Nagoya Math. J., **237** (2020), 39–78
DOI 10.1017/nmj.2018.10

# LITTLEWOOD–PALEY CHARACTERIZATIONS OF ANISOTROPIC WEAK MUSIELAK–ORLICZ HARDY SPACES

BO LI, RUIRUI SUN, MINFENG LIAO AND BAODE LI*

**Abstract.** Let $A$ be an expansive dilation on $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ an anisotropic growth function. In this article, the authors introduce the anisotropic weak Musielak–Orlicz Hardy space $WH^{\varphi}_A(\mathbb{R}^n)$ via the nontangential grand maximal function and then obtain its Littlewood–Paley characterizations in terms of the anisotropic Lusin-area function, $g$-function or $g^*_\lambda$-function, respectively. All these characterizations for anisotropic weak Hardy spaces $WH^p_A(\mathbb{R}^n)$ (namely, $\varphi(x, t) := t^p$ for all $t \in [0, \infty)$ and $x \in \mathbb{R}^n$ with $p \in (0, 1]$) are new. Moreover, the range of $\lambda$ in the anisotropic $g^*_\lambda$-function characterization of $WH^{\varphi}_A(\mathbb{R}^n)$ coincides with the best known range of the $g^*_\lambda$-function characterization of classical Hardy space $H^p(\mathbb{R}^n)$ or its weighted variants, where $p \in (0, 1]$.

## CONTENTS



## §1. Introduction

Weak function spaces play an important role in harmonic analysis. For example, in order to show that a linear operator maps $L^p(\mathbb{R}^n)$ to itself for any $p \in (1, \infty)$, it is sufficient to show that it maps the (smaller) Lorentz space $L^{p,1}(\mathbb{R}^n)$ into the (larger) weak Lebesgue space $WL^p(\mathbb{R}^n)$ for the same range of $p$'s. It is now well known that Hardy space $H^p(\mathbb{R}^n)$ is a good

Received March 23, 2017. Revised February 7, 2018. Accepted February 7, 2018.
2010 Mathematics subject classification. 42B25, 42B30, 42B35, 46E30.
This project is partially supported by the National Natural Science Foundation of China (Grant Nos. 11461065 and 11661075).
*Corresponding author.

substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$ in the study for the boundedness of operators and, moreover, when studying the boundedness of operators in the critical case, the weak Hardy spaces $WH^p(\mathbb{R}^n)$ naturally appear and prove to be a good substitute of Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$. For example, if $\delta \in (0, 1]$, $T$ is a $\delta$-Calderón–Zygmund operator and $T^*(1) = 0$, where $T^*$ denotes the adjoint operator of $T$, it is known that $T$ is bounded on $H^p(\mathbb{R}^n)$ for any $p \in (n/(n + \delta), 1]$ (see [2]), but $T$ may be not bounded on $H^{n/(n+\delta)}(\mathbb{R}^n)$; however, Liu [42] proved that $T$ is bounded from $H^{n/(n+\delta)}(\mathbb{R}^n)$ to $WH^{n/(n+\delta)}(\mathbb{R}^n)$.

Many fields in analysis require the study of specific function spaces. In harmonic analysis, one soon encounters the Lebesgue spaces, the Hardy spaces, various forms of the Lipschitz spaces, the BMO spaces and the Sobolev spaces. From the original definitions of these spaces, it may not appear that they are very closely related. There exist, however, various unified approaches to their study. The Littlewood–Paley theory, which arises naturally from the consideration of the Dirichlet problem, provides one of the most successful unifying perspectives on these function spaces (see [21] for more details). Recall that the classical Hardy spaces, which is defined via the nontangential grand maximal function, can also be equivalently characterized, respectively, via the Lusin-area function, $g$-function or $g_\lambda^*$-function (see, for example, [23, 53]).

On the other hand, as a generalization of $L^p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum and Orlicz [4] and Orlicz [46]. Since then, the theory of Orlicz-type spaces themselves has been well developed and these spaces have been widely used in many branches of analysis (see, for example, [3, 24, 26, 31, 43, 45]). Moreover, as a development of the theory of Orlicz spaces, the Orlicz–Hardy spaces and their dual spaces were studied by Strömberg [50] and Janson [27] and the Orlicz–Hardy spaces associated with divergence form elliptic operators by Jiang and Yang [28]. A survey on the real-variable theory of Orlicz-type function spaces associated with operators is recently given in [15].

Let $\mathcal{A}_q(\mathbb{R}^n)$ with $q \in [1, \infty]$ denote the class of classical Muckenhoupt weights (see, for example, [22, 23] for their definitions and properties) and let $\psi$ be a Musielak–Orlicz function (see [33]) satisfying that $\psi(x, \cdot)$ is an Orlicz function uniformly in $x \in \mathbb{R}^n$ and $\psi(\cdot, t)$ is a Muckenhoupt $\mathcal{A}_\infty(\mathbb{R}^n)$ weight uniformly in $t \in (0, \infty)$. It is known that, as a natural generalization of Orlicz functions, Musielak–Orlicz functions may vary in the spatial variables (see,

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

for example, [17, 33, 38, 44]). Recently, Ky [33] introduced a new Musielak–Orlicz Hardy space $H^{\psi}(\mathbb{R}^n)$ via the nontangential grand maximal function. It is worth noticing that some special Musielak–Orlicz Hardy spaces appear naturally in the study of the products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [6, 7, 34]), and the endpoint estimates for both the div-curl lemma and the commutators of singular integral operators (see [5, 6, 32, 34]). More applications are referred to [25, 35, 39, 40, 52, 54]. We refer the reader to [54] for a complete survey of the real-variable theory of Musielak–Orlicz Hardy spaces.

Let $A$ be an expansive dilation on $\mathbb{R}^n$ and $\varphi$ an anisotropic Musielak–Orlicz function satisfying some growth conditions (see Definition 2.3 below). In order to find an appropriate general space which includes the classical weak Hardy space of Fefferman and Soria [19], the classical weighted weak Hardy space of Quek and Yang [48], the anisotropic weak Hardy space of Ding and Lan [18], the classical weak Musielak–Orlicz Hardy space of Liang *et al.* [41] and the anisotropic weak Musielak–Orlicz Hardy space of Zhang *et al.* [55] and Qi *et al.* [47], we introduce the anisotropic weak Musielak–Orlicz Hardy space $WH^{\varphi}_A(\mathbb{R}^n)$ which includes all of the above mentioned weak spaces (see Remark 2.8 below for more details). Then the Littlewood–Paley characterizations of $WH^{\varphi}_A(\mathbb{R}^n)$ are obtained in Theorems 2.10–2.12 below.

Precisely, this article is organized as follows.

In Section 2, we recall some notions concerning expansive dilations, anisotropic Muckenhoupt weights and anisotropic growth functions. Then we introduce anisotropic weak Musielak–Orlicz Hardy spaces $WH^{\varphi}_A(\mathbb{R}^n)$ via nontangential grand maximal functions and establish their Littlewood–Paley characterizations, respectively, in terms of the anisotropic Lusin-area function, $g$-function or $g^*_\lambda$-function in Theorems 2.10–2.12 below, the proofs of which are given in Sections 3 and 4.

Section 3 is devoted to establishing the anisotropic Lusin-area function characterization of $WH^{\varphi}_A(\mathbb{R}^n)$. Let $q \in (q(\varphi), \infty]$, where $q(\varphi)$ denotes the critical weight index of $\varphi$. Here, we point out that the $q$-atomic characterization of $WH^{\varphi}_A(\mathbb{R}^n)$ (see Lemma 3.9 below) plays an important role in establishing the anisotropic Lusin-area function characterization of $WH^{\varphi}_A(\mathbb{R}^n)$ (see Theorem 2.10 below) and the anisotropic $q$-atomic characterization of $WH^{\varphi}_A(\mathbb{R}^n)$ is an anisotropic extension of Liang *et al.* [41, Theorem 3.5], which is new even when $q \in (1, \infty]$ and $WH^{\varphi}_A(\mathbb{R}^n)$ is reduced to the anisotropic weak Hardy space $WH^p_A(\mathbb{R}^n)$, where $p \in (0, 1]$ (see Remark 3.10

below for more details). Zhang *et al.* [55, Theorem 1] obtained the atomic characterization of $WH_A^{\widetilde{\varphi}}(\mathbb{R}^n)$ with respect to *a particular anisotropic growth function*, that is, the anisotropic $p$-growth function $\widetilde{\varphi}$ of uniformly lower type $p$ and of uniformly upper type $p$, where $p \in (0, 1]$. In this article, motivated by Liang *et al.* [41, Theorem 3.5], we obtain the atomic characterization of $WH_A^{\varphi}(\mathbb{R}^n)$ with respect to *a general anisotropic growth function* $\varphi$ of uniformly lower type $p$ and of uniformly upper type 1. Hence, we cannot directly use the method with respect to the uniformly upper $p$ property of $\varphi$ in [55, Theorem 1]. We overcome this difficulty via using a more general superposition principle of weak type estimates (see Lemma 3.2 below) and establishing a more subtle estimate of Schwartz function on weighted anisotropic Campanato space (see Lemma 3.5 below). Next, using some ideas from [36], the discrete anisotropic Calderón reproducing formula (see Lemma 3.12 below) and the method used in the proof the atomic characterization of $WH_A^{\varphi}(\mathbb{R}^n)$, we establish the anisotropic Lusin-area function characterization of $WH_A^{\varphi}(\mathbb{R}^n)$ (see Theorem 2.10 below). This characterization is new even when $WH_A^{\varphi}(\mathbb{R}^n)$ is reduced to the anisotropic weak Hardy space $WH_A^{p}(\mathbb{R}^n)$, where $p \in (0, 1]$ (see Remark 2.13 below for more details). We point out that, since the space variant $x$ and the time variant $t$ appeared in $\varphi(x, t)$ are inseparable, the dual method for estimating the atoms in the classical case does not work any more in the present setting. Instead, we use a method from Li *et al.* (see [36] for more details).

