J. Austral. Math. Soc. **21** (Series A) (1976), 376–380.

ON THE COMMUTATIVITY OF SOME CLASS OF RINGS

ABDULLAH HARMANCI

(Received 14 January 1975; revised 7 July 1975)

1. Introduction

Throughout, R will denote an associative ring with center Z. For elements x, y of R and k a positive integer, we define inductively $[x, y]_0 = x$, $[x, y] = [x, y]_1 = xy - yx$, $[x, y, y, \cdots, y]_k = [[x, y, y, \cdots, y]_{k-1}, y]$. A ring R is said to satisfy the k-th Engel condition if $[x, y, y, \cdots, y]_k = 0$. By an integral domain we mean a nonzero ring without nontrivial zero divisors. The purpose of this note is to generalize Theorem 1 in Ikeda-C. Koc (1974) and Herstein (1962) and Theorem 3.1.3 in Herstein (1968). The result is the following:

THEOREM. *Let k be a fixed nonnegative integer. Suppose R is a ring satisfying*

(1) $[x, y, y, \cdots, y]_k - [x, y, y, \cdots, y]_k^2 f(x, y) \in Z$ *for all* $x, y \in R$, *where $f(x, y)$ is a polynomial with integer coefficients which does not depend on x and y,*

or

(2) $[[x, y, y, \cdots, y]_k, z^m] = 0$ *for all* $x, y \in R$ *where m is a fixed positive integer. Then*

(*i*) *The commutator ideal $C(R)$ of R lies in the prime radical $P(R)$ of R,*

(*ii*) $[x, y, y, \cdots, y]_k^2 = [x, y, y, \cdots, y]_k$ *implies* $[x, y, y, \cdots, y]_k = 0$,

(*iii*) *$P(R)$ is locally nilpotent.*

2. Lemmata

We begin with

LEMMA 1. *Let R be a ring such that for each $x, y \in R$ there exists a polynomial $f_{x,y}(x, y)$ with integer coefficients which depend on x and y such that*

(3) $[x, y, y, \cdots, y]_k - [x, y, y, \cdots, y]_k^2 f_{x,y}(x, y) \in Z$. *Then the idempotents of R lie in the center Z of R.*

PROOF. Let e be a nonzero idempotent in R and x be any element of R. Then $[ex, e] = exe - ex$, $[ex, e, \cdots, e]_k = (-1)^{k+1}[ex, e]$, $[ex, e, \cdots, e]_k^2 = 0$ and $(ex, e, \cdots, e]_k - [ex, e, \cdots, e]_k^2 f_{ex,e}(ex, e) \in Z$ imply $[ex, e] \in Z$. Similarly $[xe, e] \in Z$. Hence $e[ex, e] = [ex, e]e = 0$ and $[xe, e]e = e[xe, e] = 0$, from which we obtain $ex = xe = exe$ for all $x \in R$. So $e \in Z$.

LEMMA 2. *Let R be a prime ring satisfying* (3). *Then R is an integral domain.*

PROOF. Suppose $xy = 0$ and $x \neq 0$. Let r be any element in R. Then

$$[yrx, y, y, \cdots, y]_k = (-1)^{k+1} y^{k+1} rx \quad \text{and} \quad [yrx, y, y, \cdots, y]_k^2 = 0$$

imply $(-1)^{k+1} y^{k+1} rx \in Z$. By taking the commutator of $(-1)^{k+1} y^{k+1} rx$ and y we obtain $y^{k+2} rx = 0$ for all $r \in R$. Hence $y^{k+2} Rx = 0$. This implies $y^{k+2} = 0$ for all y in the right annihilator of x, which is a right ideal. Since R is prime, Lemma 1.1 of Herstein (1969) implies $y = 0$. This completes the proof.

LEMMA 3. *Suppose R is an integral domain satisfying* (3). *Then the center of R cannot be zero.*

PROOF. We assume that $Z = (0)$ and obtain a contradiction. If R is commutative then R must be a zero integral domain which is a contradiction since R is a nonzero ring. Suppose R is not commutative. By using the fact that any integral domain satisfying the k-th Engel condition is commutative Herstein (1962), we can find x, y in R such that $a = [x, y, y, \cdots, y]_k \neq 0$. Hence $a = a^2 f_{x,y}(x, y)$ which implies that $af_{x,y}(x, y)$ is an identity, and so lies in the center which is zero. It follows that $a = 0$. This contradiction proves the lemma.

