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BIFURCATION OF NONSYMMETRIC SOLUTIONS FOR SOME DUFFING EQUATIONS

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For some symmetric Duffing equation, the existence of bifurcation of nonsymmetric, periodic solutions from symmetric periodic solutions is proved by using the change of index of symmetric, periodic solutions for variation of parameters.

1. INTRODUCTION

For the Duffing equation

(1) $$\ddot{x} + k\dot{x} + x^3 = B\cos t, \quad \left(\cdot = \frac{d}{dt}\right)$$

where $x = x(t)$ is real valued for $t \in R$, $R = (-\infty, \infty)$ and k and B are positive constants, it is known numerically that nonsymmetric, periodic solutions appear as a result of bifurcations from symmetric solutions for variation of B and for fixed k (see [**3**]); however, mathematically, it is hard to prove such a bifurcation. Equation (1) may be called a symmetric equation, because if $x(t)$ is a solution of (1), then $-x(t+\pi)$ is also a solution of (1) as long as this is defined, and the solution $x(t)$ is called a symmetric solution if $x(t+\pi) \equiv -x(t)$.

Although Loud [**7**] obtained a sufficient condition for the bifurcation, his theorem has not provided us any example, because his condition is actually not simple. It is important to obtain any example in terms of Duffing equations.

First of all we shall consider the existence of symmetric solutions for 2-dimensional symmetric systems which include (1) as a special case. In Theorem 1 it is shown that the existence of periodic solutions implies the existence of symmetric solutions by using Massera's fixed point theorem. Therefore (1) has at least one symmetric solution. Moreover it is also shown that the three dimensional analogy of Theorem 1 does not hold.

In Theorem 2, we shall consider the equation with parametric excitation:

(2) $$\ddot{x} + \varepsilon\lambda\dot{x} + \big(4n^2 + \varepsilon p(t)\big)x + g(x) = \sigma e(t),$$

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where ε, λ and σ are positive constants, n is a positive integer, $g(x)$ is analytic for $x \in R$, $g(-x) \equiv -g(x)$, $g'(0) = 0$, $g'(x) \geqslant 0$ for $x \in R$, $p(t)$ is continuous and π-periodic, $e(t)$ is continuous and 2π-periodic with least period 2π and $e(t+\pi) \equiv -e(t)$. Moreover we shall assume that

$$\lim_{|x|\to\infty} \frac{g(x)}{x} = \infty. \tag{3}$$

Equation (2) is symmetric and has a physical meaning [**8**, p.498]. It will be shown that (2) undergoes bifurcation for variation of λ and for appropriately fixed ε and σ under some additional assumption on $p(t)$. Our method is to use the idea that if the index of periodic solutions changes for variation of λ, then other periodic solutions bifurcate, which is along the same line as in [**11**, Theorem 4]. We never use any bifurcation equations.

Clearly, if $x(t)$ is a symmetric solution of (2), then its solution orbit $\Omega := \{(x(t),\ \dot{x}(t)) \in R^2;\ 0 \leqslant t \leqslant 2\pi\}$ is symmetric with respect to the origin. In Theorem 3 we shall show that if $x(t)$ is a 2π-periodic solution of (2), then the symmetry of Ω with respect to the origin implies the symmetry of $x(t)$ under the assumption of analyticity of $p(t)$ and $e(t)$.

2. Symmetric solutions

We shall consider the 2-dimensional system:

$$\dot{x} = f(t,x,y), \quad \dot{y} = g(t,x,y), \tag{4}$$

where $f(t,x,y)$ and $g(t,x,y)$ are continuous for $(t,x,y) \in R^3$, Lipschitz continuous for $(x,y) \in R^2$ and for each fixed $t \in R$, and $f(t+\pi,-x,-y) \equiv -f(t,x,y)$, $g(t+\pi,-x,-y) \equiv -g(t,x,y)$. Clearly it follows that $f(t+2\pi,x,y) \equiv f(t,x,y)$ and $g(t+2\pi,x,y) \equiv g(t,x,y)$. Then (4) is symmetric, because if $(x(t),\ y(t))$ is a solution, then $(-x(t+\pi),\ -y(t+\pi))$ is also a solution as long as this is defined. It is seen that (2) is a special case of (4), by setting $y = \dot{x}$.