In Section 4, motivated by [41, Theorems 4.8 and 4.13], Folland and Stein [20] and Aguilera and Segovia [1], the anisotropic $g$-function or $g_\lambda^*$-function characterizations of $WH_A^{\varphi}(\mathbb{R}^n)$ is established, respectively, via the above anisotropic Lusin-area function characterization of $WH_A^{\varphi}(\mathbb{R}^n)$, the anisotropic weak Musielak–Orlicz Fefferman–Stein vector-valued inequality (see Lemma 4.3 below) and the anisotropic weak Musielak–Orlicz Peetre's inequality (see Lemma 4.5 below). This method is different from that used by Liang *et al.* in the proof of [41, Theorem 4.8], in which a subtle pointwise upper estimate (see [41, (4.26)]) via the vector-valued Hardy–Littlewood maximal function was used. However, such a pointwise upper estimate is still unknown and we do not know whether it holds true or not in the present setting due to its anisotropic structure. We point out that the range of $\lambda$ in the anisotropic $g_\lambda^*$-function characterization of $WH_A^{\varphi}(\mathbb{R}^n)$ coincides with the best known range of the $g_\lambda^*$-function characterization of classical Hardy space $H^p(\mathbb{R}^n)$ or its weighted variants, where $p \in (0, 1]$ (see, for example, [1, Theorem 2] and [41, Theorem 4.13]). The anisotropic $g$-function or $g_\lambda^*$-function characterization of $WH_A^\varphi(\mathbb{R}^n)$ is new even when $WH_A^\varphi(\mathbb{R}^n)$ is reduced to the anisotropic weak Hardy space $WH_A^p(\mathbb{R}^n)$, where $p \in (0, 1]$ (see Remark 2.13 below for more details).

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. Throughout the whole paper, we denote by $C$ a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $D \lesssim F$ means that $D \leqslant CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. For any sets $E, F \subset \mathbb{R}^n$, we use $E^\complement$ to denote the set $\mathbb{R}^n \setminus E$, $\chi_E$ its *characteristic function* and $E + F$ the *algebraic sum* $\{x + y : \ x \in E, y \in F\}$. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *maximal integer* not larger than $a$. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. For example, $L^p(\mathbb{R}^n)$ is simply denoted by $L^p$. Denote by $\mathcal{S}$ the *space of all Schwartz functions* and $\mathcal{S}'$ the *space of all tempered distributions*. For any subset $E$ of $\mathbb{R}^n$, $t \in (0, \infty)$ and measurable function $f$, let $\varphi(E, t) := \int_E \varphi(x, t)\, dx$ and $\{|f| > t\} := \{x \in \mathbb{R}^n : |f(x)| > t\}$.

## §2. Notions and main results

In Section 2, we introduce the anisotropic weak Musielak–Orlicz Hardy spaces via the nontangential grand maximal function and then present their Littlewood–Paley characterizations.

First we recall the notion of expansive dilations on $\mathbb{R}^n$; see [8, p. 5]. A real $n \times n$ matrix $A$ is called an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of $A$. Let $\lambda_-$ and $\lambda_+$ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leqslant \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

In the case when $A$ is diagonalizable over $\mathbb{C}$, we can even take $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments.

It was proved in [8, p. 5, Lemma 2.2] that, for a given dilation $A$, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where $P$ is some nondegenerate $n \times n$ matrix, such that $\Delta \subset r\Delta \subset A\Delta$, and by a scaling, one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the $n$-dimensional Lebesgue measure of the set $\Delta$. For any $k \in \mathbb{Z}$, let $B_k := A^k\Delta$. Then $B_k$ is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. Throughout the whole paper, let $\sigma$ be the *minimum positive*

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

*integer* such that $2B_0 \subset B_\sigma$. Then, for any $k, j \in \mathbb{Z}$ with $k \leqslant j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.1}$$

$$B_k + (B_{k+\sigma})^{\complement} \subset (B_k)^{\complement}. \tag{2.2}$$

DEFINITION 2.1. A *quasinorm*, associated with an expansive dilation $A$, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$, for simplicity, denoted by $\rho$, satisfying

(i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$, here and hereafter, $\mathbf{0}_n := (0, \ldots, 0)$;
(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where, as above, $b = |\det A|$;
(iii) $\rho(x + y) \leqslant H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of $x$ and $y$.

In the standard dyadic case $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of quasinorms associated with $A$, here and hereafter, $\mathrm{I}_{n\times n}$ always denotes the $n \times n$ *unit matrix* and $|\cdot|$ the *Euclidean norm* in $\mathbb{R}^n$.

It was proved in [8, p. 6, Lemma 2.4] that all quasinorms associated with a given dilation $A$ are equivalent. Therefore, for a given expansive dilation $A$, in what follows, for convenience, we always use the *step quasinorm* $\rho$ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \quad \text{if } x \neq \mathbf{0}_n, \text{or else } \rho(\mathbf{0}_n) := 0.$$

By (2.1) and (2.2), we know that, for any $x, y \in \mathbb{R}^n$,

$$\rho(x + y) \leqslant b^\sigma(\max\{\rho(x), \rho(y)\}) \leqslant b^\sigma[\rho(x) + \rho(y)].$$

Furthermore, $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [16], where $dx$ denotes the *$n$-dimensional Lebesgue measure*.

DEFINITION 2.2. Let $q \in [1, \infty)$. A function $\varphi(\cdot, t) : \mathbb{R}^n \to [0, \infty)$ is said to satisfy the *anisotropic uniform Muckenhoupt condition* $\mathbb{A}_q(A)$, denoted by $\varphi \in \mathbb{A}_q(A)$, if there exists a positive constant $C$ such that, for all $t \in (0, \infty)$, when $q \in (1, \infty)$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{b^{-k}\int_{x+B_k}\varphi(y,t)\,dy\right\}\left\{b^{-k}\int_{x+B_k}[\varphi(y,t)]^{-1/(q-1)}\,dy\right\}^{q-1} \leqslant C$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

and, when $q=1$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{b^{-k}\int_{x+B_k}\varphi(y,t)\,dy\right\}\left\{\operatorname*{ess\,sup}_{y\in x+B_k}[\varphi(y,t)]^{-1}\,dy\right\}\leqslant C.$$

The minimal constant $C$ as above is denoted by $C_{(q,A,n,\varphi)}$.

Define $\mathbb{A}_\infty(A):=\bigcup_{1\leqslant q<\infty}\mathbb{A}_q(A)$ and, for any $\varphi\in\mathbb{A}_\infty(A)$, let

$$q(\varphi):=\inf\{q\in[1,\infty):\varphi\in\mathbb{A}_q(A)\}. \tag{2.3}$$

If $\varphi\in\mathbb{A}_\infty(A)$ is independent of $t\in[0,\infty)$, then $\varphi$ is just an *anisotropic Muckenhoupt* $\mathcal{A}_\infty(A)$ *weight* in [11]. Obviously, $q(\varphi)\in[1,\infty)$. If $q(\varphi)\in(1,\infty)$, by a discussion similar to [12, p. 3072], it is easy to know $\varphi\notin\mathbb{A}_{q(\varphi)}(A)$. Moreover, there exists a $\varphi\in(\cap_{q>1}\mathbb{A}_q(A))\setminus\mathbb{A}_1(A)$ such that $q(\varphi)=1$; see Johnson and Neugebauer [29, p. 254, Remark].