LEMMA 4. *Let R be an integral domain satisfying* (1) *with finite center Z. Then R is commutative.*

PROOF. We first note that $[xy, x, x, \cdots, x]_k = x[y, x, x, \cdots, x]_k$ and the non-zero elements Z^* of Z form a finite cyclic group with identity 1, say. Then 1 is also an identity of R. Let $x \neq 0 \in R$. If x is in Z, then x has an inverse. Suppose x is not in Z. Then we can find at least one y in R such that $[y, x] \neq 0$. In this case, if $[y, x, x, x]_k \neq 0$, then

$$0 \neq [xy, x, x, \cdots, x]_k = x[y, x, x, \cdots, x],$$

$$x[y, x, x, \cdots, x]_k^2 - [xy, x, x, \cdots, x]_k^2 f(xy, y) \in Z$$

imply x has an inverse. Suppose $[y, x, x, \cdots, x]_k = 0$. Let T denote the subring of R generated by x and xy. If T satisfies the k-th Engel condition, it must be commutative. This leads to $[x, y] = 0$ since x, xy are in T and T is an integral domain. This contradiction gives rise to the existence of some a and b in T

such that $[a, b, b, \cdots, b]_k \neq 0$. By considering a and b we conclude that x has an inverse in this case also. So far we have proved that each nonzero element x of R has an inverse. This shows that R is a division ring. On the other hand, the division ring R satisfies the polynomial identity

$$([x, y, y, \cdots, y]_k - [x, y, y, \cdots, y]_k^2 f(x, y))z = z([x, y, y, \cdots, y]_k - [x, y, y, \cdots, y]_k^2 f(x, y)).$$

Hence R is finite dimensional over its center Z, which is finite Kaplansky (1948). It follows that R is a finite integral domain. Thus R is commutative by Wedderburn's theorem.

LEMMA 5. *Let R be an integral domain satisfying* (1) *with an infinite center Z. Then R is commutative.*

PROOF. Decompose $f(x, y)$ into homogeneous parts $\Sigma_1^n f_i(x, y)$ and let $t_i, i = 1, 2, \cdots, n$, denote the degree of $f_i(x, y)$. Since Z has infinitely many elements, we can find $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1} \in Z$ such that the determinant

$$D = \begin{vmatrix} \lambda_1^{k+1} & \lambda_1^{t_1+2k+2} \cdots & \lambda_1^{t_n+2k+2} \\ \lambda_2^{k+1} & \lambda_2^{t_1+2k+2} \cdots & \lambda_2^{t_n+2k+2} \\ & {}_2 \quad \cdot\,\cdot & \\ \lambda_{n+1}^{k+1} & \lambda_{n+1}^{t_1+2k+2} \cdots & \lambda_{n+1}^{t_n+2k+2} \end{vmatrix}$$

is non-zero. For any $\lambda \in Z$, we may replace x and y by λx and λy respectively in (1). Using the fact that $D \neq 0$ and R is an integral domain we obtain $D[x, y, y, \cdots, y]_k \in Z$, and so $[x, y, y, \cdots, y]_{k+1} = 0$ for all x, y in R. Thus R is commutative because it is an integral domain satisfying the $k+1$-st Engel condition Herstein (1962).

By combining Lemma 2, Lemma 4 and Lemma 5 we obtain Lemma 6

LEMMA 6. *Every prime ring satisfying* (1) *is commutative.*

LEMMA 7. *Let R be a prime ring satisfying the polynomial identity* (2). *Then R is an integral domain.*

PROOF. Let $x \neq 0$, y and r be any elements of R such that $xy = 0$. From (2) we obtain $[[yrx, y, y, \cdots, y]_k, y^m] = 0$ implying $y^{m+k+1}rx = 0$ for all r in R. Since R is prime and $x \neq 0$, it follows that $y^{m+k+1} = 0$ for all y in the right annihilator of x. Hence $y = 0$ by Lemma 1.1 of Herstein (1969).

