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Anisotropic flow, entropy, and L^p-Minkowski problem

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Abstract. We provide a natural simple argument using anistropic flows to prove the existence of weak solutions to Lutwak's L^p-Minkowski problem on S^n which were obtained by other methods.

1 Introduction

For $\alpha > 0$ and nonnegative $f \in L^1(\mathbb{S}^n)$ with positive integral, we are interested in finding a weak solution to the Monge–Ampére equation

(1.1) $$u^{\frac{1}{\alpha}} \det(\bar{\nabla}^2_{ij} u + u\bar{g}_{ij}) = f,$$

or in other words, a weak solution to Lutwak's L^p-Minkowski problem on S^n when $-n-1 < p < 1$ for $p = 1 - \frac{1}{\alpha}$ where $\bar{\nabla}$ is the Levi-Civita connection of $\mathbb{S}^n$, $\bar{g}_{ij}$, with $\bar{g}$ being the induced round metric on the unit sphere. By a weak (Alexandrov) solution, we mean the following: Given a nontrivial finite Borel measure μ on $\mathbb{S}^n$ (for example, $d\mu = f\, d\theta$ for the Lebesgue measure θ on S^n and the f in (1.1)), find a convex body $\Omega \subset \mathbb{R}^{n+1}$ with $o \in \Omega$ such that

(1.2) $$d\mu = u^{\frac{1}{\alpha}}\, dS_\Omega,$$

where $u(x) = \max_{z\in\Omega}\langle x, z\rangle$ is the support function and S_Ω is the surface area measure of Ω (see [45]). If $\partial\Omega$ is C^2_+, then

$$dS_\Omega = \det(\bar{\nabla}^2_{ij} u + u\bar{g}_{ij})d\theta = K^{-1}d\theta,$$

where $K(x)$ is the Gaussian curvature at the point of $\partial\Omega$ where $x \in S^n$ is the exterior unit normal (see [45]). Concerning the regularity of the solution of (1.1), if $f \in C^{0,\beta}(S^n)$ and u are positive, then u is $C^{2,\beta}$ according to Caffarelli's regularity theory in [15, 16]. On the other hand, even if f is positive and continuous for $\alpha > \frac{1}{n}$, there might exist weak solution where $u(x) = 0$ for some $x \in S^n$ and u is not even C^1 according to Example 4.2 in [7]. Moreover, even if $f \in C^{0,\beta}(S^n)$ is positive, it is possible that $u(x) = 0$ for some $x \in S^n$ for $\alpha > \frac{1}{n}$, but Choi, Kim, and Lee [19] still managed to obtain some regularity results in this case.

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The case $\alpha = \frac{1}{n+2}$ of the Monge–Ampére equation (1.1) is the critical case when the left-hand side of (1.1) is invariant under linear transformations of Ω, and the case $\alpha = 1$ is the so-called logarithmic Minkowski problem posed by Firey [23]. Setting $p = 1 - \frac{1}{\alpha} < 1$, the Monge–Ampére equation (1.1) is Lutwak's L^p-Minkowski problem

$$u^{1-p} \det(\bar{\nabla}^2_{ij} u + u\bar{g}_{ij}) = f. \tag{1.3}$$

In this notation, (1.2) reads as

$$d\mu = u^{1-p}\, dS_\Omega; \tag{1.4}$$

that equation makes sense for any $p \in \mathbb{R}$. Within the rapidly developing L^p-Brunn–Minkowski theory (where $p = 1$ is the classical case originating from Minkowski's oeuvre) initiated by Lutwak [39–41], if $p > 1$ and $p \neq n+1$, then Hug, Lutwak, Yang, and Zhang [30] (improving on Chou and Wang [20]) prove that (1.4) has an Alexandrov solution if and only if the μ is not concentrated onto any closed hemisphere, and the solution is unique. We note that there are examples in [25] (see also [30]) and show that if $1 < p < n+1$, then it may happen that the density function f is a positive continuous in (1.3) and $o \in \partial K$ holds for the unique Alexandrov solution, and actually Bianchi, Böröczky, and Colesanti [7] exhibit an example that $o \in \partial K$ even if the density function f is a positive continuous in (1.3) assuming $-n-1 < p < 1$.

In the case $p \in (0,1)$ (or equivalently, $\alpha > 1$), if the measure μ is not concentrated onto any great subsphere of S^n, then Chen, Li, and Zhu [17] prove that there exists an Alexandrov solution $K \in \mathcal{K}^n_o$ of (1.4) using a variational argument (see also [8]). We note that for $p \in (0,1)$ and $n \geq 2$, no complete characterization of L^p-surface area measures is known (see [12] for the case $n = 1$, and [8, 43] for partial results about the case when $n \geq 2$ and the support of μ is contained in a great subsphere of S^n).

Concerning the case $p = 0$ (or equivalently, $\alpha = 1$), the still open logarithmic Minkowski problem (1.3) or (1.4) was posed by Firey [23] in 1974. The paper [11] characterized even measures μ such that (1.4) has an even solution for $p = 0$ by the so-called subspace concentration condition (see (a) and (b) in Theorem 1.1). In general, Chen, Li, and Zhu [18] proved that if a nontrivial finite Borel measure μ on S^{n-1} satisfies the same subspace concentration condition, then (1.4) has a solution for $p = 0$. On the other hand, Böröczky and Hegedus [10] provide conditions on the restriction of the μ in (1.4) to a pair of antipodal points.

If $-n-1 < p < 0$ (or equivalently, $\frac{1}{n+2} < \alpha < 1$) and $f \in L_{\frac{n+1}{n+1+p}}(S^n)$ in (1.3), then (1.3) has a solution according to [8]. For a rather special discrete measure μ satisfying that μ is not concentrated on any closed hemisphere and any n unit vectors in the support of μ are independent, Zhu [47] solves the L^p-Minkowski problem (1.4) for $p < 0$. The $p = -n-1$ (or equivalently, $\alpha = \frac{1}{n+2}$) case of the L^p-Minkowski problem is the critical case because its link with the $\mathrm{SL}(n)$ invariant centro-affine curvature whose reciprocal is $u^{n+2}\det(\bar{\nabla}^2_{ij} u + u\bar{g}_{ij})$ (see [29] or [38]). For positive results concerning the critical case $p = -n-1$, see, for example, [28, 34], and for obstructions for a solution, see, for example, [20, 22].

In the super-critical case $p < -n-1$ (or equivalently, $\alpha < \frac{1}{n+2}$), there is a recent important work by Li, Guang, and Wang [27] proving that for any positive C^2 function f, there exists a C^4 solution of (1.3). See also [22] for non-existence examples.

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The main contribution of this paper is to provide a very natural argument based on anisotropic flows developed by Andrews [4] to handle the case $-n-1<p<1$, or equivalently, the case $\frac{1}{n+2}<\alpha<\infty$.

Entropy functional. For any convex body Ω, a fixed positive function f on $\mathbb{S}^n$ and $\alpha\in(0,\infty)$, define

$$\mathcal{E}_{\alpha,f}(\Omega):=\sup_{z\in\Omega}\mathcal{E}_{\alpha,f}(\Omega,z), \tag{1.5}$$

where

$$\mathcal{E}_{\alpha,f}(\Omega,z):=\begin{cases}\frac{\alpha}{\alpha-1}\log\left(\fint_{\mathbb{S}^n}u_z(x)^{1-\frac{1}{\alpha}}f(x)d\theta(x)\right), & \alpha\neq 1,\\ \fint_{\mathbb{S}^n}\log(u_z(x))f(x)d\theta(x), & \alpha=1.\end{cases} \tag{1.6}$$

Here, $u_z(x):=\sup_{y\in\Omega}\langle y-z,x\rangle$ is the *support function* of Ω in direction x with respect to z_0 and$\fint_{\mathbb{S}^n}h(x)\,d\theta(x)=\frac{1}{\omega_n}\int_{\mathbb{S}^n}h(x)$ with ω_n being the surface area of $\mathbb{S}^n$ and θ is the Lebesgue measure on S^n. When $\alpha=1$ and $f(x)\equiv 1$, then the above quantity agrees with the entropy in [26], first introduced by Firey [23] for the centrally symmetric Ω. General integral quantities were studied by Andrews in [2, 4]. Here, we shall assume that$\fint_{\mathbb{S}^n}f(x)\,d\theta(x)=1$, namely, $\frac{1}{\omega_n}f(x)d\theta(x)$ is a probability measure. For the special case $f\equiv 1$, $\mathcal{E}_{\alpha,f}(\Omega)$ becomes the entropy $\mathcal{E}_\alpha(\Omega)$ in [6].