Definition 1: The solution $(x(t),\ y(t))$ is symmetric if $x(t+\pi) \equiv -x(t)$ and $y(t+\pi) \equiv -y(t)$. The term 'nonsymmetric solutions' means 2π-periodic solutions, which are not symmetric.

The existence of symmetric solutions was investigated by several authors for special equations (for example, see [**1**] and [**6**]). The following result may be more general.

Theorem 1. *Assume that every solution of (4) exists for $0 \leqslant t \leqslant 2\pi$. If (4) has a $2n\pi$-periodic solution for some integer n, then there exists at least one symmetric solution.*

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PROOF OF THEOREM 1: Since solutions are unique for intial values, we shall denote the solution $(x(t), y(t))$ of (4) with $x(0) = \xi$ and $y(0) = \eta$ for $(\xi, \eta) \in R^2$ by $(x(t, \xi, \eta),\ y(t, \xi, \eta))$, and furthermore set

$$T(\xi, \eta) = \left(-x(\pi, \xi, \eta),\ -y(\pi, \xi, \eta)\right). \tag{5}$$

Then we can verify that $(x(t, \xi, \eta),\ y(t, \xi, \eta))$ is symmetric if and only if $T(\xi, \eta) = (\xi, \eta)$, and moreover that

$$\begin{aligned} x(t + 2\pi, \xi, \eta) &\equiv -x\left(t + \pi,\ -x(\pi, \xi, \eta),\ -y(\pi, \xi, \eta)\right) \\ y(t + 2\pi, \xi, \eta) &\equiv -y\left(t + \pi,\ -x(\pi,\ \xi, \eta),\ -y(\pi, \xi, \eta)\right), \end{aligned}$$

by putting $t = -\pi$. This implies that

$$T^2(\xi, \eta) = \left(x(2\pi, \xi, \eta),\ y(2\pi, \xi, \eta)\right). \tag{6}$$

Since solutions are unique for initial values, T is a continuous, one to one mapping from R^2 into R^2, and moreover T is orientation preserving, because T is isotopic to the mapping of minus identity. Since (4) has $2n\pi$-periodic solutions, it follows from (6) that T has n-periodic points. Therefore Massere's fixed point theorem guarantees that T has a fixed point (see [**9**, Theorem 3.1] or [**13**, p.369]). This completes the proof. $\square$

The three dimensional analogy of Theorem 1 does not hold.

EXAMPLE 1. Let us consider the 3-dimensional system

$$\dot{x} = y,\ \dot{y} = -z^2 y - x + \cos t,\ \dot{z} = (\cos 2t) z. \tag{7}$$

Since (7) is invariant under the transformation: $(x, y, z, t) \to (-x, -y, -z, t + \pi)$, it is symmetric. By integration of the third equation of (7), we obtain that $z(t) = C \exp(\sin 2t)/2$, where C is a constant, and hence $z(t)$ is symmetric if and only if $C = 0$. When $C = 0$, that is, $z(t) \equiv 0$, (7) is reduced to the equation $\ddot{x} + x = \cos t$, whose solutions are all unbounded. Therefore (7) has no symmetric solution. On the other hand there exist 2π-periodic solutions. In fact, substituting $z = C \exp(\sin 2t)/2$ for $C \neq 0$ into (7), we see that (7) is reduced to the equation

$$\ddot{x} + \dot{x} C^2 \exp(\sin 2t) + x = \cos t. \tag{8}$$

Since the homogeneous part of (8), $\ddot{x} + \dot{x} C^2 \exp(\sin 2t) + x = 0$, has no nontrivial 2π-periodic solution, (8) has one and only one 2π-periodic solution.

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3. Nonsymmetric solutions

We shall consider (2).

