IMPROVED SECULAR STABILITY LIMITS FOR ROTATING WHITE DWARFS

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In the special case of the Maclaurin spheroids, it has been known for some time that the m = 2 barlike modes become secularly unstable for $t \equiv T/|W| \gtrsim 0.1376$ where T is the total rotational kinetic energy and W is the total gravitational energy of the spheroid. "Secular" here means that the instability depends on dissipative processes and grows on a long dissipative time scale. In particular, the Dedekind-like bar mode, which has zero eigenfrequency at t = 0.1376 as viewed in the nonrotating frame, is unstable due to gravitational radiation (Chandrasekhar 1970).

These results can be obtained with such relative ease and elegance by use of the tensor virial equation (TVE) method (cf Chandrasekhar 1969) that the method was generalized (Tassoul and Ostriker 1968) and applied to more realistic rotating stellar models, first to white dwarfs (Ostriker and Tassoul 1969) and then to polytropes (Tassoul and Ostriker 1970, Ostriker and Bodenheimer 1973). Durisen (1975) extended the mass range covered by Ostriker and Tassoul to find that secular instability imposes an upper mass limit on stable rapidly rotating white dwarfs of about 2.5 $M_{\odot}$. Remarkably, for all equations of state and rotation laws considered by these investigators, the secular stability limit was found to occur at $t = t_V = 0.138 \pm 0.003$. Because the TVE method is equivalent to a particular choice of trial eigenfunction in the variational principle of Lynden-Bell and Ostriker (1967), this universal result was accepted as reasonably accurate and physically significant. Consequently, t = 0.14 has often been cited as a secular stability limit in a wide range of applications to rotating stars (e.g. Shapiro and Lightman 1976, Endal and Sofia 1978, Larson 1979).

Lagrangian perturbation theory for rotating stars has been considerably refined in the past four years, and the TVE method has been subjected to severe criticism (Friedman and Schutz 1975 a and b, Hunter 1977). Particularly, Friedman and Schutz (1978 a and b) have shown that the TVE method provides neither a necessary nor sufficient condition for secular instability when stars are differentially

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rotating. They have shown that the Lagrangian displacement ξ_V implicitly assumed by the TVE method, namely the Maclaurin spheroid bar mode,

$$\xi_V = (r, ir, 0)\, e^{2i\varphi} \tag{1}$$

in cylindrical coordinates (r, φ, z), contains a "trivial" displacement component for differential rotation which invalidates the variational principle. A "trivial" displacement is one that violates Kelvin's circulation theorem and so reduces to a relabeling of fluid elements which does not conserve the canonical energy in the variational principle.

Bardeen, Friedman, Schutz, and Sorkin (1977, hereafter BFSS) have suggested use of the following trial eigenfunction free of trivials,

$$\xi_B = \left[rA,\ ir\,(A + B),\ 0\right] e^{2i\varphi} \tag{2}$$

where A(r) and B(r) satisfy

$$\begin{aligned} r\frac{dA}{dr} &= (1 - A)\,\frac{r}{\sigma}\frac{d\sigma}{dr} + 2B \\ r\frac{dB}{dr} &= -4B - \frac{r}{\Omega}\frac{d\Omega}{dr}\,(A + 2B) \end{aligned} \tag{3}$$

with A = 1 at r = the equatorial radius R and B = 0 at r = 0. The quantity $\sigma(r)$ is the surface density of the star as viewed along the rotation axis. Then

$$\int_{Volume} \xi_B^{*}\; C(\xi_B)\; dV \leq 0 \tag{4}$$

provides a sufficient condition for secular instability of the Dedekind-like bar mode. The operator C is given by

$$\begin{aligned} C(x) = {}& \rho\,(v\cdot\nabla)^2 x - \rho\,(x\cdot\nabla)(v\cdot\nabla)\,v + \left(\frac{1}{\rho}\nabla P\right)\nabla\cdot\rho x \\ & - \nabla(\Gamma_1 P\,\nabla\cdot x) - \nabla(x\cdot\nabla P) - \rho\,\nabla\,\delta\Phi\ , \end{aligned} \tag{5}$$

where v, ρ, P, and Γ_1 are the zero-order stellar velocity field, mass density, pressure, and adiabatic $d\ln P/d\ln\rho$ and where $\delta\Phi$ is the Eulerian perturbation in the gravitational potential. BFSS applied their criterion to thin disks and obtained a secular stability limit t_B about 10-25% higher than t_V. Clement (1979) has found that for differentially rotating main sequence stars the BFSS criterion gives a t_B = 0.10 to 0.11 or 30-40% lower than t_V.

Because of their central concentration, white dwarfs must be differentially rotating to attain large t-values without surpassing critical rotation at the equator, and so the secular stability limits obtained previously by the TVE method are invalid. We are currently applying the BFSS criterion to cold rotating white-dwarf models generated by the self-consistent field method (Ostriker and Mark 1968, Ostriker and Bodenheimer 1968). As Clement notes, the evaluation of the integral in equation (4) is "straightforward albeit laborious" except for the determination of the perturbed potential $\delta\Phi$. We must solve Poisson's equation

$$\nabla^2 \delta\Phi = 4\pi G\, \delta\rho \quad , \tag{6}$$

where $\delta\rho$ is the Eulerian perturbation in density given by

$$\delta\rho = -\underset{\sim}{\nabla} \cdot \rho \underset{\sim}{\xi} . \tag{7}$$

To accomplish this, we parallel the Green's function method of Ostriker and Mark by expressing $\delta\rho$ as an expansion in even powers of the spherical radius and in associated Legendre polynomials with m = 2 whose argument is cos θ where θ is polar angle. The integration of equations (3), on the other hand, is simple. We find, as Clement did, that a shooting integration method from r = 0 can be iterated Newton-Raphson style until both boundary conditions are satisfied. Our resulting A(r) and B(r) are similar in form to his.