For positive $f\in C^\infty(\mathbb{S}^n)$, consider the anisotropic flow for convex hypersurfaces $\tilde{X}(\cdot,\tau):M_\tau\to\mathbb{R}^{n+1}$:

$$\frac{\partial}{\partial\tau}\tilde{X}(x,\tau)=-f^\alpha(\nu)\tilde{K}^\alpha(x,\tau)\,\nu(x,\tau), \tag{1.7}$$

where $\nu(x,\tau)$ is the unit exterior normal at $\tilde{X}(x,\tau)$ of $\tilde{M}_\tau=\tilde{X}(M,\tau)$, and $\tilde{K}(x,\tau)$ is the Gauss curvature of $\tilde{M}_\tau$ at $\tilde{X}(x,\tau)$. Andrews [4] proved that flow (1.7) contracts to a point under finite time if the initial hypersurface M_0 is strictly convex. Under a proper normalization, the normalized anisotropy flow of (1.7) is

$$\frac{\partial}{\partial t}X(x,t)=-\frac{f^\alpha(\nu)K^\alpha(x,t)}{\fint_{\mathbb{S}^n}f^\alpha K^{\alpha-1}}\,\nu(x,t)+X(x,t). \tag{1.8}$$

The basic observation is that a critical point for entropy $\mathcal{E}_{\alpha,f}(\Omega)$ defined in (1.5) under volume normalization is a solution to equation (1.1). The entropy is monotone along flow (1.8). One may view (1.1) is an "optimal solution" to this variational problem as the flow (1.8) provides a natural path to reach it. This approach was devised in [5] with the aim to obtain convergence of the normalized flow (1.8). The main arguments in [5] follows those in [6, 26] where convergence of isotropic flows by power of Gauss curvature (i.e., $f=1$) was established. Unfortunately, the entropy point estimate in [6, 26] fails for general anisotropic flows except $\frac{1}{n+2}<\alpha\leq\frac{1}{n}$ [4]. The convergence was obtained in [5] assuming M_0 and f are invariant under a subgroup G of $O(n+1)$ which has no fixed point. We note that an inverse Gauss curvature flow argument was considered by Bryan, Ivaki, and Scheuer [14] to produce a origin-symmetric solution to (1.1).

Since we are only interested in finding a weak solution to (1.2), one only needs certain "weak" convergence of the flow (1.8). The key steps are to control diameter

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with entropy under appropriate conditions on measure $\mu = f d\theta$ on $\mathbb{S}^n$ and use monotonicity of entropy to produce a solution to (1.2). The following is our main result.

Theorem 1.1 *For $\alpha > \frac{1}{n+2}$ and finite nontrivial Borel measure μ on $\mathbb{S}^n$, $n \geq 1$, there exists a weak solution of (1.2) provided the following holds:*

(i) *If $\alpha > 1$ and μ is not concentrated onto any great subsphere $x^\perp \cap \mathbb{S}^n$, $x \in \mathbb{S}^n$.*
(ii) *If $\alpha = 1$ and μ satisfies that for any linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \leq \ell \leq n$, we have*
  (a) $\mu(L \cap \mathbb{S}^n) \leq \dfrac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$;
  (b) *equality in (a) for a linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \leq d \leq n$ implies the existence of a complementary linear $(n+1-\ell)$-subspace $\widetilde{L} \subset \mathbb{R}^{n+1}$ such that* $\operatorname{supp} \mu \subset L \cup \widetilde{L}$.
(iii) *If $\frac{1}{n+2} < \alpha < 1$ and $d\mu = f\, d\theta$ for nonnegative $f \in L^{\frac{n+1}{n+2-\frac{1}{\alpha}}}(\mathbb{S}^n)$ with $\int_{\mathbb{S}^n} f > 0$.*

Let us briefly discuss what is known about uniqueness of the solution of the L^p-Minkowski problem (1.4). If $p > 1$ and $p \neq n$, then Hug, Lutwak, Yang, and Zhang [30] proved that the Alexandrov solution of the L^p-Minkowski problem (1.4) is unique. However, if $p < 1$, then the solution of the L^p-Minkowski problem (1.3) may not be unique even if f is positive and continuous. Examples are provided by Chen, Li, and Zhu [17, 18] if $p \in [0, 1)$, and Milman [42] shows that for any $C \in \mathcal{K}_{(0)}$, one finds $q \in (-n, 1)$ such that if $p < q$, then there exist multiple solutions to the L^p-Minkowski problem (1.4) with $\mu = S_{C,p}$; or in other words, there exists $K \in \mathcal{K}_{(0)}$ with $K \neq C$ and $S_{K,p} = S_{C,p}$. In addition, Jian, Lu, and Wang [33] and Li, Liu, and Lu [37] prove that for any $p < 0$, there exists positive even C^∞ function f with rotational symmetry such that the L^p-Minkowski problem (1.3) has multiple positive even C^∞ solutions. We note that in the case of the centro-affine Minkowski problem $p = -n$, Li [36] even verified the possibility of existence of infinitely many solutions without affine equivalence, and Stancu [46] related unique solution in the cases $p = 0$ and $p = -n$.

The case when f is a constant function in the L^p-Minkowski problem (1.3) has received a special attention since [23]. When $p = -(n+1)$, (1.3) is self-similar solution of affine curvature flow. It is proved by Andrews that all solutions are centered ellipsoids. If $n = 2$ and $p = 2$, the uniqueness was proved by Andrews [3]. For general n and $p > -(n+1)$, through the work of Lutwak [40], Guan-Ni [26], and Andrews, Guan, and Ni [6], Brendle, Choi, and Daskalopoulos [13] finally classified that the only solutions are centered balls. See also [21, 32, 44] for other approaches. Stability versions of these results have been obtained by Ivaki [31], but still no stability version is known in the case $p \in [0, 1)$ if we allow any solutions of (1.3) not only even ones.

Concerning recent versions of the L^p-Minkowski problem, see [9].

The paper is structured as follows: The required diameter bounds are discussed in Section 2. Section 3 verifies the main properties of the Entropy, Section 4 proves our main result (Theorem 4.1) about flows, and finally Theorem 1.1 is proved in Section 5 via weak approximation.

2 Entropy and diameter estimates

For $\delta \in [0,1)$ and linear i-subspace L of $\mathbb{R}^{n+1}$ with $1 \le \dim L \le n$, we consider the collar

$$\Psi(L \cap \mathbb{S}^n, \delta) = \{x \in \mathbb{S}^n : \langle x, y\rangle \le \delta \text{ for } y \in L^\perp \cap \mathbb{S}^n\}.$$

Let $B(1) \subset \mathbb{R}^{n+1}$ be the unit ball centered at the origin.

Theorem 2.1 *Let $\alpha > \frac{1}{n+2}$, let $\fint_{\mathbb{S}^n} f = 1$ for a bounded measurable function f on $\mathbb{S}^n$ with $\inf f > 0$, and let $\Omega \subset \mathbb{R}^{n+1}$ be a convex body such that $|\Omega| = |B(1)|$ and $\operatorname{diam}\Omega = D$. For any $\delta, \tau \in (0,1)$, we have*

(i) *if $\alpha > 1$, and $\fint_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f \le 1 - \tau$ for any $z \in S^n$, then*

$$\exp\left(\frac{\alpha-1}{\alpha}\mathcal{E}_{\alpha,f}(\Omega)\right) \ge \gamma_1 \tau \delta^{1-\frac{1}{\alpha}} D^{1-\frac{1}{\alpha}},$$

where $\gamma_1 > 0$ depends on n and α;

(ii) *if $\alpha = 1$, and*

$$\fint_{\Psi(L \cap \mathbb{S}^n, \delta)} f < \frac{(1-\tau)i}{n+1},$$

for any linear i-subspace L of $\mathbb{R}^{n+1}$, $i = 1, \dots, n$, then

$$\mathcal{E}_{1,f}(\Omega) \ge \tau \log D + \log \delta - 4\log(n+1);$$

(iii) *if $\frac{1}{n+2} < \alpha < 1$, $p = 1 - \frac{1}{\alpha}$ (where $-n-1 < p < 0$), $\tau \le \frac{1}{2}\fint_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}}$ and*

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f^{\frac{n+1}{n+1+p}} \le \tau^{\frac{n+1}{n+1+p}}, \tag{2.1}$$

for any $z \in S^{n-1}$, then

$$\textit{either } D \le 16n^2/\delta^2, \quad \textit{or } \ D \le \left(\frac{1}{2}\fint_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}}\right)^{\frac{2}{p}}.$$

Moreover, if $\tau \le \frac{1}{2}\exp\left(\frac{\alpha-1}{\alpha}\mathcal{E}_{\alpha,f}(\Omega)\right)$, then

$$\textit{either } D \le 16n^2/\delta^2, \quad \textit{or } \ D \le \left(\frac{1}{2}\exp\left(\frac{\alpha-1}{\alpha}\mathcal{E}_{\alpha,f}(\Omega)\right)\right)^{\frac{2}{p}}.$$

Remark 2.2 We note that for any $\alpha \ge 1$, bounded f with $\inf f > 0$ and $\fint_{\mathbb{S}^n} f = 1$, and $\tau \in (0,1)$, there exists $\delta \in (0,1)$ such that conditions in (i) and (ii) hold. In the case of $1 > \alpha > \frac{1}{n+2}$, (iii) holds if in addition that $\tau \le \frac{1}{2}\exp\left(\frac{1-\alpha}{\alpha}\mathcal{E}_{\alpha,f}(\Omega)\right)$ for the convex body $\Omega \subset \mathbb{R}^{n+1}$.