Theorem 2. *Assume that* $\int_0^\pi p(t)\,dt = 0$ *and either* $\int_0^\pi p(t)\cos 4nt\,dt \neq 0$ *or* $\int_0^\pi p(t)\sin 4nt\,dt \neq 0$.

Then there exist a positive constant ε_0, *a positive number* $\delta(\varepsilon)$ *for each* $0 < \varepsilon < \varepsilon_0$, *and a positive constant* $\gamma(\varepsilon,\sigma)$ *for* $0 < \varepsilon < \varepsilon_0$ *and for* $0 < \sigma < \delta(\varepsilon)$, *such that (2) has a symmetric solution* $x(t,\lambda)$ *for* λ *close to* $\gamma(\varepsilon,\sigma)$ *and a nonsymmetric solution* $x^*(t,\lambda)$ *for either* $\lambda < \gamma(\varepsilon,\sigma)$ *or* $\lambda > \gamma(\varepsilon,\sigma)$, *where* $|\lambda - \gamma(\varepsilon,\sigma)|$ *is sufficiently small, and* $x^*(t,\lambda)$ *approaches* $x(t,\gamma(\varepsilon,\sigma))$ *uniformly on* $[0,2\pi]$ *as* λ *approaches* $\gamma(\varepsilon,\sigma)$.

We shall make preparations for proof of Theorem 2. Let $x(t)$ be an existing symmetric solution of (2) and consider the variational equation of (2) with respect to $x(t)$:

$$\dot{\xi} = \eta,\ \dot{\eta} = -\varepsilon\lambda\eta - \Big(4n^2 + \varepsilon p(t) + g'\big(x(t)\big)\Big)\xi. \tag{9}$$

Setting $U(t)$ to be the fundamental matrix of (9), we shall use the symbols μ_1 and μ_2 in order to denote eigenvalues of $U(\pi)$ and call these the half multipliers of $x(t)$. Then the usual multipliers of $x(t)$ are the eigenvalues of $U(2\pi)$, that is, μ_1^2 and μ_2^2.

Definition 2: The symmetric solution $x(t)$ is simple if $\mu_1 \neq -1$ and $\mu_2 \neq -1$.

The following fact may be proved by the same argument as in [**2**, p.348, Theorem 1.1].

Lemma 1. *If (2) has a simple, symmetric solution* $x_0(t)$ *for* $\lambda = \lambda_0$ *and* $\sigma = \sigma_0$ *and for fixed* ε, *then there exists a simple, symmetric solution* $x(t,\lambda,\sigma)$ *for* (λ,σ) *close to* (λ_0,σ_0), *which is analytic for* (λ,σ), *unique in a neighbourhood of* $x_0(t)$ *and* $x(t,\lambda_0,\sigma_0) = x_0(t)$.

Proof of Lemma 1: Let us denote the solution $x(t)$ of (2) with $x(0) = \xi$ and $\dot{x}(0) = \eta$ by $x(t,\xi,\eta;\ \lambda,\sigma)$, where ε is fixed. $x(t,\xi,\eta;\ \lambda,\sigma)$ is symmetric if and only if (ξ,η) is a solution of the equation.

$$F(\xi,\eta;\ \lambda,\sigma) = \xi + x(\pi,\xi,\eta;\ \lambda,\sigma) = 0$$
$$G(\xi,\eta;\ \lambda,\sigma) = \eta + \dot{x}(\pi,\xi,\eta;\ \lambda,\sigma) = 0.$$

Moreover we have that
$$\frac{\partial(F,G)}{\partial(\xi,\eta)} = (1+\mu_1)(1+\mu_2) \neq 0$$
for $\lambda = \lambda_0$ and $\sigma = \sigma_0$. Then the same argument as in [**2**] using the implicit function theorem completes the proof of Lemma 1. $\square$

When $\sigma = 0$, (2) has the trivial solution $x(t) \equiv 0$. We shall compute the half multipliers of $x(t) \equiv 0$, say $\mu_1(\varepsilon,\lambda)$ and $\mu_2(\varepsilon,\lambda)$.