Now let us recall some notions for Orlicz functions; see, for example, [33]. A function $\phi:[0,\infty)\to[0,\infty)$ is called an *Orlicz function*, if it is nondecreasing, $\phi(0)=0$, $\phi(t)>0$ for any $t\in(0,\infty)$ and $\lim_{t\to\infty}\phi(t)=\infty$. Observe that, different from the classical Orlicz functions being convex, the Orlicz functions in this article may not be convex. An Orlicz function $\phi$ is said to be of *lower* (resp. *upper*) *type* $p$ with $p\in(-\infty,\infty)$, if there exists a positive constant $C$ such that, for all $t\in[0,\infty)$ and $s\in(0,1)$ (resp. $s\in[1,\infty)$),

$$\phi(st)\leqslant Cs^p\phi(t).$$

Given a function $\varphi:\mathbb{R}^n\times[0,\infty)\to[0,\infty)$ such that, for any $x\in\mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, $\varphi$ is said to be of *uniformly lower* (resp. *upper*) *type* $p$ with $p\in(-\infty,\infty)$, if there exists a positive constant $C$ such that, for all $x\in\mathbb{R}^n$, $t\in[0,\infty)$ and $s\in(0,1)$ (resp. $s\in[1,\infty)$),

$$\varphi(x,st)\leqslant Cs^p\varphi(x,t).$$

The *critical uniformly lower type index* and the *critical uniformly upper type index* of $\varphi$ are, respectively, defined by

$$i(\varphi):=\sup\{p\in(-\infty,\infty):\varphi\text{ is of uniformly lower type }p\} \tag{2.4}$$

and

$$I(\varphi):=\inf\{p\in(-\infty,\infty):\varphi\text{ is of uniformly upper type }p\}. \tag{2.5}$$

Observe that $i(\varphi)$ and $I(\varphi)$ may not be attainable, namely, $\varphi$ may not be of uniformly lower type $i(\varphi)$ or of uniformly upper type $I(\varphi)$ (see [38]).

DEFINITION 2.3. [37, Definition 3] A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called an *anisotropic growth function* if the following conditions are satisfied:

(i) $\varphi$ is a *Musielak–Orlicz function*, namely:

  (a) the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;

  (b) the function $\varphi(\cdot,t)$ is a Lebesgue measurable function on $\mathbb{R}^n$ for all $t \in [0,\infty)$;

(ii) $\varphi \in \mathbb{A}_\infty(A)$;

(iii) $\varphi$ is of uniformly lower type $p$ for some $p \in (0,1]$ and of uniformly upper type 1.

Clearly,

$$\varphi(x,t) := w(x)\Phi(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in [0,\infty) \tag{2.6}$$

is an anisotropic growth function if $w$ is a classical or an anisotropic Muckenhoupt $\mathcal{A}_\infty$ weight (see, for example, [11]) and $\Phi$ is an Orlicz function of lower type $p$ for some $p \in (0,1]$ and of upper type 1. More examples of growth functions can be found in [32–34, 38].

REMARK 2.4. By [37, Lemma 11] (see also [33, Lemma 4.1]), without loss of generality, we may always assume that an anisotropic growth function $\varphi$ is of uniformly lower type $p$ for some $p \in (0,1]$ and of uniformly upper type 1 such that $\varphi(x,\cdot)$ is continuous and strictly increasing for any given $x \in \mathbb{R}^n$.

Denote the space of all *Schwartz functions* on $\mathbb{R}^n$ by $\mathcal{S}$, namely, the set of all $C^\infty$ functions $\phi$ satisfying that, for any $\alpha \in \mathbb{N}^n$ and $\ell \in \mathbb{N}$,

$$\|\phi\|_{\alpha,\ell} := \sup_{x \in \mathbb{R}^n} |\partial^\alpha \phi(x)|[1+\rho(x)]^\ell < \infty.$$

The dual space of $\mathcal{S}$, namely, the space of all *tempered distributions*, which equipped with the weak-$*$ topology, is denoted by $\mathcal{S}'$.

REMARK 2.5. By [8, p. 11, Lemma 3.2], we know that the Schwartz function space $\mathcal{S}$, which equipped with the pseudonorms $\{\|\cdot\|_{\alpha,\ell}\}_{\alpha\in\mathbb{N}^n,\ell\in\mathbb{N}}$, is equivalent to the classical Schwartz function space, which equipped with

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

the pseudonorms $\{\|\cdot\|^*_{\alpha,\ell}\}_{\alpha\in\mathbb{N}^n,\ell\in\mathbb{N}}$, where, for any $\alpha\in\mathbb{N}^n$, $\ell\in\mathbb{N}$ and $\phi\in\mathcal{S}$,

$$\|\phi\|^*_{\alpha,\ell} := \sup_{x\in\mathbb{R}^n} |\partial^\alpha\phi(x)|(1+|x|^2)^{\ell/2}.$$

For any $m\in\mathbb{N}$, let

$$\mathcal{S}_m := \left\{\phi\in\mathcal{S} : \sup_{x\in\mathbb{R}^n,|\alpha|\leqslant m+1} [1+\rho(x)]^{m+2}|\partial^\alpha\phi(x)|\leqslant 1\right\}.$$

Then, for any $m\in\mathbb{N}$ and $f\in\mathcal{S}'$, the *nontangential grand maximal function* $f^*_m$ of $f$ is defined by setting, for all $x\in\mathbb{R}^n$,

$$f^*_m(x) := \sup_{\phi\in\mathcal{S}_m} \sup_{k\in\mathbb{Z},y\in x+B_k} |f*\phi_k(y)|, \tag{2.7}$$

where, for any $k\in\mathbb{Z}$, $\phi_k(\cdot) := b^k\phi(A^k\cdot)$. When

$$m = m(\varphi) := \left\lfloor \left(\frac{q(\varphi)}{i(\varphi)} - 1\right)\log_{(\lambda_-)} b\right\rfloor, \tag{2.8}$$

where $q(\varphi)$ and $i(\varphi)$ are as in (2.3) and (2.4), respectively, we denote $f^*_m$ simply by $f^*$.

Recall that the *weak Musielak–Orlicz space* $WL^\varphi$ is defined to be the set of all measurable functions $f$ such that, for some $\lambda\in(0,\infty)$,

$$\sup_{t\in(0,\infty)} \varphi(\{|f|>t\}, t/\lambda) < \infty$$

equipped with the quasinorm

$$\|f\|_{WL^\varphi} := \inf\left\{\lambda\in(0,\infty) : \sup_{t\in(0,\infty)} \varphi\left(\{|f|>t\}, \frac{t}{\lambda}\right)\leqslant 1\right\}.$$

Now, we introduce the anisotropic weak Musielak–Orlicz Hardy space $WH^\varphi_{A,m}$ as follows.

DEFINITION 2.6. For any $m\in\mathbb{N}$ and anisotropic growth function $\varphi$ as in Definition 2.3, the *anisotropic weak Musielak–Orlicz Hardy space* $WH^\varphi_{A,m}$ is defined as the set of all $f\in\mathcal{S}'$ such that $f^*_m\in WL^\varphi$ equipped with the quasinorm

$$\|f\|_{WH^\varphi_{A,m}} := \|f^*_m\|_{WL^\varphi}.$$

When $m := m(\varphi)$, $WH^\varphi_{A,m}$ is denoted simply by $WH^\varphi_A$.

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

REMARK 2.7. By Lemma 3.9 below, we know that, for any $m \in \mathbb{N} \cap [m(\varphi), \infty)$, $WH^{\varphi}_{A} = WH^{\varphi}_{A,m}$ with equivalent quasinorms. For simplicity, from now on, we denote simply by $WH^{\varphi}_{A}$ the anisotropic weak Musielak–Orlicz Hardy space $WH^{\varphi}_{A,m}$ with $m \in [m(\varphi), \infty) \cap \mathbb{N}$.

REMARK 2.8.

(i) Observe that, when $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, and $\varphi$ is as in (2.6) with a classical Muckenhoupt $\mathcal{A}_\infty$ weight $w$ and an Orlicz function $\Phi$, $WH^{\varphi}_{A}$ is just a weighted weak Orlicz–Hardy space which includes the classical weak Hardy space of Fefferman and Soria [19] ($\Phi(t) := t$ for all $t \in [0, \infty)$ and $\omega \equiv 1$ in this context) and the classical weighted weak Hardy space of Quek and Yang [48] ($\Phi(t) := t^p$ for all $t \in [0, \infty)$ with $p \in (0, 1]$ in this context).

(ii) When $\varphi$ is as in (2.6) with taking $\omega \equiv 1$ and $\Phi(t) := t^p$ for all $t \in [0, \infty)$, where $p \in (0, 1]$, $WH^{\varphi}_{A}$ becomes the anisotropic weak Hardy space of Ding and Lan [18], and more generally, when $\Phi$ is an Orlicz function, the space $WH^{\varphi}_{A}$ is probably new.