Proof Given $\alpha > \frac{1}{n+2}$, bounded f with $\inf f > 0$ and $\fint_{\mathbb{S}^n} f = 1$, and $\tau \in (0,1)$, the existence of suitable $\delta \in (0,1)$ follows from the fact that the Lebesgue measure is a Borel measure.

Now, we assume that the conditions in (i)–(iii) hold. We may assume that the centroid of Ω is the origin; thus, Kannan, Lovász, and Simonovics [35] yield the existence of an o-symmetric ellipsoid such that

$$E \subset \Omega \subset (n+1)E, \text{ and hence } -\Omega \subset (n+1)\Omega. \tag{2.2}$$

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Let u be the support function of Ω, and let $R = \max\{\|y\| : y \in \Omega\} \geq D/2$ and $z_0 \in \mathbb{S}^n$ such that $Rz_0 \in \partial\Omega$. We observe that the definition of the entropy yields

$$\fint_{\mathbb{S}^n} f u^{1-\frac{1}{\alpha}} \leq \exp\left(\frac{1-\alpha}{\alpha}\,\mathcal{E}_{\alpha,f}(\Omega)\right) \text{ if } \alpha > 1;$$
$$\fint_{\mathbb{S}^n} f \log u \leq \mathcal{E}_{0,f}(\Omega);$$
$$\fint_{\mathbb{S}^n} f u^{1-\frac{1}{\alpha}} \geq \exp\left(\frac{1-\alpha}{\alpha}\,\mathcal{E}_{\alpha,f}(\Omega)\right) \text{ if } \frac{1}{n+2} < \alpha < 1.$$

Case 1: $\alpha > 1$.

According to the condition in (i), we may choose $\zeta \in \{+1, -1\}$ such that

$$\fint_{\Phi} f \geq \frac{\tau}{2} \text{ for } \Phi = \{x \in \mathbb{S}^n : \langle x, \zeta z_0\rangle > \delta\},$$

and hence $\frac{R\zeta z_0}{n+1} \in \Omega$ by (2.2). Since $u_\sigma(x) \geq \langle \frac{R\zeta z_0}{n+1}, x\rangle \geq \frac{R\delta}{n+1}$ for $x \in \Phi$, we have

$$\fint_{\mathbb{S}^n} f u^{1-\frac{1}{\alpha}} \geq \fint_{\Phi} f \left(\frac{R\delta}{n+1}\right)^{1-\frac{1}{\alpha}} \geq \frac{\tau}{2}\cdot\left(\frac{D\delta}{2(n+1)}\right)^{1-\frac{1}{\alpha}}.$$

Case 2: $\alpha = 1$.

To simplify notation, we consider the Borel probability measure $\mu(A) = \fint_A f$ on S^n. Let $e_1, \ldots, e_{n+1} \in \mathbb{S}^n$ be the principal directions associated with the ellipsoid E in (2.2), and let $r_1, \ldots, r_{n+1} > 0$ be the half axes of E with $r_i e_i \in \partial E$ where we may assume that $r_1 \leq \cdots \leq r_{n+1}$. In particular, (2.2) yields that

$$(n+1)^{n+1}\prod_{i=1}^{n+1} r_i = \frac{|(n+1)E|}{|B(1)|} \geq \frac{|\Omega|}{|B(1)|} = 1. \tag{2.3}$$

We observe that for any $v \in \mathbb{S}^n$, there exists e_i such that $|\langle v, e_i\rangle| \geq \frac{1}{\sqrt{n+1}} > \frac{\delta}{n+1}$. For $i = 1, \ldots, n+1$, we define

$$B_i = \left\{v \in \mathbb{S}^n : |\langle v, e_i\rangle| \geq \frac{\delta}{n+1} \text{ and } |\langle v, e_j\rangle| < \frac{\delta}{n+1} \text{ for } j > i\right\}.$$

In particular, $B_i \subset \Psi(L_i \cap \mathbb{S}^n, \delta)$ for $i = 1, \ldots, n$ and $L_i = \text{lin}\{e_1, \ldots, e_i\}$.

It follows that $\mathbb{S}^n$ is partitioned into the Borel sets $B_1, \ldots, B_{n+1}$, and as $B_i \subset \Psi(L_i \cap \mathbb{S}^n, \delta)$ for $i = 1, \ldots, n$, we have

$$\mu(B_1) + \cdots + \mu(B_i) \leq \frac{i(1-\tau)}{n+1} \text{ for } i = 1, \ldots, n, \tag{2.4}$$

$$\mu(B_1) + \cdots + \mu(B_{n+1}) = 1. \tag{2.5}$$

For $\zeta = \frac{1-\tau}{n+1}$, we have $0 < \zeta < \frac{1}{n+1}$, and define

$$\beta_i = \mu(B_i) - \zeta \text{ for } i = 1, \ldots, n, \tag{2.6}$$

$$\beta_{n+1} = \mu(B_{n+1}) - \zeta - \tau, \tag{2.7}$$

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where (2.4) and (2.5) yield

$$\beta_1 + \cdots + \beta_i \le 0 \quad \text{for } i = 1, \ldots, m-1, \tag{2.8}$$

$$\beta_1 + \cdots + \beta_{n+1} = 0. \tag{2.9}$$

As $r_i e_i \in \Omega$, it follows from the definition of B_i that $u(x) \ge \langle x, r_i e_i \rangle \ge r_i \cdot \frac{\delta}{n+1}$ for $x \in B_i$, $i = 1, \ldots, n+1$. We deduce from applying (2.3), (2.5)–(2.9), $r_1 \le \cdots \le r_{n+1}$, and $\zeta < \frac{1}{n+1}$ that

$$\begin{aligned}
\int_{\mathbb{S}^n} \log u \, d\mu &= \sum_{i=1}^{n+1} \int_{B_i} \log u \, d\mu \\
&\ge \sum_{i=1}^{n+1} \mu(B_i) \log r_i + \sum_{i=1}^{n+1} \mu(B_i) \log \frac{\delta}{n+1} = \sum_{i=1}^{n+1} \mu(B_i) \log r_i + \log \frac{\delta}{n+1} \\
&= \sum_{i=1}^{n+1} \beta_i \log r_i + \sum_{i=1}^{n+1} \zeta \log r_i + \tau \log r_{n+1} + \log \frac{\delta}{n+1} \\
&\ge \sum_{i=1}^{n+1} \beta_i \log r_i + \zeta \log \frac{1}{(n+1)^{n+1}} + \tau \log r_{n+1} + \log \frac{\delta}{n+1} \\
&= (\beta_1 + \cdots + \beta_{n+1}) \log r_{n+1} + \sum_{i=1}^{n} (\beta_1 + \cdots + \beta_i)(\log r_i - \log r_{i+1}) \\
&\quad - (n+1)\zeta \log(n+1) + \tau \log r_{n+1} + \log \frac{\delta}{n+1} \\
&\ge -\log(n+1) + \tau \log r_{n+1} + \log \frac{\delta}{n+1}.
\end{aligned}$$

Now, $D \le (n+1) \operatorname{diam} E = 2(n+1) r_{n+1} \le (n+1)^2 r_{n+1}$ and $\tau < 1$, and hence

$$\begin{aligned}
-\log(n+1) + \tau \log r_{n+1} + \log \frac{\delta}{n+1} &\ge -\log(n+1) + \tau \log \frac{D}{(n+1)^2} + \log \frac{\delta}{n+1} \\
&= \log(\delta D^\tau) - (2 + 2\tau) \log(n+1) \\
&\ge \tau \log D + \log \delta - 4 \log(n+1).
\end{aligned}$$

In particular, we conclude that

$$\mathcal{E}_{1,f}(\Omega) \ge \int_{\mathbb{S}^n} f \log u = \int_{\mathbb{S}^n} \log u \, d\mu \ge \tau \log D + \log \delta - 4 \log(n+1).$$

Case 3: $\frac{1}{n+2} < \alpha < 1$.