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LEMMA 2.

$$\mu_1(\varepsilon,\lambda) = 1 - \frac{\varepsilon}{2}\left(\frac{1}{2n}\sqrt{a^2+b^2} + \lambda\pi\right) + O(\varepsilon^2),$$
(10)
$$\mu_2(\varepsilon,\lambda) = 1 + \frac{\varepsilon}{2}\left(\frac{1}{2n}\sqrt{a^2+b^2} - \lambda\pi\right) + O(\varepsilon^2),$$

where

$$a = \int_0^\pi p(t)\cos 4nt\,dt \text{ and } b = \int_0^\pi p(t)\sin 4nt\,dt.$$

PROOF OF LEMMA 2: Since $g'(0) = 0$, the variational system of (2) with respect to $x(t) \equiv 0$ is the following:

(11)
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \big(A - \varepsilon B(t)\big)\begin{pmatrix}x\\ y\end{pmatrix},$$

where

$$A = \begin{pmatrix}0 & 1\\ -4n^2 & 0\end{pmatrix} \text{ and } B(t) = \begin{pmatrix}0 & 0\\ p(t) & \lambda\end{pmatrix}.$$

Letting $U(t,\varepsilon)$ be the fundamental matrix of (11), we may set

$$U(t,\varepsilon) = U_0(t) + \varepsilon U_1(t) + V(t,\varepsilon),$$

where $U_0(0) = I$ for the 2×2 unit matrix I, $U_1(0) = 0$ and $V(0,\varepsilon) = 0$, and hence it follows that

$$\begin{aligned}\dot{U}_0 &= AU_0,\\ \dot{U}_1 &= AU_1 - B(t)U_0,\\ \dot{V} &= \big(A - \varepsilon B(t)\big)V - \varepsilon^2 B(t)U_1.\end{aligned}$$

Immediately we may compute $U_0(t)$ and show that $U_0(\pi) = I$. $U_1(t)$ and $V(t,\varepsilon)$ may also be obtained by the variational formula, and it may be shown that

$$U_1(\pi) = \begin{pmatrix}\dfrac{b}{4n} - \dfrac{\pi\lambda}{2} & -\dfrac{a}{8n^2}\\ -\dfrac{a}{2} & -\dfrac{b}{4n} - \dfrac{\pi\lambda}{2}\end{pmatrix}$$

and

$$V(\pi) = O(\varepsilon^2).$$

It follows that

$$U(\pi,\varepsilon) = \begin{pmatrix}1 + \varepsilon\left(\dfrac{b}{4n} - \dfrac{\pi\lambda}{2}\right) & -\dfrac{a\varepsilon}{8n^2}\\ -\dfrac{a\varepsilon}{2} & 1 - \varepsilon\left(\dfrac{b}{4n} + \dfrac{\pi\lambda}{2}\right)\end{pmatrix} + O(\varepsilon^2),$$

which implies (10). This completes the proof. ∎

Lemma 3 is exactly as stated in [**5**], and Lemma 4 follows immediately from [**12**].

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LEMMA 3. *In the equation*

$$\ddot{x} + k\dot{x} + a(t)x = 0,$$

where k is a positive constant and $a(t)$ is continuous for $t \in R$ and satisfies

$$\alpha \leqslant a(t) \leqslant \beta$$

for positive constants α and β, assume that

$$k > \sqrt{\beta} - \sqrt{\alpha}.$$

Then for any solution $x(t)$, both $x(t)$ and $\dot{x}(t)$ approach zero exponentially as $t \to \infty$.

LEMMA 4. *Consider the equation*

$$\ddot{x} + k\dot{x} + a(t)x + g(x) = b(t), \tag{12}$$

where k is a positive constant, $a(t)$ and $b(t)$ are continuous, 2π-periodic for $t \in R$, and $g(x)$ is continuous for $x \in R$ and satisfies (3). Then for each positive constant k_0 there exists a positive constant B_0 such that for $k > k_0$, every 2π-periodic solution of (12) satisfies

$$|x(t)| \leqslant B_0 \quad \text{for} \quad 0 \leqslant t \leqslant 2\pi,$$

where B_0 only depends on $\max_{0 \leqslant t \leqslant 2\pi} \left\{ |a(t)| + |b(t)| \right\}$.

