# PARACOMPACTNESS AND STRONG SCREENABILITY

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Throughout this paper, a space means a $T_1$-space. A space is called *fully normal* if every open covering $\mathfrak{V}$ of it has a $\Delta$-refinement $\mathfrak{W}$, that is, an open covering for which the stars $(x, \mathfrak{W})$ form a covering which refines $\mathfrak{V}$. A space is called *paracompact* if every open covering $\mathfrak{V}$ of it has a locally finite ( = neighborhood finite) open covering $\mathfrak{W}$ which refines $\mathfrak{V}$. It is well known that paracompactness is identical with full normality in a Hausdorff space ([3], [7]). Recently R. H. Bing [1] has introduced new concepts into spaces: collectionwise normality and strong screenability. A collection $\mathfrak{H}$ of subsets $H_\alpha$ $(\alpha \in A)$ of a space is called *discrete* if (1) the closures $\overline{H}_\alpha$ are mutually disjoint, (2) every union of the form $\bigcup_{\beta \in B} \{\overline{H}_\beta;\ \beta \in B \subset A\}$ is closed. A space is called *collectionwise normal* if every discrete collection $\{H_\alpha\}$ of closed subsets is covered by $\{G_\alpha\}$ of open sets which are mutually disjoint. A space is called *strongly screenable* if every open covering of it can be refined by an open one which can be decomposed into a sequence of discrete collections. He proved that both every strongly screenable regular space and every fully normal space are collectionwise normal. Thus each of full normality, paracompactness and strong screenability of a regular space always implies collectionwise normality. But more is true: Paracompactness is identical with strong screenability in a regular space. To show this is the main purpose of this paper. Thus we obtain the following scheme:

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               |→ fully normal spaces                      →|
               |          ↑ ↓                                |  collectionwise
metric spaces—|→ paracompact Hausdorff spaces             →|— normal spaces
               |          ↑ ↓                                |
               |→ strongly screenable regular spaces       →|
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THEOREM 1. *Every point-wise paracompact*[1], *collectionwise normal space $R$ is strongly screenable.*

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[1] A space is called *point-wise paracompact* if every open covering of it can be refined by a point-finite open one.



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LEMMA 1. *If* $\{H_\alpha;\ \alpha\in A\}$ *is a discrete collection of sets in* $R$, *every collection of sets* $\{K_\alpha;\ \alpha\in A\}$ *such that* $H_\alpha\supset K_\alpha$ *for every* $\alpha\in A$ *is also discrete in* $R$.

This can easily be seen and hence the proof is omitted.

LEMMA 2. *If a space* $R$ *is collectionwise normal, every discrete collection of sets* $\{H_\alpha;\ \alpha\in A\}$ *in* $R$ *can be covered by a discrete collection of open sets* $\{G_\alpha;\ \alpha\in A\}$ *such that* $\bar H_\alpha\subset G_\alpha$ *for every* $\alpha\in A$.

*Proof.* Let $\{E_\alpha;\ \alpha\in A\}$ be a mutually disjoint collection of open sets such that $E_\alpha\supset\bar H_\alpha$ for every $\alpha\in A$. Since a collectionwise normal space is normal, there is an open set $G$ such that $\bigcup_{\alpha\in A}E_\alpha\supset\bar G\supset G\supset\bigcup_{\alpha\in A}\bar H_\alpha$. Setting $G_\alpha=G\cap E_\alpha$, $\bar H_\alpha\subset G_\alpha$ and $\bar G_\alpha\subset\bar G\cap(R-\bigcup_{\alpha'\in A-\alpha}E_{\alpha'})\subset(\bigcup_{\alpha'\in A}E_{\alpha'})\cap(R-\bigcup_{\alpha'\in A-\alpha}E_{\alpha'})=E_\alpha$. Thus $\bar G_\alpha$'s are mutually disjoint. Let $B$ be an arbitrary subset of indices of $A$ and then $\overline{\bigcup_{\alpha\in B}\bar G_\alpha}\subset\bar G\cap(R-\bigcup_{\alpha\in A-B}E_\alpha)\subset(\bigcup_{\alpha\in A}E_\alpha)\cap(R-\bigcup_{\alpha\in A-B}E_\alpha)=\bigcup_{\alpha\in B}E_\alpha$. Thus if $p$ is in $\overline{\bigcup_{\alpha\in B}\bar G_\alpha}$, $p$ is in one and only one $E_{\alpha(p)}$ such that $\alpha(p)\in B$. Since $E_{\alpha(p)}\cap(\bigcup_{\alpha\in B-\alpha(p)}\bar G_\alpha)=\phi$, $p\notin\overline{\bigcup_{\alpha\in B-\alpha(p)}\bar G_\alpha}$ and hence $p\in\bar G_{\alpha(p)}$, which shows $\overline{\bigcup_{\alpha\in B}\bar G_\alpha}=\bigcup_{\alpha\in B}\bar G_\alpha$. Q.E.D.