In this case, $-(n+1) < 1 - \frac{1}{\alpha} < 0$. We may assume that

$$D \ge 16 n^2 / \delta^2,$$

and we consider

$$\begin{aligned}
\Phi_0 &= \left\{ x \in \mathbb{S}^n : u(x) > \sqrt{2R} \right\}, \\
\Phi_1 &= \left\{ x \in \mathbb{S}^n : u(x) \le \sqrt{2R} \right\}.
\end{aligned}$$

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Concerning Φ_0, we have

$$\fint_{\Phi_0} f \cdot u^{1-\frac{1}{\alpha}} \le (2R)^{\frac{1}{2}(1-\frac{1}{\alpha})} \int_{\Phi_0} f \le D^{\frac{1}{2}(1-\frac{1}{\alpha})} = D^{\frac{p}{2}}. \tag{2.10}$$

On the other hand, we have $\pm\frac{R}{(n+1)} z_0 \in \Omega$ by (2.2), thus any $x \in \Phi_1$ satisfies

$$\sqrt{2R} \ge u(x) \ge \left|\left\langle x, \frac{R}{n+1} z_0 \right\rangle\right|,$$

and hence $|\langle x, z_0\rangle| \le (n+1)\sqrt{\frac{2}{R}} \le \frac{4n}{\sqrt{D}} \le \delta$; or in other words,

$$\Phi_1 \subset \Psi(z_0^\perp \cap \mathbb{S}^n, \delta).$$

It follows from $|\Omega| = |B(1)|$ and the Blaschke–Santaló inequality (*cf.* [45]) that

$$\int_{\mathbb{S}^n} u^{-(n+1)} \le (n+1)|B(1)| = \omega_n, \text{ and hence } \fint_{\mathbb{S}^n} u^{-(n+1)} \le 1.$$

For $p = 1 - \frac{1}{\alpha} \in (-n-1, 0)$, Hölder's inequality and $\int_{\Phi_1} f^{\frac{n+1}{n+1+p}} < \tau^{\frac{n+1}{n+1+p}}$ yield

$$\fint_{\Phi_1} f \cdot u^{1-\frac{1}{\alpha}} \le \left(\fint_{\Phi_1} f^{\frac{n+1}{n+1+p}}\right)^{\frac{n+1+p}{n+1}} \left(\fint_{\Phi_1} u_\sigma^{-(n+1)}\right)^{\frac{|p|}{n+1}} \le \left(\fint_{\Phi_1} f^{\frac{n+1}{n+1+p}}\right)^{\frac{n+1+p}{n+1}} \le \tau.$$

Finally, adding the last estimate to (2.10) yields

$$\exp\left(\frac{\alpha-1}{\alpha}\mathcal{E}_{\alpha,f}(\Omega)\right) \le \fint_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}} \le D^{\frac{p}{2}} + \tau,$$

and hence the conditions either $\tau \le \frac{1}{2}\fint_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}}$ or $\tau \le \frac{1}{2}\exp\left(\frac{1-\alpha}{\alpha}\mathcal{E}_{\alpha,f}(\Omega)\right)$ on τ implies (iii). ∎

3 Anisotropic flows and monotonicity of entropies

The following theorem was proved by Andrews in [4] (see also for a discussion of contracting of non-homogeneous fully nonlinear anisotropic curvature flows in [24]).

Theorem 3.1 [4] *For any $\alpha > 0$ and positive $f \in C^\infty(\mathbb{S}^n)$ and any initial smooth, strictly convex hypersurface $\tilde{M}_0 \subset \mathbb{R}^{n+1}$, the hypersurfaces $\tilde{M}_\tau$ given by the solution of (1.7) exist for a finite time T and converge in Hausdorff distance to a point $p \in \mathbb{R}^{n+1}$ as τ approaches T.*

Assuming

$$\fint_{\mathbb{S}^n} f = 1, \quad |\Omega_0| = |B(1)|,$$

solution (1.7) yields a smooth convex solution to the normalized flow (1.8) with volume preserved.

Set

$$h_z(x,t) \doteqdot f(x) u_z^{-\frac{1}{\alpha}}(x,t) K(x,t), \quad d\sigma_t(x) = \frac{u_z(x,t)}{K(x,t)} d\theta(x). \tag{3.1}$$

Note that $\fint_{\mathbb{S}^n} d\sigma_t(x) = \fint_{\mathbb{S}^n} d\theta(x) = 1$.

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Since the un-normalized flow (1.7) shrinks to a point in finite time, we may assume that it is the origin. Then the support function $u(x,t)$ is positive for the normalized flow (1.8).

Lemma 3.2 (a) *The entropy $\mathcal{E}_{\alpha,f}(\Omega_t)$ defined in (1.5) is monotonically decreasing,*

$$\mathcal{E}_{\alpha,f}(\Omega_{t_2}) \le \mathcal{E}_{\alpha,f}(\Omega_{t_1}), \quad \forall t_1 \le t_2 \in [0,\infty). \tag{3.2}$$

(b) *There is $D > 0$ depending only on* $\inf f, \sup f, \alpha, \Omega_0$ *such that*

$$\operatorname{diam}\Omega_t = D(t) \le D, \ \forall t \ge 0. \tag{3.3}$$

(c) $\forall t_0 \in [0,\infty)$,

$$\mathcal{E}_{\alpha,f}(\Omega_{t_0},0) \ge \mathcal{E}_{\alpha,f,\infty} + \int_{t_0}^{\infty}\left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\,d\sigma_t}{\fint_{\mathbb{S}^n} h(x,t)\,d\sigma_t \cdot \fint_{\mathbb{S}^n} h^{\alpha}(x,t)\,d\sigma_t} - 1\right)dt, \tag{3.4}$$

where

$$h(x,t) = h_0(x,t),\ \mathcal{E}_{\alpha,f,\infty} \doteqdot \lim_{t\to\infty}\mathcal{E}_{\alpha,f}(\Omega_t).$$

Proof (a) We follow argument in [26]. For each $T_0 >$ fixed, pick $T > T_0$. Let $a^T = (a_1^T,\dots,a_{n+1}^T)$ be an interior point of Ω_T. Set $u^T = u - e^{t-T}\sum_{i=1}^{n+1} a_i^T x_i$; it satisfies equation

$$\frac{\partial}{\partial t}u^T(x,t) = -\frac{f^\alpha(x)K^\alpha(x,t)}{\fint_{\mathbb{S}^n} f^\alpha K^{\alpha-1}} + u^T(x,t). \tag{3.5}$$

Note that since a^T is an interior point of Ω_T and $u(x,T)$ is the support function of Ω_T with respect to a^T, $u^T(x,T) > 0, \forall x \in \mathbb{S}^n$. We claim

$$u^T(x,t) > 0,\ \forall t \in [0,T).$$

Suppose $u^T(x_0,t') \le 0$ for some $0 < t' < T, x_0 \in \mathbb{S}^n$, and equation (3.5) implies $u^T(x_0,t) < 0$ for all $t > t'$, which contradicts to $u^T(x,T) > 0$.

Set $a^T(t) = e^{t-T}a^T$. By the claim, $a^T(t)$ is in the interior of Ω_t, $\forall t \le T$. Denote

$$d\sigma_{T,t} = u^T(x,t)K^{-1}(x,t)d\theta,$$

we rewrite equation (3.3) as

$$\frac{\partial}{\partial t}u_{a^T(t)}(x,t) = -\frac{f^\alpha(x)K^\alpha(x,t)}{\fint_{\mathbb{S}^n} h^\alpha_{a^T(t)}(x,t)\,d\sigma_{T,t}} + u_{a^T(t)}(x,t). \tag{3.6}$$

We have

$$\frac{\partial}{\partial t}\mathcal{E}_{\alpha,f}(\Omega_t,a^T(t)) = \frac{-\fint_{\mathbb{S}^n} h^{\alpha+1}_{a^T(t)}(x,t)\,d\sigma_{T,t}}{\fint_{\mathbb{S}^n} h_{a^T(t)}(x,t)\,d\sigma_{T,t}\cdot\fint_{\mathbb{S}^n} h^\alpha_{a^T(t)}(x,t)\,d\sigma_{T,t}} + 1.$$

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Thus, $\forall t < T$,

$$\begin{aligned} &\mathcal{E}_{\alpha,f}(\Omega_t, a^T(t)) - \mathcal{E}_{\alpha,f}(\Omega_T, a^T) \\ &= \int_t^T \fint_{\mathbb{S}^n} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}_{a^T(t)}(x,t)\, d\sigma_{T,t}}{\fint_{\mathbb{S}^n} h_{a^T(t)}(x,t)\, d\sigma_{T,t} \cdot \fint_{\mathbb{S}^n} h^{\alpha}_{a^T(t)}(x,t)\, d\sigma_{T,t}} - 1 \right) dt \geq 0. \end{aligned} \tag{3.7}$$

Therefore,

$$\mathcal{E}_{\alpha,f}(\Omega_t) \geq \mathcal{E}_{\alpha,f}(\Omega_T, a^T), \ \forall t < T.$$

Since a^T is arbitrary, (3.2) is proved.

(b) The boundedness of $D(t)$ follows from Theorem 2.1 combined with the estimate $\mathcal{E}_{\alpha,1}(\Omega_t) \leq \mathcal{E}_{\alpha,1}(B(1))$ from (a) (see also [6, 26]). The only nontrivial case is when $\frac{1}{n+2} < \alpha < 1$ because we have to choose a τ independent of t. However, we may choose any $\tau \in (0,1)$ with $\tau \leq \frac{1}{2} \exp\left(\frac{1-\alpha}{\alpha}\, \mathcal{E}_{\alpha,f}(B(1))\right)$ according to $\mathcal{E}_{\alpha,1}(\Omega_t) \leq \mathcal{E}_{\alpha,1}(B(1))$.

