to show that D is not a v-multiplication ring. Let G denote the group of divisibility of D, H the group of divisibility of R, and Z_i (= additive group of integers) the group of divisibility of V_i. Since R is a unique factorization domain, H is a lattice group [**1**-(*b*), p. 32, Theorem 1]. The representation $D = R \cap \{V_i | i = 1, 2, \ldots\}$ yields a canonical equi-order embedding of G in the lattice group $H \oplus (\Pi Z_i)$, where $H \oplus (\Pi Z_i)$ is ordered coordinatewise. Moreover, the fact that R and each of the V_i's are quotient rings of D implies that this embedding $\phi: G \to H \oplus (\Pi Z_i)$ is a v-embedding by 1.6. Let $G^\wedge$ denote the subsemi-group of $H \oplus (\Pi Z_i)$ consisting of all elements of $H \oplus (\Pi Z_i)$ which are the infimums of a finite number of elements of $\phi(G)$. By 1.5, D is a v-multiplication ring if and only if $G^\wedge$ is a group.

If g is a positive element of G and $\phi(g) = (h, t_1, t_2, \ldots)$ with $h > 0$, then we observe that there exists a positive integer n such that $t_i > 0$ for $i > n$. For if g is the image of $d \in D$, then $d \in k(x_1, \ldots, x_n, y, z)$ for some n. Since $h > 0$, d is then in the maximal ideal of

$$R \cap k(x_1, \ldots, x_n, y, z) = k(x_1, \ldots, x_n)[y, z]_{(y,z)}.$$

Thus, d has strictly positive value in each V_i for $i > n$, which means that

$t_i > 0$ for $i > n$. It follows that the infimum in $H \oplus (\Pi Z_i)$ of finitely many positive elements of $\phi(G)$ of the form $(h, t_1, t_2, \ldots)$ with $h > 0$ also has the property that its ith coordinate is > 0 for all i greater than some n.

Let now $\bar{y}$, $\bar{z}$ denote the images of y, z in G, and let $e = \inf\{\phi(\bar{y}), \phi(\bar{z})\}$ in $H \oplus (\Pi Z_i)$. Then $e = (0, 1, 1, \ldots)$. Consider $\phi(\bar{y}) - e$ in $H \oplus (\Pi Z_i)$, and observe that $\phi(\bar{y}) - e = (h, 0, 0, \ldots)$ with $h > 0$. The preceding paragraph shows that $\phi(\bar{y}) - e \notin G^\wedge$ even though $\phi(\bar{y})$ and e are in $G^\wedge$. Thus, $G^\wedge$ is not a group.

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