(iii) When $A := 2\mathrm{I}_{n\times n}$ and $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, $\varphi$ is reduced to the isotropic growth function of Liang *et al.* [41] and $WH^{\varphi}_{A}$ is just the weak Musielak–Orlicz Hardy space of Liang *et al.* [41].

(iv) When $\varphi$ is an anisotropic $p$-growth function with $i(\varphi) = I(\varphi) = p$, where $p \in (0, 1]$, $WH^{\varphi}_{A}$ is reduced to the anisotropic weak Musielak–Orlicz Hardy space of Zhang *et al.* [55] and Qi *et al.* [47].

Recall that a tempered distribution $f$ is said to *vanish weakly at infinity* if, for any $\phi \in \mathcal{S}$, $f * \phi_k \to 0$ in $\mathcal{S}'$ as $k \to -\infty$. Denote by $\mathcal{S}'_0$ the set of all $f \in \mathcal{S}'$ vanishing weakly at infinity.

DEFINITION 2.9. Let $\psi \in \mathcal{S}$ such that, for any $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leqslant m(\varphi)$, $\int_{\mathbb{R}^n} \psi(x) x^\alpha \, dx = 0$, where $m(\varphi)$ is as in (2.8). For any $f \in \mathcal{S}'$ and $\lambda \in (0, \infty)$, the *anisotropic Littlewood–Paley Lusin-area function* $S(f)$, *g-function* $g(f)$ and $g^*_\lambda$*-function* $g^*_\lambda(f)$ of $f$, associated with $\psi$, are defined by setting, respectively, for all $x \in \mathbb{R}^n$,

$$S(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} b^{-k} \int_{x+B_k} |f * \psi_{-k}(y)|^2 \, dy \right\}^{1/2}, \tag{2.9}$$

$$g(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} |f * \psi_{-k}(x)|^2 \right\}^{1/2}$$

and

$$g_\lambda^*(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} b^{-k} \int_{\mathbb{R}^n} \left[ \frac{b^k}{b^k + \rho(x-y)} \right]^\lambda |f * \psi_{-k}(y)|^2 \, dy \right\}^{1/2}.$$

The *anisotropic weak Musielak–Orlicz Hardy space* $WH^\varphi_{A,S}$ is defined as the set of all $f \in \mathcal{S}'_0$ such that

$$\|f\|_{WH^\varphi_{A,S}} := \|S(f)\|_{WL^\varphi} < \infty.$$

Similarly, the *anisotropic weak Musielak–Orlicz Hardy space* $WH^\varphi_{A,g}$ or $WH^\varphi_{A,g^*_\lambda}$ can also be defined with $S(f)$ replaced by $g(f)$ or $g^*_\lambda(f)$, respectively.

The main results of this section are the following three theorems.

THEOREM 2.10. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3. Then*

$$WH^\varphi_A = WH^\varphi_{A,S}$$

*with equivalent quasinorms.*

THEOREM 2.11. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3. Then*

$$WH^\varphi_A = WH^\varphi_{A,g}$$

*with equivalent quasinorms.*

THEOREM 2.12. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3, $q \in [1,\infty)$, $\varphi \in \mathbb{A}_q(A)$ and $\lambda \in (2q/p, \infty)$. Then there exists a positive constant $C := C_{(\varphi,q)}$, depending on $\varphi$ and $q$, such that, for any $f \in \mathcal{S}'$,*

$$\frac{1}{C}\|g^*_\lambda(f)\|_{WL^\varphi} \leqslant \|f\|_{WH^\varphi_A} \leqslant C\|g^*_\lambda(f)\|_{WL^\varphi} \tag{2.10}$$

*and, furthermore, $WH^\varphi_A = WH^\varphi_{A,g^*_\lambda}$ with equivalent quasinorms.*

REMARK 2.13.

(i) When $A := 2\mathrm{I}_{n\times n}$ and $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, $\varphi$ is reduced to the isotropic growth function of Liang *et al.* [41] and Theorems 2.10, 2.11 and 2.12 are reduced to Theorems 4.5, 4.8 and 4.13 of Liang *et al.* [41], respectively.

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

(ii) When $\varphi$ is an anisotropic $p$-growth function with $i(\varphi) = I(\varphi) = p$, where $p \in (0, 1]$, Theorems 2.10, 2.11 and 2.12 contain the corresponding results of Qi *et al.* [47, Theorems 1 and 2].
(iii) When $\varphi$ is as in (2.6) with taking $\omega \equiv 1$ and $\Phi(t) := t^p$ for all $t \in [0, \infty)$, where $p \in (0, 1]$, Theorems 2.10, 2.11 and 2.12 are also new.

COROLLARY 2.14. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3. Then $WH^{\varphi}_{A,S}$ is well defined. Precisely, if $\psi_1, \psi_2 \in \mathcal{S}$ are as in Definition 2.9, then $WH^{\varphi}_{A,S_{\psi_1}} = WH^{\varphi}_{A,S_{\psi_2}}$ with equivalent quasinorms, where $S_{\psi_1}$ or $S_{\psi_2}$ is defined as in (2.9) via replacing $\psi$ by $\psi_1$ or $\psi_2$, respectively. The above result also holds true with $WH^{\varphi}_{A,S}$ replaced by $WH^{\varphi}_{A,g}$ or $WH^{\varphi}_{A,g^*_\lambda}$, respectively.*

## §3. Proof of Theorem 2.10

To obtain the anisotropic Lusin-area function characterization of $WH^{\varphi}_{A}$, we begin with recalling some notation and establishing several technical lemmas.

Throughout the whole paper, let $\mathcal{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}$ be the *collection of all dilated balls.*

LEMMA 3.1. [51, pp. 7–8] *Let $q \in [1, \infty)$ and $\varphi \in \mathbb{A}_q(A)$. Then there exists a positive constant $C$ such that, for any $E \subset B \in \mathcal{B}$ and $t \in (0, \infty)$,*

$$\frac{1}{C}\frac{|B|^{1/q}}{|E|^{1/q}} \leqslant \frac{\varphi(B, t)}{\varphi(E, t)} \leqslant C\frac{|B|^q}{|E|^q}.$$

The following lemma is an anisotropic variant of well-known superposition principle of weak type estimates, the proof of which is similar to that of [14, Lemma 7.13]. When $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, and $\varphi(x, t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$ with $p \in (0, 1)$, it goes back to the well-known superposition principle of weak type estimates obtained by Stein *et al.* [49, Lemma 1.8] and, independently, by Kalton [30, Theorem 6.1].

LEMMA 3.2. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3 satisfying $I(\varphi) \in (0, 1)$, where $I(\varphi)$ is as in (2.5). Assume that $\{a_j\}_{j\in\mathbb{Z}_+}$ is a sequence of measurable functions and $\{\lambda_j\}_{j\in\mathbb{Z}_+} \subset \mathbb{C}$ such that there exists a sequence $\{x_j + B_{l_j}\}_{j\in\mathbb{Z}_+}$ of dilated balls, where $l_j \in \mathbb{Z}$, it satisfies that*

$$\sum_{j\in\mathbb{Z}_+} \varphi\left(x_j + B_{l_j}, \frac{|\lambda_j|}{\|\chi_{x_j+B_{l_j}}\|_{L^\varphi}}\right) < \infty.$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

*Moreover, if there exist positive constants $C$ and $\eta_0$, where $C$ is independent of $\eta_0$, such that, for any $j \in \mathbb{Z}_+$,*

$$\varphi(\{|\lambda_j a_j| > \eta_0\}, \eta_0) \leqslant C\varphi\left(x_j + B_{l_j}, \frac{|\lambda_j|}{\|\chi_{x_j+B_{l_j}}\|_{L^\varphi}}\right), \tag{3.1}$$

*then there exists a positive constant $\widetilde{C}$, independent of $\eta_0$, such that*

$$\varphi\left(\left\{\sum_{j\in\mathbb{Z}_+} |\lambda_j a_j| > \eta_0\right\}, \eta_0\right) \leqslant \widetilde{C} \sum_{j\in\mathbb{Z}_+} \varphi\left(x_j + B_{l_j}, \frac{|\lambda_j|}{\|\chi_{x_j+B_{l_j}}\|_{L^\varphi}}\right).$$

REMARK 3.3. It is worth pointing out that the assumption of Lemma 3.2 is weaker than that of [14, Lemma 7.13] and hence also the conclusion, but it is just enough for later use. Precisely, we only need some constant $\eta_0 \in (0, \infty)$ such that the condition (3.1) holds while, in [14, Lemma 7.13], the corresponding condition must hold for any $\eta \in (0, \infty)$. Luckily, the proof of Lemma 3.2 is similar to that of [14, Lemma 7.13], the details being omitted.

The following lemma is a property of anisotropic growth functions, the proof of which is similar to that of [41, Lemma 3.3(ii)], the details being omitted here.