*Proof of the theorem.* For simplicity, let $d_{\mathfrak D}(p)$ denote the degree of $\mathfrak D=\{D_\lambda;\ \lambda\in\Lambda\}$ at $p$ (where $\mathfrak D$ is an arbitrary collection of subsets of $R$), i.e. the number of sets of $\mathfrak D$ which contain $p$. Let $\{V\}$ be an arbitrary open covering of $R$ and then, from point-wise paracompactness of $R$, there is a point-finite open covering $\mathfrak U=\{U_\alpha;\ \alpha\in A\}$ which refines $\{V\}$. Let

$$F_i=\{p;\ d_{\mathfrak U}(p)=i\}\qquad(i=1,2,\ldots),$$

and it is clear $\bigcup_{i=1}^{\infty}F_i=R$ from point-finiteness of $\mathfrak U$.

We shall show, by induction, that $(P_i)$: there is a collection $\mathfrak U_i$ of open sets which refines $\mathfrak U$ and covers $\bigcup_{j=1}^{i}F_j$ and can be decomposed into a finite number of discrete collections.

Let $F_\alpha=R-\bigcup_{\alpha'\neq\alpha}U_{\alpha'}$ and it can easily be seen that $F_\alpha\subset U_\alpha$, $\bigcup_{\alpha\in A}F_\alpha=F_1$ and $F_\alpha$'s are mutually disjoint closed sets. Moreover $\{F_\alpha;\ \alpha\in A\}$ is discrete: Let $B$ be an arbitrary subset of indices of $A$. If $p$ is in $\overline{\bigcup_{\alpha\in B}F_\alpha}$, $p$ is in one and only one $U_{\alpha(p)}$, $\alpha(p)\in B$, by virtue of the inequality $\overline{\bigcup_{\alpha\in B}F_\alpha}\subset R-\bigcup_{\alpha\in A-B}U_\alpha$. Since $U_{\alpha(p)}\cap(\bigcup_{\alpha\in B-\alpha(p)}F_\alpha)=\phi$, $p$ is in $F_{\alpha(p)}$, which shows that $\overline{\bigcup_{\alpha\in B}F_\alpha}=\bigcup_{\alpha\in B}F_\alpha$. Since $R$ is collectionwise normal, there is, by Lemma 2, a discrete collection $\{G'_\alpha;\ \alpha\in A\}$

of open sets such that $G'_\alpha \supset F_\alpha$ for every $\alpha$. Let $G_\alpha = G'_\alpha \cap U_\alpha$ and then $\mathfrak{U}_1 = \{G_\alpha ; \alpha \in A\}$ is also discrete by Lemma 1 and clearly refines $\mathfrak{U}$ and covers $F_1$. Thus $(P_1)$ is valid.

Now we put the inductive assumption that $(P_i)$ is valid for $i = n$. Let

$$A_{n+1} = \{\alpha^{n+1}\} = \{\{\alpha_1, \ldots, \alpha_{n+1}\} ; \alpha_j \ (1 \leqq j \leqq n+1) \in A, \\ \alpha_j\text{'s are all different from each other}\}$$

and $\mathfrak{V}_{n+1} = \{V_{\alpha^{n+1}}, \alpha^{n+1} \in A_{n+1}\}$ be the collection of open sets of type $\bigcap_{j=1}^{n+1} U_{\alpha_j}$ such that $\{\alpha_1, \ldots, \alpha_{n+1}\} \in A_{n+1}$. Let

$$F_{\alpha^{n+1}} = \{p ; p \in V_{\alpha^{n+1}}, d_{\mathfrak{V}_{n+1}}(p) = 1\}$$

and then it can be seen, from the point-finiteness of $\mathfrak{V}_{n+1}$, that $\{F_{\alpha^{n+1}} ; \alpha^{n+1} \in A_{n+1}\}$ is discrete in $G_{n+1} = \bigcup_{\alpha^{n+1} \in A_{n+1}} V_{\alpha^{n+1}}$ and each $F_{\alpha^{n+1}}$ is closed in $G_{n+1}$ by the analogous argument to the preceding one.