(c) $\forall \varepsilon > 0$, $\forall t_0$ fixed, pick $T > T_0 > t_0$. As $\mathcal{E}_{\alpha,f}(\Omega_T)$ is bounded by (a), $\exists a^T$ inside Ω_T such that $\mathcal{E}_{\alpha,f}(\Omega_T) \leq \mathcal{E}_{\alpha,f}(\Omega_T, a^T) + \varepsilon$. By (3.7),

$$\begin{aligned} &\mathcal{E}_{\alpha,f}(\Omega_{t_0}, a^T(t_0)) - \mathcal{E}_{\alpha,f}(\Omega_T) \\ &\geq \int_{t_0}^{T_0} \fint_{\mathbb{S}^n} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}_{a^T(t)}(x,t)\, d\sigma_{T,t}}{\fint_{\mathbb{S}^n} h_{a^T(t)}(x,t)\, d\sigma_{T,t} \cdot \fint_{\mathbb{S}^n} h^{\alpha}_{a^T(t)}(x,t)\, d\sigma_{T,t}} - 1 \right) dt - \varepsilon. \end{aligned}$$

As $|a^T| \leq D$, $\forall T$, let $T \to \infty$,

$$a^T(t) \to 0, \ u^T(x,t) \to u(x,t), \quad \text{uniformly for } 0 \leq t \leq T_0, x \in \mathbb{S}^n.$$

We obtain $\forall t_0 < T_0$,

$$\mathcal{E}_{\alpha,f}(\Omega_{t_0}, 0) - \mathcal{E}_{\alpha,f,\infty} \geq \int_{t_0}^{T_0} \fint_{\mathbb{S}^n} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{\fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot \fint_{\mathbb{S}^n} h^{\alpha}(x,t)\, d\sigma_t} - 1 \right) dt - \varepsilon.$$

Then let $T_0 \to \infty$, as $\varepsilon > 0$ is arbitrary, we obtain (3.4). ∎

4 Weak convergence

The goal of this section is to prove the following statement.

Theorem 4.1 *For a C^∞ function $f : \mathbb{S}^n \to (0,\infty)$ and $\alpha > \frac{1}{n+2}$ with $\fint_{\mathbb{S}^n} f = 1$, there exist $\lambda > 0$ and a convex body $\Omega \subset \mathbb{R}^{n+1}$ with $o \in \Omega$ whose support function u is a (possibly weak) solution of the Monge–Ampère equation*

$$u^{\frac{1}{\alpha}} \det(\bar{\nabla}^2_{ij} u + u \bar{g}_{ij}) = f \tag{4.1}$$

and Ω satisfies that

$$\mathcal{E}_{\alpha,f}(\lambda\Omega) \leq \mathcal{E}_{\alpha,f}(B(1)), \quad |\lambda\Omega| = |B(1)|, \tag{4.2}$$

where $C^{-1} < \lambda < C$ for a $C > 1$ depending only on the α, τ, δ in Theorem 2.1 such that f satisfies the conditions in Theorem 2.1.

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From now on, we will assume that the f in Theorem 4.1 satisfies the corresponding condition in Theorem 2.1 and $\Omega_0 = B(1)$ in (1.8). We note that for any $z \in B(1)$, $v_z \le 2$ for the support function v_z of $B(1)$ at z, and hence if $\alpha > \frac{1}{n+2}$, then

$$\mathcal{E}_{\alpha,f_k}(B(1)) \le \begin{cases} \frac{\alpha}{\alpha-1} \cdot \log 2^{1-\frac{1}{\alpha}}, & \text{if } \alpha \ne 1, \\ \log 2, & \text{if } \alpha = 1. \end{cases} \tag{4.3}$$

The following is a consequence of Theorem 2.1 and Lemma 3.2.

Lemma 4.2 *There exist $C_{\alpha,\tau,\delta} > 0$, $D_{\alpha,\tau,\delta} > 0$, and $c_{\alpha,\tau,\delta} \in \mathbb{R}$ depending only on constants α, τ, δ in Theorem 2.1 such that, along (1.8), we have*

$$D(t) \le D_{\alpha,\tau,\delta},\ \mathcal{E}_{\alpha,f}(\Omega_t, 0) \ge c_{\alpha,\tau,\delta},\ \frac{1}{C_{\alpha,\tau,\delta}} \le \fint_{\mathbb{S}^n} h(x,t) d\sigma_t \le C_{\alpha,\tau,\delta}. \tag{4.4}$$

Proof For each $\alpha > \frac{1}{n+2}$ fixed with condition on f as in Theorem 2.1, $\mathcal{E}_{\alpha,f}(\Omega_t)$ is bounded from below in terms of the diameter $D(t)$. Since $|\Omega_t| = |B(1)|$, we have $D(t) \ge 2$ by the Isodiametric Inequality (*cf.* [45]). By Theorem 2.1, $\mathcal{E}_{\alpha,f}(\Omega_t)$ is bounded from below by a constant $c_{\alpha,\tau,\delta} > 0$, and hence $\mathcal{E}_{\alpha,f,\infty} \ge c_{\alpha,\tau,\delta}$. It follows from Lemma 3.2 that $\mathcal{E}_{\alpha,f}(\Omega_t) \le \mathcal{E}_{\alpha,f}(B(1))$, and this estimate combined with (4.3) and Theorem 2.1 yields $D(t) \le D_{\alpha,\tau,\delta}$ where $D_{\alpha,\tau,\delta}$ depends only on constants in condition on f in Theorem 2.1. Finally, the inequalities follow from Lemma 3.2. ■

Set

$$\eta(t) = \fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t. \tag{4.5}$$

We note that $\fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t$ is monotone and bounded from below and above by Lemma 4.2, and hence we have

$$C_{\alpha,\tau,\delta} \ge \lim_{t\to\infty} \fint_{\mathbb{S}^n} h(x,t) = \eta \ge \frac{1}{C_{\alpha,\tau,\delta}}. \tag{4.6}$$

By Lemma 3.2 and Corollary 4.2,

$$\int_0^\infty \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{\fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot \fint_{\mathbb{S}^n} h^{\alpha}(x,t)\, d\sigma_t} - 1 \right) dt < \infty. \tag{4.7}$$

Since the integrand is nonnegative, $\exists t_k \to \infty$ such that

$$\lim_{k\to\infty} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t_k)\, d\sigma_{t_k}}{\fint_{\mathbb{S}^n} h(x,t_k)\, d\sigma_{t_k} \cdot \fint_{\mathbb{S}^n} h^{\alpha}(x,t_k)\, d\sigma_{t_k}} - 1 \right) = 0. \tag{4.8}$$

This implies

$$\lim_{k\to\infty} \frac{\left(\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t_k)\, d\sigma_{t_k}\right)^{\frac{1}{1+\alpha}}}{\fint_{\mathbb{S}^n} h(x,t_k)\, d\sigma_{t_k}} = \lim_{k\to\infty} \frac{\left(\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t_k)\, d\sigma_{t_k}\right)^{\frac{\alpha}{1+\alpha}}}{\fint_{\mathbb{S}^n} h^{\alpha}(x,t_k)\, d\sigma_{t_k}} = 1. \tag{4.9}$$

After considering a subsequence, we may assume that

$$\Omega_{t_k} \to \Omega, \quad u(x,t_k) \to u(x), \tag{4.10}$$

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where u is the support function of Ω. In view of (4.9) and (4.6),

$$\lim_{k\to\infty}\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t_k)\,d\sigma_{t_k} = \eta^{1+\alpha},\quad \lim_{k\to\infty}\fint_{\mathbb{S}^n} h^{\alpha}(x,t_k)\,d\sigma_{t_k} = \eta^{\alpha}. \tag{4.11}$$

The following lemma is crucial for the weak convergence, which is a refined form of the classical Hölder inequality.[1]

Lemma 4.3 *Let $p,\ q \in \mathbb{R}^+$ with $\frac{1}{p}+\frac{1}{q}=1$, and set $\beta = \min\{\frac{1}{p},\frac{1}{q}\}$. Let (M,μ) be a measurable space; $\forall F \in L^p,\ G \in L^q$,*

$$\int_M |FG| d\mu \le \|F\|_{L^p}\|G\|_{L^q}\left(1-\beta\int_M\left(\frac{|F|^{\frac{p}{2}}}{(\int_M|F|^p d\mu)^{\frac{1}{2}}}-\frac{|G|^{\frac{q}{2}}}{(\int_M|G|^q d\mu)^{\frac{1}{2}}}\right)^2\right). \tag{4.12}$$

Proof We first prove the following *Claim*. $\forall s,t\in\mathbb{R}$,

$$e^{\frac{s}{p}+\frac{t}{q}} \le \frac{e^s}{p}+\frac{e^t}{q}-\beta(e^{\frac{s}{2}}-e^{\frac{t}{2}})^2. \tag{4.13}$$

We may assume $t\ge s$, set $\tau = t-s$, and (4.13) is equivalent to

$$e^{\frac{\tau}{q}} \le \frac{1}{p}+\frac{e^\tau}{q}-\beta(1-e^{\frac{\tau}{2}})^2,\ \forall\tau\ge 0. \tag{4.14}$$