LEMMA 3.4. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3. Then, for any $f \in WL^\varphi$ satisfying $\|f\|_{WL^\varphi} > 0$,*

$$\sup_{t\in(0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\|f\|_{WL^\varphi}}\right) = 1.$$

Let $s \in \mathbb{N}$ and $\mathcal{P}_s$ denote the *linear space* of polynomials of degrees not bigger than $s$. Recall that a locally integrable function $f$ on $\mathbb{R}^n$ is said to belong to the *weighted anisotropic Campanato space* $\mathcal{L}_{p_0,\varphi(\cdot,1),s}$ if

$$\|f\|_{\mathcal{L}_{p_0,\varphi(\cdot,1),s}} := \sup_{B\in\mathcal{B}} \frac{1}{[\varphi(B, 1)]^{1/p_0}} \int_B |f(x) - P^s_B f(x)|\, dx < \infty,$$

where $P^s_B f$ denotes the unique $P \in \mathcal{P}_s$ such that, for any polynomial $R$ on $\mathbb{R}^n$ with order not bigger than $s$, $\int_B [f(x) - P(x)]R(x)\, dx = 0$.

LEMMA 3.5. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3, $p_0 \in (0, i(\varphi))$, $q_0 \in (q(\varphi), \infty)$ and $s \in \mathbb{N}$ such that $s > (q_0/p_0 - 1) \log_{(\lambda_-)} b - 1$, where $i(\varphi)$ and $q(\varphi)$ are as in (2.4) and (2.3), respectively. If $\phi \in \mathcal{S}$, then $\phi \in \mathcal{L}_{p_0,\varphi(\cdot,1),s}$.*

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

*Proof.* We show this lemma by borrowing some ideas from the proof of [40, Proposition 2.3]. For any $\phi \in \mathcal{S}$, $x \in \mathbb{R}^n$ and dilated ball $B := x_0 + B_k \in \mathcal{B}$, where $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, let

$$p_B(x) := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant s} \frac{\partial^\alpha \phi(x_0)}{\alpha!}(x - x_0)^\alpha \in \mathcal{P}_s.$$

Clearly, $B \subset B_{k_0}$, where $k_0 := \sigma + 1 + \lfloor \log_b(b^k + \rho(x_0)) \rfloor$. Then, by [8, p. 51, (8.9)], there exists a positive constant $C$, depending only on $s$, such that, for any $\mathbf{B} \in \mathcal{B}$ and $f \in L^1(\mathbf{B})$,

$$\sup_{x \in \mathbf{B}} |P^s_{\mathbf{B}} f(x)| \leqslant C \frac{1}{|\mathbf{B}|} \int_{\mathbf{B}} |f(x)|\, dx. \tag{3.2}$$

From this and Taylor's theorem, we deduce that, for any $x \in B$, there exists $\xi := \xi(x) \in B$ such that

$$\begin{aligned}
&\int_B |\phi(x) - P^s_B \phi(x)|\, dx \\
&\quad \leqslant \int_B |\phi(x) - p_B(x)|\, dx + \int_B |P^s_B(p_B - \phi)(x)|\, dx \\
&\quad \lesssim \int_B |\phi(x) - p_B(x)|\, dx \\
&\quad \sim \int_B \left| \sum_{\alpha \in \mathbb{N}^n, |\alpha| = s+1} \frac{\partial^\alpha \phi(\xi)}{\alpha!}(x - x_0)^\alpha \right| dx \\
&\quad \lesssim \int_B \frac{1}{[1 + \rho(\xi)]^M} |x - x_0|^{s+1}\, dx, \qquad (3.3)
\end{aligned}$$

where the constant $M$ is arbitrary for the moment and will be fixed later.

Now, if $k_0 \leqslant 10\sigma$, namely, $B \subset B_{k_0} \subset B_{10\sigma}$, then, by Lemma 3.1, (3.3) with taking $M = 0$,

$$\begin{aligned}
|x| &\lesssim [\rho(x)]^{\log_b(\lambda_-)} \chi_{\{y \in \mathbb{R}^n : \rho(y) < 1\}}(x) \\
&\quad + [\rho(x)]^{\log_b(\lambda_+)} \chi_{\{y \in \mathbb{R}^n : \rho(y) \geqslant 1\}}(x) \quad \text{(see [8, p. 11, Lemma 3.2])} \qquad (3.4)
\end{aligned}$$

and $s > (q_0/p_0 - 1)\log_{(\lambda_-)} b - 1$, we see that

$$
\begin{aligned}
&\frac{1}{[\varphi(B,1)]^{1/p_0}}\int_B |\phi(x) - P_B^s\phi(x)|\,dx \\
&\quad\lesssim \left(\frac{|B_{10\sigma}|^{q_0}}{|B|^{q_0}}\right)^{1/p_0}\frac{1}{[\varphi(B_{10\sigma},1)]^{1/p_0}}\int_B |\phi(x) - P_B^s\phi(x)|\,dx \\
&\quad\lesssim b^{-k(q_0/p_0)}\int_B |x - x_0|^{s+1}\,dx \\
&\quad\sim b^{-k(q_0/p_0)}\int_{B_k} |x|^{s+1}\,dx \\
&\quad\lesssim b^{-k(q_0/p_0)}\int_{B_k} \{[\rho(x)]^{\log_b(\lambda_-)}\chi_{\{y\in\mathbb{R}^n:\rho(y)<1\}}(x) \\
&\qquad + [\rho(x)]^{\log_b(\lambda_+)}\chi_{\{y\in\mathbb{R}^n:\rho(y)\geqslant 1\}}(x)\}^{s+1}\,dx \\
&\quad\lesssim b^{k[1+(s+1)\log_b(\lambda_-)-(q_0/p_0)]} + b^{k[1+(s+1)\log_b(\lambda_+)-(q_0/p_0)]} \\
&\quad\lesssim b^{10\sigma[1+(s+1)\log_b(\lambda_-)-(q_0/p_0)]} + b^{10\sigma[1+(s+1)\log_b(\lambda_+)-(q_0/p_0)]} \lesssim 1.
\end{aligned}\tag{3.5}
$$

If $k_0 > 10\sigma$ and $\rho(x_0) \leqslant b^{k+2\sigma}$, then $|B| \sim |B_{k_0}|$. From this, Lemma 3.1, $|B_{10\sigma}| < |B_{k_0}|$, (3.2) and Remark 2.5, we deduce that

$$
\begin{aligned}
&\frac{1}{[\varphi(B,1)]^{1/p_0}}\int_B |\phi(x) - P_B^s\phi(x)|\,dx \\
&\quad\lesssim \left(\frac{|B_{k_0}|^{q_0}}{|B|^{q_0}}\right)^{1/p_0}\frac{1}{[\varphi(B_{k_0},1)]^{1/p_0}}\int_B |\phi(x) - P_B^s\phi(x)|\,dx \\
&\quad\lesssim \frac{1}{[\varphi(B_{10\sigma},1)]^{1/p_0}}\int_B |\phi(x) - P_B^s\phi(x)|\,dx \\
&\quad\lesssim \int_B |\phi(x) - P_B^s\phi(x)|\,dx \\
&\quad\lesssim \int_B |\phi(x)|\,dx \\
&\quad\lesssim \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}}\,dx \lesssim 1.
\end{aligned}\tag{3.6}
$$

If $k_0 > 10\sigma$ and $\rho(x_0) > b^{k+2\sigma}$, then, for any $x \in B$, it holds true that $b^k \lesssim \rho(x) \sim \rho(x_0)$. From this, Lemma 3.1, (3.3) with taking $M = (1/p_0)(q_0 - 1/q_0)+[1 + (s+1)\log_b(\lambda_+) - q_0/p_0]$, (3.4) and $s > (q_0/p_0 - 1)\log_{(\lambda_-)} b - 1$,