Now we shall prove Theorem 2.

PROOF OF THEOREM 2: First let us take a number λ_1 such that

$$0 < \lambda_1 < \frac{1}{2n\pi}\sqrt{a^2 + b^2}, \tag{13}$$

where a and b are the constants of Lemma 2, and consider (2) for the case where $\lambda = \lambda_1$ and $\sigma = 0$. Then the trivial solution $x(t) \equiv 0$ has the half multipliers μ_1 and μ_2 such that $0 < \mu_1 < 1 < \mu_2$, where $0 < \varepsilon < \varepsilon_0$ for a small positive constant ε_0, because of (10) and (13). Furthermore, by Lemma 1 we can take a positive constant $\delta(\varepsilon)$ for each $0 < \varepsilon < \varepsilon_0$ such that if $\lambda = \lambda_1$ and $0 < \sigma < \delta(\varepsilon)$ then (2) has a unique, symmetric solution in a neighbourhood of $x(t) \equiv 0$ with the half multipliers μ_1 and μ_2 such that $0 < \mu_1 < 1 < \mu_2$. In the following we shall fix ε and σ such that $0 < \varepsilon < \varepsilon_0$ and $0 < \sigma < \delta(\varepsilon)$, and increase λ from λ_1 towards plus infinity. Again by Lemma 1, (2) has a unique, symmetric solution for λ close to λ_1, which is analytic for λ and denoted by $x(t, \lambda)$ as long as it is analytically defined for $\lambda \geqslant \lambda_1$. Moreover we shall denote the half multipliers of $x(t, \lambda)$ by $\mu_1(\lambda)$ and $\mu_2(\lambda)$, and hence $0 < \mu_1(\lambda_1) < 1 < \mu_2(\lambda_1)$.

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We shall show that there exists a number $\lambda_2 > \lambda_1$ such that $x(t,\lambda)$ exists for $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, $0 < \mu_1(\lambda) < 1$, $\mu_2(\lambda) > 0$ for $\lambda_1 \leqslant \lambda \leqslant \lambda_2$ and $0 < \mu_1(\lambda_2) < 1$, $0 < \mu_2(\lambda_2) < 1$. By Abel's equality we obtain

$$\mu_1(\lambda)\mu_2(\lambda) = e^{-\varepsilon\lambda\pi} < 1. \tag{14}$$

First, if $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are not real for some $\lambda' > \lambda_1$, then $|\mu_1(\lambda')| = |\mu_2(\lambda')| < 1$ by (14), and hence our assertion above follows from the continuity of $\mu_1(\lambda)$ and $\mu_2(\lambda)$ for λ. Secondly we may assume that $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are real numbers as long as $x(t,\lambda)$ is defined for $\lambda > \lambda_1$. In fact $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are positive as long as $x(t,\lambda)$ is defined for $\lambda > \lambda_1$. By Lemma 1, $x(t,\lambda)$ is actually defined for $\lambda > \lambda_1$, and hence $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are defined and positive for $\lambda > \lambda_1$. We shall consider the variational equation of (2) with respect to $x(t,\lambda)$:

$$\ddot{\xi} + \varepsilon\lambda\dot{\xi} + a(t,\lambda)\xi = 0, \tag{15}$$

where $a(t,\lambda) = 4n^2 + \varepsilon p(t) + g'(x(t,\lambda))$.

Because of Lemma 4, $x(t,\lambda)$ is uniformly bounded for $0 \leqslant t \leqslant 2\pi$ and for $\lambda \geqslant \lambda_1$, and hence $g'(x(t,\lambda))$ is too. Since $g'(x) \geqslant 0$, we can take positive constants $\alpha < \beta$ such that

$$\alpha \leqslant a(t,\lambda) \leqslant \beta \text{ for } 0 \leqslant t \leqslant 2\pi \text{ and for } \lambda \geqslant \lambda_1.$$

For a positive number λ_2 such that $\varepsilon\lambda_2 > \sqrt{\beta} - \sqrt{\alpha}$, the assertion of Lemma 3 implies that $0 < \mu_1(\lambda_2) < 1$ and $0 < \mu_2(\lambda_2) < 1$.