Let $G_n = \bigcup_{U^n \in \mathfrak{U}_n} U^n$ and then $G_n \supset \{p ; d_{\mathfrak{U}}(p) \leqq n\} = \bigcup_{j=1}^{n} F_j$ by the induction assumption. Since $d_{\mathfrak{V}_{n+1}}(p) \geqq 1$ implies $d_{\mathfrak{U}}(p) \geqq n+1$ and conversely and $G_{n+1}$ is nothing but $\{p ; d_{\mathfrak{V}_{n+1}}(p) \geqq 1\}$, we have

$$G_{n+1} \supset R - G_n.$$

Hence $H_{\alpha^{n+1}} = F_{\alpha^{n+1}} \cap (R - G_n)$ is closed in $R - G_n$ and then so in $R$. Since $\overline{\bigcup_{\alpha^{n+1} \in A_{n+1}} H_{\alpha^{n+1}}}$ is contained in closed $R - G_n$ and, by Lemma 1, $\{H_{\alpha^{n+1}} ; \alpha^{n+1} \in A_{n+1}\}$ is discrete in $G_{n+1}$, $\{H_{\alpha^{n+1}}\}$ is discrete in $R$. Hence there is a collection $\{U'_{\alpha^{n+1}} ; \alpha^{n+1} \in A_{n+1}\}$ of open sets which is discrete in $R$ such that $U'_{\alpha^{n+1}} \supset H_{\alpha^{n+1}}$ for every $\alpha^{n+1} \in A_{n+1}$. Let $U_{\alpha^{n+1}} = U'_{\alpha^{n+1}} \cap V_{\alpha^{n+1}}$ and then, by Lemma 1, $\mathfrak{W}_{n+1} = \{U_{\alpha^{n+1}} ; \alpha^{n+1} \in A_{n+1}\}$ is also discrete in $R$ and refines $\mathfrak{V}_{n+1}$. Setting

$$\mathfrak{U}_{n+1} = \mathfrak{U}_n \cup \mathfrak{W}_{n+1},$$

it is easily seen that $\mathfrak{U}_{n+1}$ satisfies $(P_{n+1})$ by the construction. Thus the induction is completed.

Let

$$\mathfrak{V} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$$

and then $\mathfrak{V}$ refines $\mathfrak{U}$ and can be decomposed into a sequence of discrete collections. Since $\mathfrak{U}_i$ covers $\bigcup_{j=1}^{i} F_j$ and $\bigcup_{j=1}^{\infty} F_j = R$, $\mathfrak{V}$ is a covering of $R$. Hence $R$ is strongly screenable. Q.E.D.

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THEOREM 2. *Every paracompact Hausdorff space is strongly screenable (and regular).*

*Proof.* A paracompact Hausdorff space is regular [3] and a priori pointwise paracompact. Adding to those, we know that every paracompact Hausdorff space is collectionwise normal ([1] or [2]). Hence the theorem is a trivial consequence of Theorem 1. Q.E.D.

THEOREM 3. *Every strongly screenable, regular space $R$ is paracompact.*

LEMMA 3. *Every countable covering $\{U_i\}$ of an arbitrary space $R$ each element of which is elementary open ( = a set which is expressible as $\{x; f(x) > 0\}$ by a suitable continuous function $f$ defined on the whole space) can be refined by a locally finite countable one, say $\{V_i\}$, each element of which is also elementary open such that $U_i \supset \bar{V}_i$ for every $i$.*

This is to be shown in [5].

*Proof of the theorem.* Let $\{V\}$ be an arbitrary open covering of $R$. Since $R$ is strongly screenable and regular, there is an open covering $\{U\}$ of $R$ which can be decomposed into a sequence of discrete collections, $\{U_{\alpha^i}; \alpha^i \in A_i\}$ $(i = 1, 2, \ldots)$, such that $(\bar{U})$ refines $\{V\}$. Since $R$ is collectionwise normal [1] and Lemma 2 holds in $R$, there is, for each $i$, a discrete collection of open sets, $\{V_{\alpha^i}; \alpha^i \in A_i\}$, such that $V_{\alpha^i} \supset \bar{U}_{\alpha^i}$ for every $\alpha^i \in A_i$. Since $\{\bar{U}\}$ refines $\{V\}$, we can assume, by Lemma 1, with no loss of generality that $\{V_{\alpha^i}; \alpha^i \in A_i\}$ refines $\{V\}$. Since closed $F_i = \bigcup_{\alpha^i \in A_i} \bar{U}_{\alpha^i}$ is contained in open $V_i = \bigcup_{\alpha^i \in A_i} V_{\alpha^i}$, there is an elementary open set $G_i$ such that $F_i \subset G_i \subset V_i$. Since $\{G_i; i = 1, 2, \ldots\}$ covers $R$, there is, by Lemma 3, a locally finite countable open covering of $R$, $\{H_i; i = 1, 2, \ldots\}$, such that $G_i \supset \bar{H}_i$ for every $i$. Let

$$H_{\alpha^i} = H_i \cap V_{\alpha^i}$$

and it can be seen that $\{H_{\alpha^i}; \alpha^i \in A_i\}$ is discrete in $R$ and refines $\{V\}$ and $\mathfrak{H} = \{H_{\alpha^i}; \alpha^i \in A_i, i = 1, 2, \ldots\}$ is an open covering of $R$.