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

we deduce that

$$
\begin{aligned}
&\frac{1}{[\varphi(B,1)]^{1/p_0}}\int_B |\phi(x)-P_B^s\phi(x)|\,dx\\
&\quad\lesssim \left(\frac{|B_{k_0}|^{q_0}}{|B|^{q_0}}\right)^{1/p_0}\frac{1}{[\varphi(B_{k_0},1)]^{1/p_0}}\int_B |\phi(x)-P_B^s\phi(x)|\,dx\\
&\quad\lesssim \left(\frac{|B_{k_0}|^{q_0}}{|B|^{q_0}}\right)^{1/p_0}\left(\frac{|B_{10\sigma}|^{1/q_0}}{|B_{k_0}|^{1/q_0}}\right)^{1/p_0}\frac{1}{[\varphi(B_{10\sigma},1)]^{1/p_0}}\\
&\qquad\times\int_B |\phi(x)-P_B^s\phi(x)|\,dx\\
&\quad\lesssim |B_{k_0}|^{(1/p_0)(q_0-1/q_0)}|B|^{-q_0/p_0}\int_B \frac{1}{[1+\rho(\xi)]^M}|x-x_0|^{s+1}\,dx\\
&\quad\lesssim |B_{k_0}|^{(1/p_0)(q_0-1/q_0)}|B|^{-q_0/p_0}\frac{1}{[1+\rho(x_0)]^M}\int_{B_k}|x|^{s+1}\,dx\\
&\quad\lesssim b^{k_0(1/p_0)(q_0-1/q_0)}b^{-k(q_0/p_0)}\frac{1}{[1+\rho(x_0)]^M}\\
&\qquad\times\left(b^{k+k(s+1)\log_b(\lambda_-)}+b^{k+k(s+1)\log_b(\lambda_+)}\right)\\
&\quad\lesssim b^{[\sigma+1+\log_b(b^k+\rho(x_0))](1/p_0)(q_0-1/q_0)}\frac{1}{[1+\rho(x_0)]^M}\\
&\qquad\times\left(b^{k[1+(s+1)\log_b(\lambda_-)-q_0/p_0]}+b^{k[1+(s+1)\log_b(\lambda_+)-q_0/p_0]}\right)\\
&\quad\lesssim [\rho(x_0)]^{(1/p_0)(q_0-(1/q_0))}\frac{1}{[1+\rho(x_0)]^M}\\
&\qquad\times\left([\rho(x_0)]^{(1+(s+1)\log_b(\lambda_-)-q_0/p_0)}+[\rho(x_0)]^{(1+(s+1)\log_b(\lambda_+)-q_0/p_0)}\right)\\
&\quad\lesssim 1.
\end{aligned}
\tag{3.7}
$$

Combining (3.5), (3.6) and (3.7), we see that $\|\phi\|_{\mathcal{L}_{p_0,\varphi(\cdot,1),s}}\lesssim 1$. This finishes the proof of Lemma 3.5. ∎

DEFINITION 3.6. For any measurable subset $E$ of $\mathbb{R}^n$, the *space* $L^q_\varphi(E)$ for $q\in[1,\infty]$ is defined as the set of all measurable functions $f$ on $E$ such that

$$
\|f\|_{L^q_\varphi(E)} := \begin{cases} \sup\limits_{t\in(0,\infty)}\left[\dfrac{1}{\varphi(E,t)}\displaystyle\int_E |f(x)|^q\varphi(x,t)\,dx\right]^{1/q}<\infty, & q\in[1,\infty);\\[2ex] \|f\|_{L^\infty(E)}<\infty, & q=\infty.\end{cases}
$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

Recall that the *Musielak–Orlicz space* $L^\varphi$ is defined as the set of all measurable functions $f$ such that, for some $\lambda \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda)\, dx < \infty$$

equipped with the Luxembourg (or called the Luxembourg–Nakano) (quasi-) norm

$$\|f\|_{L^\varphi} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda)\, dx \leqslant 1 \right\}.$$

DEFINITION 3.7. Let $\varphi$ be an anisotropic growth function as in Definition 2.3.

(i) An anisotropic triplet $(\varphi, q, s)$ is said to be *admissible*, if $q \in (q(\varphi), \infty]$ and $s \in [m(\varphi), \infty) \cap \mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.3) and (2.8), respectively.

(ii) For an admissible anisotropic triplet $(\varphi, q, s)$, a measurable function $a$ is called an *anisotropic* $(\varphi, q, s)$*-atom* associated with some dilated ball $B \in \mathcal{B}$ if it satisfies the following three conditions:

   (a) $\operatorname{supp} a \subset B$;
   (b) $\|a\|_{L^q_\varphi(B)} \leqslant \|\chi_B\|^{-1}_{L^\varphi}$;
   (c) $\int_{\mathbb{R}^n} a(x) x^\alpha\, dx = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant s$.

Now, via anisotropic $(\varphi, q, s)$-atom as in Definition 3.7, we introduce the definition of anisotropic weak Musielak–Orlicz atomic Hardy space as follows, which is motivated by Liang *et al.* [41, Definition 3.7].

DEFINITION 3.8. For an admissible anisotropic triplet $(\varphi, q, s)$ as in Definition 3.7, the *anisotropic weak Musielak–Orlicz atomic Hardy space* $WH^{\varphi,q,s}_{A,\mathrm{at}}$ is defined as the space of all $f \in \mathcal{S}'$ satisfying that there exist a sequence of anisotropic $(\varphi, q, s)$-atoms,$\{a_i^k\}_{k\in\mathbb{Z},i}$, associated with dilated balls $\{B_i^k\}_{k\in\mathbb{Z},i}$, and a positive constant $C$ such that $\sum_i \chi_{B_i^k}(x) \leqslant C$ for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and $f = \sum_{k\in\mathbb{Z}} \sum_i \lambda_i^k a_i^k$ in $\mathcal{S}'$, where $\lambda_i^k := \widetilde{C} 2^k \|\chi_{B_i^k}\|_{L^\varphi}$ for any $k \in \mathbb{Z}$ and $i$ with $\widetilde{C}$ being a positive constant independent of $f$.

Moreover, define

$$\|f\|_{WH^{\varphi,q,s}_{A,\mathrm{at}}} := \inf \left\{ \inf \left\{ \lambda \in (0, \infty) : \sup_{k\in\mathbb{Z}} \left\{ \sum_i \varphi\left( B_i^k, \frac{2^k}{\lambda} \right) \right\} \leqslant 1 \right\} \right\},$$

where the first infimum is taken over all decompositions of $f$ as above.

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

The following is the atomic characterization of $WH^{\varphi}_A$.

LEMMA 3.9. *Let $(\varphi, q, s)$ be an admissible anisotropic triplet as in Definition 3.7. If $m \in [m(\varphi), \infty) \cap \mathbb{N}$, where $m(\varphi)$ is as in (2.8), then*

$$WH^{\varphi}_{A,m} = WH^{\varphi,q,s}_{A,\mathrm{at}}$$

*with equivalent quasinorms.*

*Proof.* **Step 1.** In this step, we prove, for any $m \in [m(\varphi), \infty) \cap \mathbb{N}$, $WH^{\varphi,q,s}_{A,\mathrm{at}} \subset WH^{\varphi}_{A,m}$.

The argument presented in this step partly follows the proof of [41, Theorem 3.5]. For any $f \in WH^{\varphi,q,s}_{A,\mathrm{at}}$, by Definition 3.8, we know that there exist a sequence of multiples of anisotropic $(\varphi, q, s)$-atoms, $\{f_i^k\}_{k\in\mathbb{Z},i}$, associated with dilated balls $\{B_i^k\}_{k\in\mathbb{Z},i}$, where $B_i^k \in \mathcal{B}$, such that $f = \sum_{k\in\mathbb{Z}} \sum_i f_i^k$ in $\mathcal{S}'$, $\|f_i^k\|_{L^q_\varphi(B_i^k)} \lesssim 2^k$ for any $k \in \mathbb{Z}$ and $i$, $\sum_i \chi_{B_i^k}(x) \lesssim 1$ for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and

$$\|f\|_{WH^{\varphi,q,s}_{A,\mathrm{at}}} \sim \inf\left\{\lambda \in (0,\infty) : \sup_{k\in\mathbb{Z}}\left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\} \leqslant 1\right\}. \tag{3.8}$$

Thus, to show $WH^{\varphi,q,s}_{A,\mathrm{at}} \subset WH^{\varphi}_{A,m}$, it suffices to prove that, for any $\alpha, \lambda \in (0,\infty)$ and $m \in [m(\varphi), \infty) \cap \mathbb{N}$,

$$\varphi\left(\{f_m^* > \alpha\}, \frac{\alpha}{\lambda}\right) \lesssim \sup_{k\in\mathbb{Z}}\left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\}, \tag{3.9}$$

where $f_m^*$ is as in (2.7).

To show (3.9), we may assume that there exists some $k_0 \in \mathbb{Z}$ such that $\alpha = 2^{k_0}$ without loss of generality. Write

$$f = \sum_{k=-\infty}^{k_0-1} \sum_i f_i^k + \sum_{k=k_0}^{\infty} \sum_i f_i^k =: F_1 + F_2.$$

For $F_1$, by repeating the estimate of $F_1$ in the proof of [41, Theorem 3.5] with $b_{i,j}$ and $2^{i_0}$ replaced by $f_i^k$ and $2^{k_0}$, respectively, we have

$$\varphi\left(\left\{(F_1)_m^* > 2^{k_0}\right\}, \frac{2^{k_0}}{\lambda}\right) \lesssim \sup_{k\in\mathbb{Z}}\left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\}. \tag{3.10}$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

Let $B_i^k := x_i^k + B_{l_i^k}$ with $x_i^k \in \mathbb{R}^n$ and $l_i^k \in \mathbb{Z}$, and $A_{k_0} := \bigcup_{k=k_0}^{\infty} \bigcup_i (x_i^k + B_{l_i^k+\sigma})$. Now we are interested in $(F_2)_m^*$. To show that

$$\varphi\left(\{(F_2)_m^* > 2^{k_0}\}, \frac{2^{k_0}}{\lambda}\right) \lesssim \sup_{k\in\mathbb{Z}} \left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\}, \tag{3.11}$$

we cut $\{(F_2)_m^* > 2^{k_0}\}$ into $A_{k_0}$ and $\{x \in (A_{k_0})^{\complement} : (F_2)_m^*(x) > 2^{k_0}\}$.