We can choose a number $\lambda_1 < \lambda_0 < \lambda_2$ such that $0 < \mu_1(\lambda_0) < 1$ and $\mu_2(\lambda_0) = 1$, by using the continuity of $\mu_1(\lambda)$ and $\mu_2(\lambda)$, in fact $\mu_1(\lambda_0) = e^{-\varepsilon\lambda\pi}$. Since $x(t,\lambda)$ is analytic for λ at $\lambda = \lambda_0$, it follows that $\mu_2(\lambda)$ is analytic for λ at $\lambda = \lambda_0$. Therefore we may assume that $\mu_1(\lambda) < 1 < \mu_2(\lambda)$ for $\lambda_0 - \delta' \leqslant \lambda < \lambda_0$ and $0 < \mu_1(\lambda) < 1$, $0 < \mu_2(\lambda) < 1$ for $\lambda_0 < \lambda \leqslant \lambda_0 + \delta'$, where δ' is a small positive number.

Since the number of existing 2π-periodic solutions of (2) is finite [**10**], we may define the index of $x(t,\lambda)$, say $I(x(\cdot,\lambda))$ (for the definition of index, see [**4**] or [**11**]). Now we shall suppose that (2) has no 2π-periodic solutions different from $x(t,\lambda)$ in a neighbourhood of $x(t,\lambda_0)$ for λ close to λ_0 and for $\lambda \neq \lambda_0$. Then, $x(t,\lambda)$ is the unique, 2π-periodic solution in a neighbourhood of $x(t,\lambda_0)$ for λ close to λ_0. Since $x(t,\lambda)$ is continuous for λ at $\lambda = \lambda_0$, $I(x(\cdot,\lambda))$ is continuous for λ at $\lambda = \lambda_0$, and hence, a constant integer for λ close to λ_0. Therefore we may assume that $I(x(\cdot,\lambda))$ is constant for $\lambda_0 - \delta' \leqslant \lambda \leqslant \lambda_0 + \delta'$. However, since $0 < \mu_1(\lambda_0 - \delta') < 1 < \mu_2(\lambda_0 - \delta')$ and since $0 < \mu_1(\lambda_0 + \delta') < 1$ and $0 < \mu_2(\lambda_0 + \delta') < 1$, it folows from [**4**] that $I(x(\cdot,\lambda_0 - \delta')) = -1$ and $I(x(\cdot,\lambda_0 + \delta')) = 1$. This is a contradiction. Therefore, (2) has a 2π-periodic solution $x^*(t,\lambda)$ different from $x(t,\lambda)$ such that $x^*(t,\lambda)$ approaches

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$x(t,\lambda_0)$ uniformly on $[0,2\pi]$ as $\lambda \to \lambda_0$. As is noted in Lemma 1, $x(t,\lambda)$ is the unique, symmetric solution in a neighbourhood of $x(t,\lambda_0)$ for λ close to λ_0, and hence $x^*(t,\lambda)$ must be nonsymmetric. Setting $\lambda_0 = \lambda(\varepsilon,\sigma)$ completes the proof. □

4. Solution orbit

We shall consider the equation:

$$\ddot{x} + f(x)\dot{x} + a(t)x + g(x) = e(t), \tag{16}$$

where $f(x)$ and $g(x)$ are analytic for $x \in R$, $f(-x) \equiv f(x)$, $g(-x) \equiv -g(x)$, $a(t)$ and $e(t)$ are analytic for $t \in R$, $a(t+\pi) \equiv a(t)$, $e(t+\pi) \equiv -e(t)$ and $e(t)$ has the least period 2π.

For the 2π-periodic solution $x(t)$ of (16) we shall call the set $\Omega = \{(x(t),\ \dot{x}(t)) \in R^2;\ 0 \leqslant t \leqslant 2\pi\}$ the solution orbit of $x(t)$.