Set

$$\xi(\tau) = \frac{1}{p}+\frac{e^\tau}{q}-\beta(1-e^{\frac{\tau}{2}})^2-e^{\frac{\tau}{q}}.$$

We have $\xi(0)=0$,

$$\xi'(\tau) = \frac{e^{\frac{\tau}{q}}}{q}\rho,\ \text{where } \rho(\tau) = e^{\frac{\tau}{p}}(1-\beta q)+q\beta e^{\frac{\tau}{2}-\frac{\tau}{q}}-1.$$

If $\beta = \frac{1}{q}$, then $\frac{1}{q}\le\frac{1}{2}$; since $\tau\ge 0$,

$$\rho(\tau) = e^{\frac{\tau}{p}}(1-\beta q)+q\beta e^{\frac{\tau}{2}-\frac{\tau}{q}}-1 = e^{\frac{\tau}{2}-\frac{\tau}{q}}-1\ge 0.$$

If $\beta=\frac{1}{p}$, then $\frac{1}{q}\ge\frac{1}{2}$; we have

$$\begin{aligned}\rho'(\tau) &= e^{\frac{\tau}{p}}\left(\frac{1-\beta q}{p}+\beta q(\frac{1}{2}-\frac{1}{q})e^{\frac{\tau}{2}-\frac{\tau}{q}}\right)\\ &\ge e^{\frac{\tau}{p}}\left(\frac{1-\beta q}{p}+\beta q(\frac{1}{2}-\frac{1}{q})\right)\\ &\ge e^{\frac{\tau}{p}}\beta q(\frac{1}{2}-\frac{1}{p})\ge 0.\end{aligned}$$

We conclude that

$$\rho(\tau)\ge 0,\ \forall\tau\ge 0.$$

In turn,

$$\xi'(\tau)\ge 0,\ \forall\tau\ge o.$$

This yields (4.14) and (4.13). The *Claim* is verified.

[1] We would like to thank referee for pointing out that the lemma was proved as Theorem 2.2 in [1]. Here, we provide a proof for completeness.

Back to the proof of the lemma. We may assume

$$F \geq 0, \; g \geq 0, \; \int F^p > 0, \; \int G^q > 0.$$

Set

$$e^s = \frac{F^p}{\int F^p}, \quad e^t = \frac{G^q}{\int G^q}.$$

Put them into (4.13) and integrate, as $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{\int FG}{(\int F^p)^{\frac{1}{p}}(\int G^q)^{\frac{1}{q}}} \leq \left(1 - \beta \int \Big(\frac{F^{\frac{p}{2}}}{(\int F^p)^{\frac{1}{2}}} - \frac{G^{\frac{q}{2}}}{(\int G^q)^{\frac{1}{2}}}\Big)^2\right).$$

■

We prove weak convergence.

Proposition 4.4 *$\forall \alpha > \frac{1}{n+2}$, suppose that (4.10) and (4.11) hold. Denote*

$$u_k = u(x, t_k), \; \sigma_{n,k} = \sigma_n(u_{ij}(x, t_k) + u(x, t_k)\delta_{ij}).$$

Then

$$\lim_{k\to\infty} \fint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - \frac{f}{\eta}| d\theta = 0, \tag{4.15}$$

where η is defined in (4.5) which is bounded from below and above in (4.6). As a consequence, there is a convex body $\Omega \subset \mathbb{R}^{n+1}$ with $o \in \Omega$,

$$|\Omega| = |B(1)|, \quad \mathcal{E}_{\alpha,f}(\Omega_t) \leq \mathcal{E}_{\alpha,f}(B(1)),$$

and its support function u satisfies

$$u^{\frac{1}{\alpha}} S_\Omega = \frac{1}{\eta} f d\theta. \tag{4.16}$$

Proof We only need to verify (4.15). By (4.11), it is equivalent to prove

$$\lim_{k\to\infty} \fint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - f\eta^{-1}(t_k)| d\theta = 0. \tag{4.17}$$

Since $D(t_k)$ is bounded,

$$\fint_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta \leq (D(t_k))^{\frac{1}{\alpha^2}} \fint_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta \leq (D(t_k))^{\frac{1}{\alpha^2}} |\partial\Omega_{t_k}| \leq C.$$

$$\begin{aligned}
\fint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - f\eta^{-1}(t_k)| d\theta &= \fint_{\mathbb{S}^n} \Big|\frac{f}{\eta(t_k) u_k^{\frac{1}{\alpha}} \sigma_{n,k}} - 1\Big| u_k^{\frac{1}{\alpha}} \sigma_{n,k} d\theta \\
&\leq \left(\fint_{\mathbb{S}^n} \Big|\frac{f}{\eta(t_k) u_k^{\frac{1}{\alpha}} \sigma_{n,k}} - 1\Big|^{1+\alpha} d\sigma_{t_k}\right)^{\frac{1}{1+\alpha}} \left(\fint_{\mathbb{S}^n} u_k^{(\frac{1}{\alpha}-1)\frac{1+\alpha}{\alpha}} d\sigma_{t_k}\right)^{\frac{\alpha}{1+\alpha}} \\
&= \left(\fint_{\mathbb{S}^n} \Big|\frac{f}{\eta(t_k) u_k^{\frac{1}{\alpha}} \sigma_{n,k}} - 1\Big|^{1+\alpha} d\sigma_{t_k}\right)^{\frac{1}{1+\alpha}} \left(\fint_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta\right)^{\frac{\alpha}{1+\alpha}} \\
&\leq C \left(\fint_{\mathbb{S}^n} |f\eta^{-1}(t_k) u_k^{-\frac{1}{\alpha}} \sigma_{n,k}^{-1} - 1|^{1+\alpha} d\sigma_{t_k}\right)^{\frac{1}{1+\alpha}}. \qquad (4.18)
\end{aligned}$$

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By (4.8), (4.11), and Lemma 4.3, with $p = \alpha + 1$, $F^{\frac{1}{1+\alpha}} = h(x, t_k)$, $G = 1$,

$$\lim_{k\to\infty} ⨍ \left((\frac{h(x,t_k)}{\eta(t_k)})^{\frac{1+\alpha}{2}} - 1 \right)^2 d\sigma_{t_k} = 0. \tag{4.19}$$

For t_k fixed, let

$$\gamma_k(x) = f\eta^{-1}(t_k)u_k^{-\frac{1}{\alpha}}\sigma_{n,k}^{-1} = h(x,t_k)\eta^{-1}(t_k)$$

and set

$$\Sigma_k = \left\{ x \in \mathbb{S}^n \mid |\gamma_k(x) - 1| \leq \frac{1}{2} \right\}.$$

It is straightforward to check that $\exists A_\alpha \geq 1$ depending only on α such that

$$A_\alpha |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1| \geq |\gamma_k(x) - 1|, \ \forall x \in \Sigma_k,$$
$$A_\alpha |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 \geq |\gamma_k(x) - 1|^{1+\alpha}, \ \forall x \in \Sigma_k^c.$$

Since $|\gamma_k^{\frac{1+\alpha}{2}}(x) - 1| \leq 2^{1+\alpha}, \ \forall x \in \Sigma_k$, let $\delta = \min\{1+\alpha, 2\}$,

$$\begin{aligned}
⨍_{\mathbb{S}^n} |\gamma_k(x) - 1|^{1+\alpha} d\sigma_{t_k} =& \frac{1}{\omega_n}\left(\int_{\Sigma_k} |\gamma_k(x) - 1|^{1+\alpha} d\sigma_{t_k} + \int_{\Sigma_k^c} |\gamma_k(x) - 1|^{1+\alpha} d\sigma_{t_k} \right) \\
\leq& \frac{A_\alpha^{1+\alpha}}{\omega_n}\left(\int_{\Sigma_k} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^{1+\alpha} d\sigma_{t_k} + \int_{\Sigma_k^c} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} \right) \\
\leq& \frac{(2A_\alpha)^{1+\alpha}}{\omega_n}\left(\int_{\Sigma_k} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^{\delta} d\sigma_{t_k} + \int_{\Sigma_k^c} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} \right) \\
\leq& (2A_\alpha)^{1+\alpha}\left(⨍_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^{\delta} d\sigma_{t_k} + ⨍_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} \right) \\
\leq& (2A_\alpha)^{1+\alpha}\left((⨍_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k})^{\frac{\delta}{2}} + ⨍_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} \right).
\end{aligned}$$

By (4.19),

$$\lim_{k\to\infty} ⨍_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} = 0.$$

Hence,

$$\lim_{k\to\infty} ⨍_{\mathbb{S}^n} |\gamma_k(x) - 1|^{1+\alpha} d\sigma_{t_k} = 0. \tag{4.20}$$

Now, (4.17) follows from (4.18)–(4.20). ■

Proof Proof of Theorem 4.1. It follows from Proposition 4.4 after a proper rescaling as η satisfies (4.6) and (4.16). ■

5 The general Monge–Ampère equations – proof of Theorem 1.1

In order to prove Theorem 1.1, we need weak approximation in the following sense.