Since $\varphi$ is of uniformly lower type $p$ and, by Lemma 3.1 with $\varphi \in \mathbb{A}_\infty(A)$, $\varphi(x_i^k + B_{l_i^k+\sigma}, 2^{k_0}/\lambda) \lesssim \varphi(B_i^k, 2^{k_0}/\lambda)$, it follows that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}\varphi\left(A_{k_0}, \frac{2^{k_0}}{\lambda}\right) &\lesssim \sum_{k=k_0}^{\infty} \sum_i \varphi\left(B_i^k, \frac{2^{k_0}}{\lambda}\right)\\ &\lesssim \sum_{k=k_0}^{\infty} 2^{(k_0-k)p} \sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right) \lesssim \sup_{k\in\mathbb{Z}} \left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\},\end{aligned} \tag{3.12}$$

which is wished.

By Definition 3.8, we assume that $f_i^k := \lambda_i^k a_i^k$, where $a_i^k$ for any $k \in \mathbb{Z}$ and $i$ is an anisotropic $(\varphi, q, s)$-atom and $\lambda_i^k := \widetilde{C} 2^k \|\chi_{B_i^k}\|_{L^\varphi}$. Since $x \in (A_{k_0})^{\complement} \subset x_i^k + (B_{l_i^k+\sigma})^{\complement}$, it follows that there exists some $j \in \mathbb{N}$ such that $x \in x_i^k + B_{l_i^k+\sigma+j+1} \setminus B_{l_i^k+\sigma+j}$. By repeating the estimate of (100) in [37, p. 12], we obtain that

$$(a_i^k)_m^*(x) \lesssim [b(\lambda_-)^{s+1}]^{-j} \|a_i^k\|_{L^q_\varphi(B_i^k)}.$$

From this, $\lambda_i^k = \widetilde{C} 2^k \|\chi_{B_i^k}\|_{L^\varphi}$, (b) of Definition 3.7(ii) and $\rho(x - x_i^k) = b^{l_i^k+\sigma+j}$, we deduce that

$$\begin{aligned}(f_i^k)_m^*(x) &\lesssim [b(\lambda_-)^{s+1}]^{-j} \lambda_i^k \|a_i^k\|_{L^q_\varphi(B_i^k)}\\ &\lesssim 2^k [b(\lambda_-)^{s+1}]^{-j} \lesssim 2^k \left[\frac{b^{l_i^k}}{\rho(x - x_i^k)}\right]^M,\end{aligned} \tag{3.13}$$

where $M := (s+1)\log_b(\lambda_-) + 1$.

Since $s \geqslant m(\varphi) = \lfloor (q(\varphi)/i(\varphi) - 1) \log_{(\lambda_-)} b \rfloor$, it follows that there exist $q_0 \in (q(\varphi), \infty)$ and $p_0 \in (0, i(\varphi))$ such that $s > (q_0/p_0 - 1) \log_{(\lambda_-)} b - 1$ and hence $p_0 M - q_0 > 0$. For any $k \in \mathbb{Z}$, $i$ and $\lambda \in (0, \infty)$, by (3.13) and Lemma 3.1 with $\varphi \in \mathbb{A}_{q_0}(A)$, we have

$$\begin{aligned}
&\varphi\left(\{x\in(A_{k_0})^{\complement}:(f_i^k)_m^*(x)>2^{k_0}\},\frac{2^{k_0}}{\lambda}\right)\\
&\lesssim\varphi\left(\{x\in\mathbb{R}^n:\rho(x-x_i^k)<2^{(k-k_0)/M}b^{l_i^k}\},\frac{2^{k_0}}{\lambda}\right).
\end{aligned}\tag{3.14}$$

Notice that $b>1$, then there exists some $\widetilde{k}\in\mathbb{Z}_+$ such that $b^{\widetilde{k}}\sim 2^{1/M}$. By Lemma 3.1 with $\varphi\in\mathbb{A}_{q_0}(A)$ and uniformly lower type $p_0$ property of $\varphi$, we see that (3.14) is bounded by a positive constant times

$$\begin{aligned}
&\varphi\left(\{x\in\mathbb{R}^n:\rho(x-x_i^k)<b^{l_i^k+\widetilde{k}(k-k_0)}\},\frac{2^{k_0}}{\lambda}\right)\\
&\quad\lesssim\varphi\left(x_i^k+B_{l_i^k+\widetilde{k}(k-k_0)},\frac{2^{k_0}}{\lambda}\right)\\
&\quad\lesssim(2^{k-k_0})^{(q_0/M)-p_0}\varphi\left(B_i^k,\frac{2^k}{\lambda}\right).
\end{aligned}\tag{3.15}$$

Therefore, we know that, for any $\lambda\in(0,\infty)$,

$$\varphi\left(\{x\in(A_{k_0})^{\complement}:(f_i^k)_m^*(x)>2^{k_0}\},\frac{2^{k_0}}{\lambda}\right)\lesssim(2^{k-k_0})^{(q_0/M)-p_0}\varphi\left(B_i^k,\frac{2^k}{\lambda}\right).$$

Because $\varphi$ is of uniformly upper type 1, we cannot use the superposition principle of weak type estimates directly. Instead, we introduce an auxiliary function $\widetilde{\varphi}$. For any $x\in\mathbb{R}^n$ and $t\in(0,\infty)$, let $\widetilde{\varphi}(x,t):=\varphi(x,t)t^{(q_0/M)-p_0}$, then $\widetilde{\varphi}$ is an anisotropic Musielak–Orlicz function of uniformly lower type $q_0/M$ and of uniformly upper type $1+q_0/M-p_0$.

Let $\widetilde{\lambda_i^k}:=2^k\|\chi_{B_i^k}\|_{L^\varphi}/\lambda$,

$$\widetilde{a_i^k}(x):=\frac{1}{\|\chi_{B_i^k}\|_{L^\varphi}}\left[\frac{b^{l_i^k}}{\rho(x-x_i^k)}\right]^M\qquad\text{and}\qquad\eta_0:=\frac{2^{k_0}}{\lambda}.$$

Then, by (3.14) and (3.15), we have

$$\begin{aligned}
&\widetilde{\varphi}(\{|\widetilde{\lambda_i^k}\widetilde{a_i^k}|>\eta_0\},\eta_0)\\
&\quad=\varphi\left(\{x\in\mathbb{R}^n:\rho(x-x_i^k)<2^{(k-k_0)/M}b^{l_i^k}\},\frac{2^{k_0}}{\lambda}\right)\left(\frac{2^{k_0}}{\lambda}\right)^{q_0/M-p_0}\\
&\quad\lesssim\varphi\left(B_i^k,\frac{2^k}{\lambda}\right)\left(\frac{2^k}{\lambda}\right)^{q_0/M-p_0}\sim\widetilde{\varphi}\left(B_i^k,\frac{2^k}{\lambda}\right).
\end{aligned}\tag{3.16}$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

By $q_0/M - p_0 < 0$ and (3.8), we see that

$$
\begin{aligned}
&\sum_{k=k_0}^{\infty}\sum_i \widetilde{\varphi}\left(B_i^k, \frac{|\widetilde{\lambda_i^k}|}{\|\chi_{B_i^k}\|_{L^\varphi}}\right)\\
&\quad \lesssim \left(\frac{2^{k_0}}{\lambda}\right)^{q_0/M-p_0} \sup_{k\in\mathbb{Z}}\left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\} < \infty. \qquad (3.17)
\end{aligned}
$$