Definition 2: The solution orbit Ω of the 2π-periodic solution $x(t)$ is symmetric with respect to the origin, if for each $t \in [0,2\pi)$, there exists a number $s \in [0,2\pi)$ such that $x(t) = -x(s)$ and $\dot{x}(t) = -\dot{x}(s)$.

Theorem 3. *Let $x(t)$ be a 2π-periodic solution of (16). If the solution orbit Ω of $x(t)$ is symmetric with respect to the origin, then $x(t)$ is symmetric.*

Proof of Theorem 3: Since $f(x)$, $g(x)$, $a(t)$ and $e(t)$ are analytic for respective arguments x and t, it is known that $x(t)$ is analytic for $t \in R$. We can take a number $0 \leqslant t_0 < 2\pi$ such that $\dot{x}(t_0) \neq 0$, because $x(t)$ is not constant, and moreover set $t_k = t_0 + (1/k)$ for some large positive interger k. By the symmetry of Ω there exists a number $s_k \in [0,2\pi)$ such that

$$x(t_k) = -x(s_k) \text{ and } \dot{x}(t_k) = -\dot{x}(s_k). \tag{17}$$

The sequence $\{s_k\}_{k=1}^{\infty}$ may be assumed to converge as $k \to \infty$, and hence we may set $s_0 = \lim_{k\to\infty} s_k$.

We shall consider the equation of (t,s) in a neighbourhood of (t_0,s_0):

$$f(t,s) = x(t) + x(s),$$

where $f(t,s)$ is analytic for (t,s). Since $f(t_0,s_0) = 0$ and $\dfrac{\partial f}{\partial t}(t_0,s_0) = \dot{x}(t_0) \neq 0$, the implicit function theorem gives us an analytic function $t = \varphi(s)$ defined for $s \in I$, where I is a neighbourhood of s_0, such that $\varphi(s_0) = t_0$ and

$$f(\varphi(s),s) = x(\varphi(s)) + x(s) = 0 \quad \text{for } s \in I. \tag{18}$$

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Moreover, since $f(t_k, s_k) = 0$ by (17), it follows that $\varphi(s_k) = t_k$; and hence that $\dot{x}(\varphi(s_k)) = -\dot{x}(s_k)$. Because both $\dot{x}(\varphi(s))$ and $\dot{x}(s)$ are analytic for $s \in I$, the unicity theorem guarantees that

$$\dot{x}(\varphi(s)) = -\dot{x}(s) \quad \text{for } s \in I. \tag{19}$$

On the other hand (18) implies that $\dot{x}(\varphi(s))\dot{\varphi}(s) = -\dot{x}(s)$ for $s \in I$. Therefore, $\dot{\varphi}(s) = 1$ for $s \in I$, because $\dot{x}(t_0) = \dot{x}(\varphi(s_0)) \neq 0$, where I is taken to be sufficiently small. Therefore $\varphi(s) = s + \omega$ for a constant $\omega = t_0 - s_0$, where $-2\pi < \omega < 2\pi$, and hence (18) implies that

$$x(s + \omega) = -x(s) \quad \text{for } s \in I. \tag{20}$$

Since $x(s)$ is analytic for $s \in R$, (20) holds not only for $s \in I$ but also for $s \in R$.

Finally we shall show that $\omega = \pi$. It may be assumed that $0 \leqslant \omega < 2\pi$ by the periodicity of $x(t)$. By combining (16) and (20) we can obtain that

$$\{a(t+\omega) - a(t)\}x(t) = -\sigma\{e(t+\omega) + e(t)\}.$$

If $\omega \neq \pi$, then $e(t+\omega) + e(t) \not\equiv 0$, and hence it follows from the analyticity of $x(t)$ that

$$x(t) = \frac{-\sigma\{e(t+\omega) + e(t)\}}{a(t+\omega) - a(t)} \quad \text{for } t \in R,$$

which immediately implies that $x(t+\pi) \equiv -x(t)$. This completes the proof of Theorem 3. ∎

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