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Lemma 5.1 *For $\delta, \varepsilon \in (0, \frac{1}{2})$ and a Borel probability measure μ on $\mathbb{S}^n$, $n \geq 1$, there exists a sequence $d\mu_k = \frac{1}{\omega_n} f_k \, d\theta$ of Borel probability measures whose weak limit is μ and $f_k \in C^\infty(\mathbb{S}^n)$ satisfies $f_k > 0$ and the following properties:*

(i) *If $\mu(\Psi(z^\perp \cap \mathbb{S}^n, 2\delta)) \leq 1 - \varepsilon$ for any $z \in S^{n-1}$, then*

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f_k \leq 1 - \varepsilon \ \text{ for any } z \in S^{n-1}. \tag{5.1}$$

(ii) *If $\mu(\Psi(L \cap \mathbb{S}^n, 2\delta)) < (1 - 2\delta) \cdot \frac{\ell}{n+1}$ for any linear ℓ-subspace L of $\mathbb{R}^{n+1}$, $\ell = 1, \ldots, n$, then*

$$\mu_k(\Psi(L \cap \mathbb{S}^n, \delta)) < (1 - \delta) \cdot \frac{\ell}{n+1}. \tag{5.2}$$

(iii) *If $d\mu = \frac{1}{\omega_n} f \, d\theta$ for $f \in L^r(\mathbb{S}^n)$ where $r > 1$, and*

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n, 2\delta)} f^r \leq \varepsilon \tag{5.3}$$

for any $z \in S^{n-1}$, then

$$\int_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f_k^r \leq 2^r \varepsilon \ \text{ for any } z \in S^{n-1}. \tag{5.4}$$

Proof For $k \geq 1$, let $\{B_{k,i}\}_{i=1,\ldots,m(k)}$ be a partition of S^n into spherically convex Borel measurable sets $B_{k,i}$ with $\operatorname{diam} B_{k,i} \leq \frac{1}{k}$ and $\theta(B_{k,i}) > 0$. For each $B_{k,i}$, we choose a C^∞ function $h_{k,i} : \mathbb{S}^n \to [0, \infty)$ such that for $M_{k,i} = \max h_{k,i}$ and the probability measure $d\tilde{\theta} = \frac{1}{\omega_n} d\theta$, we have:

- $h_{k,i} = 0$ if $x \notin B_{k,i}$;
- $M_{k,i} \leq (1 + \frac{1}{k}) \cdot \frac{\mu(B_{k,i})}{\tilde{\theta}(B_{k,i})}$;
- $\theta(\{x \in B_{k,i} : h_{k,i}(x) < M_{k,i}\}) < \frac{1}{k} \theta(B_{k,i})$;
- $\int_{B_{k,i}} h_{k,i} \, d\tilde{\theta} = \mu(B_{k,i})$.

We consider the positive C^∞ function $\tilde{f}_k = \frac{1}{k} + \sum_{i=1}^{m(k)} h_{k,i}$, and hence $f_k = \left(\fint_{\mathbb{S}^n} \tilde{f}_k\right)^{-1} \tilde{f}$ satisfies that the probability measure $d\mu_k = f_k \, d\tilde{\theta}$ tends weakly to μ, and for large $k \geq 1/\delta$, μ_k satisfies (i), and if (ii) holds, then μ_k also satisfies (5.2).

Turning to (iii), we assume that $d\mu = f \, d\tilde{\theta}$ for $f \in L^r(\mathbb{S}^n)$ where $r > 1$, and f satisfies (5.3). For any large k and $i = 1, \ldots, m(k)$, we deduce from the Hölder inequality that

$$\begin{aligned}
\fint_{B_{k,i}} \tilde{f}_k^r = \fint_{B_{k,i}} \left(h_{k,i} + \frac{1}{k}\right)^r &\leq 2^{r-1} \fint_{B_{k,i}} h_{k,i}^r + 2^{r-1} \fint_{B_{k,i}} \frac{1}{k^r} \\
&\leq 2^{r-1} \tilde{\theta}(B_{k,i}) M_{k,i}^r + 2^{r-1} \fint_{B_{k,i}} \frac{1}{k^r} \\
&\leq 2^{r-1} \left(1 + \frac{1}{k}\right)^r \tilde{\theta}(B_{k,i}) \left(\frac{\int_{B_{k,i}} f}{\tilde{\theta}(B_{k,i})}\right)^r + 2^{r-1} \fint_{B_{k,i}} \frac{1}{k^r} \\
&\leq 2^{r-1} \left(1 + \frac{1}{k}\right)^r \fint_{B_{k,i}} f^r + 2^{r-1} \fint_{B_{k,i}} \frac{1}{k^r}.
\end{aligned}$$

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Summing this estimate up for large k and all $B_{k,i}$ with $B_{k,i} \cap \Psi(z^\perp \cap \mathbb{S}^n, \delta) \neq \emptyset$, and using that $\fint_{\mathbb{S}^n} \tilde{f}_k \geq 2^{-1/2}$ for large k, we deduce that

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n,\delta)} f_k^r \leq \sqrt{2} \fint_{\Psi(z^\perp \cap \mathbb{S}^n,\delta)} \tilde{f}_k^r \leq \sqrt{2} \cdot 2^{r-1} \left(1 + \frac{1}{k}\right)^r \fint_{\Psi(z^\perp \cap \mathbb{S}^n, 2\delta)} f^r + \sqrt{2} \cdot \frac{2^{r-1}}{k^r} \leq 2^r \varepsilon.$$

■

For $\alpha > 0$ and $p = 1 - \frac{1}{\alpha}$, the L^p-surface area $dS_{\Omega,p} = u^{1-p} dS_\Omega$ was introduced in the seminal works [39–41] for a convex body $\Omega \subset \mathbb{R}^{n+1}$ with $o \in \Omega$ and support function u. Since the surface area measure is weakly continuous for $p < 1$, and if $K \subset \mathbb{R}^{n+1}$ is an at most n-dimensional compact convex set, then $S_{K,p} \equiv 0$ for $p < 1$, we have the following statement.

Lemma 5.2 *If convex bodies $\Omega_m \subset \mathbb{R}^{n+1}$ tend to a compact convex set $K \subset \mathbb{R}^{n+1}$ where $o \in \Omega_m$, K, and $\liminf_{m\to\infty} S_{\Omega_m,p} > 0$, then $\mathrm{int}K \neq \emptyset$ and $S_{\Omega_m,p}$ tends weakly to $S_{K,p}$.*

For the reader's sake, let us recall Theorem 1.1.

Theorem 5.3 *For $\alpha > \frac{1}{n+2}$ and finite nontrivial Borel measure μ on $\mathbb{S}^n$, $n \geq 1$, there exists a weak solution of (1.2) provided the following holds:*

(i) *If $\alpha > 1$ and μ is not concentrated onto any great subsphere $x^\perp \cap \mathbb{S}^n$, $x \in \mathbb{S}^n$.*
(ii) *If $\alpha = 1$ and μ satisfies that for any linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \leq \ell \leq n$, we have:*
  (a) $\mu(L \cap \mathbb{S}^n) \leq \dfrac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$;
  (b) *equality in (a) for a linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \leq d \leq n$ implies the existence of a complementary linear $(n+1-\ell)$-subspace $\widetilde{L} \subset \mathbb{R}^{n+1}$ such that* $\mathrm{supp}\,\mu \subset L \cup \widetilde{L}$.
(iii) *If $\frac{1}{n+2} < \alpha < 1$, assume $d\mu = f\,d\theta$ for nonnegative $f \in L^{\frac{n+1}{n+2-\frac{1}{\alpha}}}(\mathbb{S}^n)$ with $\int_{\mathbb{S}^n} f > 0$.*

Proof Let $\alpha > \frac{1}{n+2}$. After rescaling, we may assume that the μ in (1.2) is a probability measure. We consider the sequence $d\mu_k = \frac{1}{\omega_n} f_k\,d\theta$ of Lemma 5.1 of Borel probability measures whose weak limit is μ and $f_k \in C^\infty(\mathbb{S}^n)$ satisfies $f_k > 0$. For each f_k, let $\Omega_k \subset \mathbb{R}^{n+1}$ be the convex body with $o \in \Omega_k$ provided by Theorem 4.1 whose support function u_k is the solution of the Monge–Ampère equation

$$u_k^{\frac{1}{\alpha}}\,dS_{\Omega_k} = f_k\,d\theta; \tag{5.5}$$

$\exists \lambda_k > 0$ under control, with $|\lambda_k \Omega| = |B(1)|$, Ω_k satisfies that

$$\mathcal{E}_{\alpha,f_k}(\lambda_k \Omega_k) \leq \mathcal{E}_{\alpha,f_k}(B(1)). \tag{5.6}$$

We also need the observations that

$$|\Omega_k| = \frac{1}{n+1} \int_{\mathbb{S}^n} u_k\,dS_{\Omega_k}, \tag{5.7}$$

and if $p = 1 - \frac{1}{\alpha}$, then

$$S_{\Omega_k,p}(\mathbb{S}^n) = \int_{\mathbb{S}^n} u_k^{1-\frac{1}{\alpha}}\,dS_{\Omega_k} = \omega_n \fint_{\mathbb{S}^n} f_k = \omega_n. \tag{5.8}$$

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We claim that if there exists $\Delta > 0$ depending on n, α, and μ such that

$$\text{diam}\Omega_k \le \Delta, \quad \text{then Theorem 5.3 holds.} \tag{5.9}$$

To prove this claim, we note that (5.9) yields the existence of a subsequence of $\{\Omega_k\}$ tending to a compact convex set Ω with $o \in \Omega$, which is a convex body by (5.8) and Lemma 5.2. Moreover, Lemma 5.2 also yields that Ω is an Alexandrov solution of (1.2), verifying the claim (5.9).