LEMMA 8. *Let R be a prime ring satisfying* (2). *Then R is commutative.*

PROOF. From the preceding Lemma it follows that R is an integral domain. Then Posner's theorem, [Theorem 5.6 of McCoy (1964)] implies that R can be embedded in a division ring R' satisfying the same identity as does R. The division ring R' satisfying (2) is commutative [Lemma 2, Ikeda-C. Koc (1974)]. Hence R is commutative, since it is a subring of R'.

3. Proof of the theorem

Let P be any prime ideal of R. Then the prime ring R/P is commutative by Lemmas 6 and 8. Hence each commutator $[x, y] = 0$ and so the commutator ideal $C(R)$ lies in P. Since P is an arbitrary prime ideal, $C(R)$ lies in the prime radical $P(R)$, thus proving (i). Since $P(R)$ is a nil ideal [Theorem 4.21 McCoy (1964)], to prove (ii) it is enough to show that $x^t = 0$ implies x is in $P(R)$. For this, assume $x^t = 0$. The commutative prime ring R/P does not contain nonzero nilpotent elements. So x lies in each prime ideal and therefore in $P(R)$, thus proving (ii). We have just proved $C(R)$ is in $P(R)$. It is well known that $P(R)$ lies in the Jacobson radical $J(R)$ of R. If $[x, y, y, \cdots, y]_k^2 = [x, y, y, \cdots, y]_k$, then $[x, y, y, \cdots, y]_k$ would be an idempotent in $J(R)$ implying that $[x, y, y, \cdots, y]_k = 0$ which proves (iii). Since $P(R)$ is nil and satisfies a polynomial identity, it is locally nilpotent [Theorem 5, Kaplansky (1948)].

4. Examples

The existence of a polynomial satisfying (3) with not necessarily integral coefficients which depend on a pair of elements of R need not imply the commutativity of R, even if R is a division ring. Therefore, some restrictions on the polynomial or on its coefficients are necessary in the hypothesis of the Theorem:

Example 1. Let R denote the ring of real quaternions and for each $x, y \in R$, we define

$$f_{x,y}(x, y) = \begin{cases} [x, y]^{-1} & \text{if } [x, y] \neq 0 \\ 0 & \text{if } [x, y] = 0 \end{cases}$$

In R, (3) is satisfied by $k = 1$ and $f_{x,y}(x, y)$ defined above, which depends on x and y but does not have integral coefficients. Indeed R is not commutative.

To fix the polynomial as in (1) again need not, in general, imply the commutativity of the ring:

Example 2. Let R denote the subring of the ring of all 3×3 matrices over the Galois field GF(2) generated by e_{12}, e_{13}, e_{23} (or e_{21}, e_{31}, e_{32}) where e_{ij}, $ij = 1,2,3$, denotes the matrix with 1 at the (i, j) entry and zeros elsewhere. It is readily verified that $[x, y]^2 = 0$ and $xy = 0$ or $e_{13}(e_{31})$ and $e_{13}(e_{31}) \in Z$. Hence (1) is satisfied by any polynomial and $k = 1$. But R is indeed non-commutative.

https://doi.org/10.1017/S1446788700018681 Published online by Cambridge University Press

References

S. A. Amitsur (1957), 'A generalisation of Hilsert's nullstellensatz', *Proc. Amer. Math. Soc.* **8**, 649–656.

S. A. Amitsur (1955), 'On rings with identities', *J. London Math. Soc.* **30**, 464–470.

I. N. Herstein (1962), 'Sugli Anelli Soddis, ad una Cond. di Engel', *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **32**, 177–180.

I. N. Herstein (1968), '*Non-commutative*', New York.

I. N. Herstein (1969), '*Topics in ring theory*', Chicago.

M. Ikeda-C. Koc (1974), 'On the commutator ideal of certain rings', *Arc. der Math.* **25**, 348–353.

N. Jacobson (1964), 'Structure of Rings', *Amer. Math. Soc. Coll. Publ.*

I. Kaplansky (1948), 'Rings with a polynomial identity', *Bull. Amer. Math. Soc.* **54**, 575–580.

N. H. McCoy (1964), '*The Theory of Rings*', MacMillan.

Department of Mathematics
Hacettepe University
Ankara
Turkey.

https://doi.org/10.1017/S1446788700018681 Published online by Cambridge University Press