At the time of writing, we have not yet obtained firm values of t_B for white dwarfs, but we can anticipate our results on the basis of Clement's work. Roughly speaking, his upper main sequence models have both $\Gamma_1 \approx 4/3$ and an effective polytropic index $n_e \approx 3$. The white-dwarf models yielding the highest secularly stable masses in Durisen's TVE work have central densities well in excess of 10^6 gm/cc and so, being relativistically degenerate, also have $\Gamma_1 \approx 4/3$ and $n_e \approx 3$. The degree of differential rotation is similar in both sets of models: 9 to 1 in $\Omega(r)$ from center to equator in Clement's $\beta = 8$ case and 7 to 1 for Durisen's n' = 0 case at high masses. For his 30 $M_\odot$ $\beta = 8$ model, Clement found $t_B = 0.10$. For white dwarfs, this corresponds to a revised upper mass limit of about 2.0 $M_\odot$ for secular stability. The secularly stable region in the total mass M -

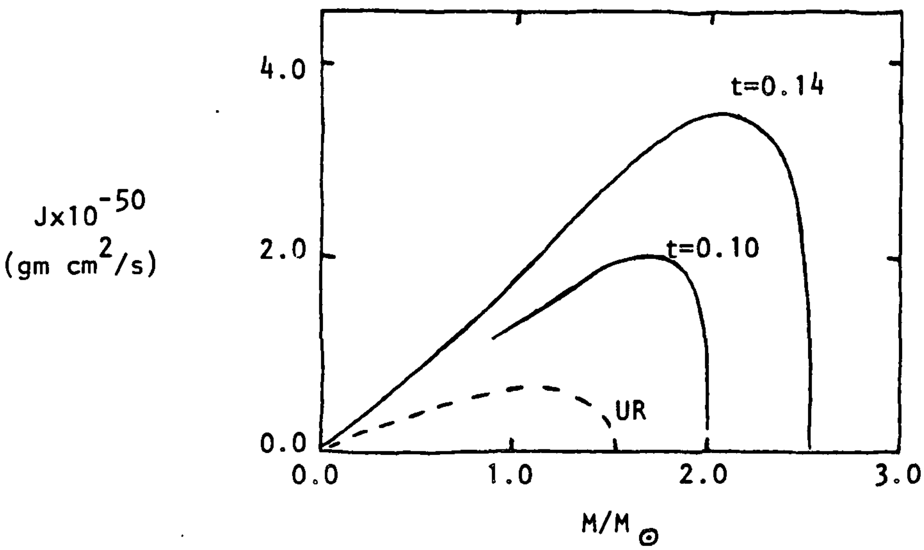

Figure 1. - The (M, J) - plane.

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total angular momentum J plane is correspondingly smaller as shown in figure 1. The t = 0.10 curve is not extended below about one solar mass, because low-mass models are nonrelativistically degenerate and the analogy to Clement's models does not apply. The t-curves in figure 1 are based on the n' = 0 angular momentum distribution described by Durisen. Also shown in dashes is the boundary of the region where uniform rotation is possible (James 1964). Estimates for the e-folding time due to gravitational radiation in the unstable region are not much affected by the change in limiting t and are still typically in the range $10^{5\pm3}$ years (see Durisen's figure 1).

Modes of higher azimuthal order, $\xi \sim e^{im\varphi}$ with $|m| > 2$, may also prove to be important for secular stability. Friedman and Schutz (1978b) discovered a generic instability of all rotating stars for sufficiently large m-values. The e-folding time scales due to gravitational radiation are $\gg 10^{10}$ years except for the lower m-values of neutron stars and for m = 2 of white dwarfs (Papaloizou and Pringle 1978). However, there are other mechanisms that could mimic gravitational radiation but have a much shorter time scale: particularly surface stresses and gravitational torques exerted by an external medium (red giant envelope, accretion disk) or by a binary companion.

For instance, suppose the pattern speed ω_p of an unstable mode is such that $\omega_p R \gg$ the sound speed c_s in an external medium of density ρ_e. The surfaces stresses induced by distortion of the stellar surface will then be $\pi/2$ out of phase with the surface displacement. Using the surface drag from the solution for supersonic flow past a wavy wall (Liepman and Roshko 1957) and using a canonical mode energy of $-m^2 \int \rho \Omega^2 |\xi|^2 \, dV$ (see equation 16 of Friedman and Schutz 1978b), we find a growth time

$$\tau_p \sim \frac{T}{\rho_e c_s R^4 \omega_p^2} \tag{8}$$

To get a feel for the order of magnitude here, let $T = 10^{48}$ ergs, $\rho_e = 10^{-6}$ gm/cc, $c_s = 10$ km/sec, $R = 10^9$ cm, and $\omega_p = 10^{-1}$ /s. Then $\tau_p = 3$ million years. The consequence of such an instability would be loss of angular momentum and rotational energy by the dwarf to the surrounding medium on this time scale.

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