We divide the rest of the argument verifying Theorem 5.3 into three cases.

Case 1: $\alpha > 1$.

Since μ is not concentrated to any great subsphere, there exist $\delta \in (0, \frac{1}{2})$ depending on μ such that $\mu\left(\Psi(z^\perp \cap \mathbb{S}^n, 2\delta)\right) \le 1 - 2\delta$ for any $z \in S^{n-1}$. It follows from Lemma 5.1 that we may assume that

$$\int_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f_k \le 1 - \delta \quad \text{for any } z \in S^{n-1}. \tag{5.10}$$

Now, Theorem 4.1 implies that $\lambda_k \ge c$ for a constant $c > 0$ depending on n, δ, and α, and in turn Theorem 4.1, (4.3), and $\frac{1}{\alpha} - 1 < 0$ yield that

$$\mathcal{E}_{\alpha,f}(\Omega_k) = \frac{\alpha}{\alpha - 1} \cdot \log \lambda_k^{\frac{1}{\alpha}-1} + \mathcal{E}_{\alpha,f}(\lambda_k \Omega_k) \le \frac{\alpha}{\alpha - 1} \cdot \log \lambda_k^{\frac{1}{\alpha}-1} + \mathcal{E}_{\alpha,f}(B(1)) \le C$$

for a constant $C > 0$ depending on n, δ, and α. Therefore, Theorem 2.1 and (5.10) imply that the sequence $\{\Omega_k\}$ is bounded, and in turn the claim (5.9) implies Theorem 5.3 if $\alpha > 1$.

Case 2: $\alpha = 1$.

The argument is by induction on $n \ge 0$ where we do not put any restriction on the probability measure μ in the case $n = 0$. For the case $n = 0$, we observe that any finite measure μ on S^0 can be represented in the form $d\mu = u\, dS_\Omega$ for a suitable segment $\Omega \subset \mathbb{R}^1$.

For the case $n \ge 1$, assuming that we have verified Theorem 5.3(ii) in smaller dimensions, we consider a Borel measure probability μ on S^n satisfying (a) and (b).

Case 2.1: *There exists a linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \le \ell \le n$ and $\mu(L \cap \mathbb{S}^n) = \frac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$.*

Let $\widetilde{L} \subset \mathbb{R}^{n+1}$ be the complementary linear $(n+1-\ell)$-subspace with supp $\mu \subset L \cup \widetilde{L}$, and hence $\mu(\widetilde{L} \cap \mathbb{S}^n) = \frac{n+1-\ell}{n+1} \cdot \mu(\mathbb{S}^n)$. It follows by induction that there exist an ℓ-dimensional compact convex set $K' \subset L$ and an $(n+1-\ell)$-dimensional compact convex set $\widetilde{K}' \subset \widetilde{L}$ such that $\mu \llcorner (L \cap S^n) = \ell\, V_{K'}$ and $\mu \llcorner (\widetilde{L} \cap S^n) = (n+1-\ell) V_{\widetilde{K}'}$. Finally, for $K = \widetilde{L}^\perp \cap (K' + L^\perp)$ and $\widetilde{K} = L^\perp \cap (\widetilde{K}' + \widetilde{L}^\perp)$, there exist $\alpha, \tilde{\alpha} > 0$ such that

$$\mu = (n+1) V_{\alpha K + \tilde{\alpha} \widetilde{K}}.$$

Case 2.2: $\mu(L \cap \mathbb{S}^n) < \frac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$ *for any linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \le \ell \le n$.*

It follows by a compactness argument that there exists $\delta \in (0, \frac{1}{2})$ depending on μ such that $\mu(\Psi(L \cap \mathbb{S}^n, 2\delta)) < (1 - 2\delta) \cdot \frac{\ell}{n+1}$ for any linear ℓ-subspace L of $\mathbb{R}^{n+1}$, $\ell = 1, \ldots, n$. We consider the sequence of probability measures $d\mu_k = \frac{1}{\omega_n} f_k\, d\theta$ of

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Lemma 5.1 tending weakly to μ such that $f_k > 0$, $f_k \in C^\infty(\mathbb{S}^n)$, and

$$\mu_k\left(\Psi\left(L \cap \mathbb{S}^n, \delta\right)\right) < (1-\delta) \cdot \frac{\ell}{n+1} \tag{5.11}$$

for any linear ℓ-subspace L of $\mathbb{R}^{n+1}$, $\ell = 1, \ldots, n$.

For each f_k, let $\Omega_k \subset \mathbb{R}^{n+1}$ with $o \in \Omega_k$ be the convex body provided by Theorem 4.1 whose support function u_k is the solution of the Monge–Ampère equation (4.1) and satisfies (4.2) with $f = f_k$ and $\lambda = \lambda_k$ where $|B(1)| = |\lambda_k \Omega_k|$ for $\lambda_k > 0$, and

$$\begin{aligned}|\Omega_k| &= \frac{1}{n+1}\int_{\mathbb{S}^n} u_k \det(\bar{\nabla}^2_{ij} u_k + u_k \bar{g}_{ij})\, d\theta = \frac{\omega_n}{n+1} \fint_{\mathbb{S}^n} u_k \det(\bar{\nabla}^2_{ij} u_k + u_k \bar{g}_{ij}) \\ &= |B(1)| \fint_{\mathbb{S}^n} f_k = |B(1)|,\end{aligned}$$

and hence $\lambda_k = 1$. In particular, (4.3) yields

$$\mathcal{E}_{1,f_k}(\lambda_k \Omega_k) \le \mathcal{E}_{1,f_k}(B(1)) \le \log 2.$$

Since $\mathcal{E}_{1,f_k}(\Omega_k)$ is bounded, (5.11) and Theorem 2.1 imply that the sequence Ω_k stays bounded, as well. Therefore, the claim (5.9) yields Theorem 5.3 if $\alpha = 1$.

Case 3: $\frac{1}{n+2} < \alpha < 1$.

We set $p = 1 - \frac{1}{\alpha} \in (-n-1, 0)$ and $r = \frac{n+1}{n+1+p} > 1$, and

$$\tau = \frac{1}{2} \cdot 2^{-\frac{|p|(n+1)}{|p|+n}}, \tag{5.12}$$

and choose $\delta \in (0, \frac{1}{2})$ such that

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n, 2\delta)} f^r \le \frac{\tau^r}{2^r}$$

for any $z \in S^{n-1}$. We deduce from Lemma 5.1 that if $z \in S^{n-1}$, then

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f_k^r \le \tau^r. \tag{5.13}$$

We deduce from (5.5), (5.7), and $|\lambda_k \Omega_k| = |B(1)| = \frac{\omega_n}{n+1}$ that

$$\fint_{\mathbb{S}^n} u_k^p f_k = \frac{n+1}{\omega_n} \int_{\mathbb{S}^n} u_k \, dS_{\Omega_k} = \frac{n+1}{\omega_n} |\Omega_k| = \lambda_k^{-n-1}. \tag{5.14}$$

In particular, (4.3) and the upper bound on the entropy yield that

$$\begin{aligned}2^p &\le \exp\left(p \cdot \mathcal{E}_{\alpha,f_k}(B(1))\right) \le \exp\left(p \cdot \mathcal{E}_{\alpha,f}(\lambda_k \Omega_k)\right) \le \fint_{\mathbb{S}^n} (\lambda_k u_k)^p f_k \\ &= \lambda_k^p \int_{\mathbb{S}^n} u_k \, dS_{\Omega_k} = \lambda_k^{p-n} \cdot \frac{n+1}{\omega_n} \cdot |\lambda_k \Omega_k| = \lambda_k^{p-n}.\end{aligned} \tag{5.15}$$

It follows from (5.15) that $\lambda_k \le 2^{\frac{|p|}{|p|+n}}$, and in turn (5.14) yields that

$$\fint_{\mathbb{S}^n} u_k^p f_k \ge 2^{-\frac{|p|(n+1)}{|p|+n}}.$$

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Therefore, $\tau \leq \frac{1}{2}\int_{\mathbb{S}^n} u_k^p f_k$ (*cf.* (5.12)), (5.13), and Theorem 2.1 yield that the sequence $\{\Omega_k\}$ is bounded, and in turn the claim (5.9) implies Theorem 5.3 if $\frac{1}{n+2} < \alpha < 1$. ∎

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