Now we shall show that $\mathfrak{H}$ is locally finite. Let $p$ be an arbitrary point of $R$ and then, from the local-finiteness of $\{H_i\}$, there is an open neighborhood $W_0(p)$ of $p$ which meets only a finite number of elements of $\{H_i\}$. If $p \notin \bar{H}_i$, let

$$W_i(p) = R - \bar{H}_i.$$

If $p \in \overline{H}_i$, let

$$W_i(p) = V_{\alpha^i(p)},$$

where $V_{\alpha^i(p)}$ denotes the element of $\{V_{\alpha^i}; \alpha^i \in A_i\}$ such that $p \in V_{\alpha^i}$: This $V_{\alpha^i(p)}$ exists, since $\overline{H}_i \subset G_i \subset V_i = \bigcup_{\alpha^i \in A_i} V_{\alpha^i}$. In both cases $W_i(p)$ meets at most one element of $\{H_{\alpha^i}; \alpha^i \in A_i\}$. Let

$$W(p) = \bigcap_{i=0}^{n} W_i(p)$$

where $n = \max\{i; W_0(p) \cap H_i \neq \phi\}$. Then $W(p)$ meets at most $n$ elements of $\mathfrak{H}$. Thus local-finiteness of $\mathfrak{H}$ is established and $R$ is paracompact. Q.E.D.

COROLLARY. *A collectionwise normal space is paracompact if and only if it is point-wise paracompact.*

THEOREM 4. *Every point-finite open covering $\{U\}$ of a collectionwise normal space $R$ has an open locally finite $\Delta$-refinement.*

*Proof.* Since $\{U\}$ is point-finite, there is, by [6, 33.4, p. 26], an open point-finite covering $\{V\}$ of $R$ such that $\{\overline{V}\}$ refines $\{U\}$. By the same argument stated in the proof of Theorem 1, $\{V\}$ can be refined by an open covering $\{W\}$ which can be decomposed into a sequence of discrete collections. Since $\{\overline{W}\}$ refines $\{U\}$, the same argument stated in the proof of Theorem 3 can be applied and $\{U\}$ has a locally finite open refinement. It is well known that every locally finite open covering of a normal space has a locally finite $\Delta$-refinement (see [3]). Q.E.D.

Duing to R. H. Bing [1], a space is called *screenable* if every open covering of it can be refined by an open one which can be decomposed into a sequence of collections whose elements are mutually disjoint. Duing to C. H. Dowker [4], a space is called *countably paracompact* if every countable open covering of it can be refined by a locally finite (countable) open one. Under these terminologies, we get the following theorem.

THEOREM 5. *Every screenable, countably paracompact, normal space $R$ is strongly screenable.*

*Proof.* Let $\{U\}$ be an arbitrary open covering of $R$ and it can be refined, from screenablity of $R$, by an open one which can be decomposed into a sequence of collections of mutually disjoint sets, $\{V_{\alpha^i}; \alpha^i \in A_i\}$. $i = 1, 2, \ldots$.

Let $V_i = \bigcup_{\alpha^i \in A_i} V_{\alpha^i}$. Since $\{V_i; i = 1, 2, \ldots\}$ covers $R$, there is a locally finite open covering $\{W_i; i = 1, 2, \ldots\}$ of $R$ such that $V_i \supset \overline{W}_i$ for every $i$ (Cf. [6]). Let $W_{\alpha^i} = W_i \cap V_{\alpha^i}$ and it can easily be seen that $\{W_{\alpha^i}; \alpha^i \in A_i, i = 1, 2, \ldots\}$ covers $R$ and refines $\{U\}$ and each $\{W_{\alpha^i}; \alpha^i \in A_i\}$ is discrete in $R$. Hence $R$ is strongly screenable. Q.E.D.

THEOREM 6. *Every screenable, point-wise paracompact, normal space is strongly screenable.*

This follows from the above theorem, since every point-wise paracompact, normal space is countably paracompact [4].

QUESTION. *Is there a screenable, normal space which is not strongly screenable?*

If this question could be answered in the negative, it could be determined, by Theorem 5, that a normal space is not necessarily countably paracompact.

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Addendum (November 2, 1954). The author learned, after writing this paper, that E. Michael has recently obtained the theorem which asserts that for every regular space the paracompactness is identical with the strong screenability. (Proc. Amer. Math. Soc. **4** (1953), 831-838.)