Thus, from (3.13), Lemma 3.2 with (3.16), (3.17) and $I(\widetilde{\varphi}) \in (0,1)$, and $q_0/M - p_0 < 0$, it follows that, for any $\lambda \in (0,\infty)$,

$$
\begin{aligned}
&\varphi\left(\left\{x \in (A_{k_0})^{\complement} : \sum_{k=k_0}^{\infty}\sum_i (f_i^k)_m^*(x) > 2^{k_0}\right\}, \frac{2^{k_0}}{\lambda}\right)\\
&\quad = \widetilde{\varphi}\left(\left\{x \in (A_{k_0})^{\complement} : \sum_{k=k_0}^{\infty}\sum_i (f_i^k)_m^*(x) > 2^{k_0}\right\}, \frac{2^{k_0}}{\lambda}\right)\\
&\qquad \times \left(\frac{2^{k_0}}{\lambda}\right)^{p_0-q_0/M}\\
&\quad \lesssim \widetilde{\varphi}\left(\left\{x \in \mathbb{R}^n : \sum_{k=k_0}^{\infty}\sum_i 2^k\left[\frac{b^{l_i^k}}{\rho(x-x_i^k)}\right]^M > 2^{k_0}\right\}, \frac{2^{k_0}}{\lambda}\right)\\
&\qquad \times \left(\frac{2^{k_0}}{\lambda}\right)^{p_0-q_0/M}\\
&\quad \sim \widetilde{\varphi}\left(\left\{\sum_{k=k_0}^{\infty}\sum_i |\widetilde{\lambda_i^k}\widetilde{a_i^k}| > \eta_0\right\}, \eta_0\right)\left(\frac{2^{k_0}}{\lambda}\right)^{p_0-q_0/M}\\
&\quad \lesssim \sum_{k=k_0}^{\infty}\sum_i \widetilde{\varphi}\left(B_i^k, \frac{|\widetilde{\lambda_i^k}|}{\|\chi_{B_i^k}\|_{L^\varphi}}\right)\left(\frac{2^{k_0}}{\lambda}\right)^{p_0-q_0/M}\\
&\quad \lesssim \sup_{k\in\mathbb{Z}}\left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\}. \qquad (3.18)
\end{aligned}
$$

Combining (3.10), (3.12) and (3.18), we finally obtain (3.9). This finishes the proof of Step 1.

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

**Step 2.** In this step, we prove, for any $m\in[m(\varphi),\infty)\cap\mathbb{N}$, $WH^{\varphi}_{A,m}\subset WH^{\varphi,q,s}_{A,\mathrm{at}}$. Since, for any $q\in(q(\varphi),\infty)$, an anisotropic $(\varphi,\infty,s)$-atom is also an anisotropic $(\varphi,q,s)$-atom, it follows that $WH^{\varphi,\infty,s}_{A,\mathrm{at}}\subset WH^{\varphi,q,s}_{A,\mathrm{at}}$. To show the desired conclusion, we only need to prove that $WH^{\varphi}_{A,m}\subset WH^{\varphi,\infty,s}_{A,\mathrm{at}}$. Since the proof of $WH^{\varphi}_{A,m}\subset WH^{\varphi,\infty,s}_{A,\mathrm{at}}$ is similar to that of [55, Theorem 1], we use the same notation as in the proof of [55, Theorem 1]. In [55, Theorem 1], $\varphi$ is an anisotropic $p$-growth function which is of uniformly lower type $p$ and of uniformly upper type $p$, but in our situation, $\varphi$ is an anisotropic growth function which is of uniformly lower type $p$ and of uniformly upper type 1. Here we just give out the necessary modifications with respect to the uniformly upper type $p$ property of $\varphi$ in [55, Theorem 1]. Without loss of generality, we may assume $\|f\|_{WH^{\varphi}_{A,m}}=1$ and the general case follows at once by the homogeneity of $\|\cdot\|_{WH^{\varphi}_{A,m}}$. By checking the proof of [55, Theorem 1], [55, (25)] can be replaced by, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}\sum_i\varphi\left(B_i^k,\frac{2^k}{\lambda}\right)&\lesssim\sum_i\varphi\left(x_i^k+B_{l_i^k-2\sigma},\frac{2^k}{\lambda}\right)\lesssim\varphi\left(\Omega_k,\frac{2^k}{\lambda}\right)\\&\lesssim\sup_{\alpha\in(0,\infty)}\varphi\left(\{f_m^*>\alpha\},\frac{\alpha}{\lambda}\right),\end{aligned}$$

which, together with taking $\lambda=\|f\|_{WH^{\varphi}_{A,m}}$ and using Lemma 3.4, further implies that

$$\sum_i\left(B_i^k,\frac{2^k}{\|f\|_{WH^{\varphi}_{A,m}}}\right)\lesssim 1.$$

Hence, we obtain

$$\sum_i\varphi\left(B_i^k,\frac{2^k}{\lambda}\right)\lesssim\|f\|_{WH^{\varphi}_{A,m}}.$$

On the other hand, we need to prove that, for any $l\in\mathbb{Z}_+$,

$$\sum_{|k|>N}f_k^l\to 0\ (N\to\infty)\quad\text{in }\mathcal{S}'\text{ uniformly},\tag{3.19}$$

where $f_k^l:=\sum_i\beta_i^{l,k}$. For any $k\in\mathbb{Z}$, let $\Omega_k:=\{f_m^*>2^k\}$. Then $\Omega_k$ is open and, by the assumption $\|f\|_{WH^{\varphi}_{A,m}}=1$, we further see that

$$\sup_{k\in\mathbb{Z}}\varphi(\Omega_k,2^k)\lesssim 1.\tag{3.20}$$

Since $s \geqslant m(\varphi) = \lfloor (q(\varphi)/i(\varphi) - 1) \log_{(\lambda_-)} b \rfloor$ and $p \in (0, i(\varphi))$ can be chosen close enough to $i(\varphi)$, then we can choose $p_0 \in (0, p)$ and $q_0 \in (q(\varphi), \infty)$ close to $p$ and $q(\varphi)$, respectively, such that $s > (q_0/p_0 - 1) \log_{(\lambda_-)} b - 1$. By [55, (24), (26) and (27)], we see that

$$\operatorname{supp} \beta_i^{l,k} \subset x_i^k + B_{l_i^k + 4\sigma} =: B_i^k, \quad \text{where } l_i^k \in \mathbb{Z},$$
$$\int_{\mathbb{R}^n} \beta_i^{l,k}(x) P(x)\, dx = 0 \quad \text{for any } P \in \mathcal{P}_s$$

and

$$\|\beta_i^{l,k}\|_{L^\infty} \lesssim 2^k.$$

From this, $\sum_i \chi_{B_i^k}(x) \lesssim 1$ for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$ (see [55, (9)]), Lemma 3.5 with $s > (q_0/p_0 - 1) \log_{(\lambda_-)} b - 1$, $\varphi(B_i^k, 1) \lesssim \varphi(x_i^k + B_{l_i^k - 2\sigma}, 1)$ (see Lemma 3.1), $\sum_i \varphi(x_i^k + B_{l_i^k - 2\sigma}, 1) \leqslant \varphi(\Omega_k, 1)$ (see [55, (8)]), the uniformly lower type $p$ property of $\varphi$ and (3.20), we deduce that, for any $\phi \in \mathcal{S}$ and $l \in \mathbb{Z}_+$,

$$\begin{aligned}
\left| \left\langle \sum_{|k|>N} f_k^l, \phi \right\rangle \right| &\leqslant \sum_{|k|>N} \sum_i |\langle \beta_i^{l,k}, \phi\rangle| \\
&= \sum_{k=-\infty}^{-N-1} \sum_i \left| \int_{B_i^k} \beta_i^{l,k}(x)\phi(x)\, dx \right| \\
&\quad + \sum_{k=N+1}^{\infty} \sum_i \left| \int_{B_i^k} \beta_i^{l,k}(x)[\phi(x) - P^s_{B_i^k}\phi(x)]\, dx \right| \\
&\lesssim \sum_{k=-\infty}^{-N-1} 2^k \int_{\Omega_k} |\phi(x)|\, dx \\
&\quad + \sum_{k=N+1}^{\infty} \sum_i 2^k \int_{B_i^k} |\phi(x) - P^s_{B_i^k}\phi(x)|\, dx \\
&\lesssim 2^{-N} \|\phi\|_{L^1} + \sum_{k=N+1}^{\infty} \sum_i 2^k [\varphi(B_i^k, 1)]^{1/p_0} \|\phi\|_{\mathcal{L}_{p_0, \varphi(\cdot,1), s}} \\
&\lesssim 2^{-N} + \left[ \sum_{k=N+1}^{\infty} \sum_i 2^{kp_0} \varphi(x_i^k + B_{l_i^k - 2\sigma}, 2^{-k} 2^k) \right]^{1/p_0}
\end{aligned}$$

$$\lesssim 2^{-N} + \left[\sum_{k=N+1}^{\infty} 2^{-k(p-p_0)}\varphi(\Omega_k, 2^k)\right]^{1/p_0}$$
$$\lesssim 2^{-N} + 2^{-N(p-p_0)/p_0} \to 0\ (N \to \infty),$$

which implies that (3.19) holds true.

Finally, by repeating the rest proof of [55, Theorem 1], we can obtain $f \in WH^{\varphi,\infty,s}_{A,\mathrm{at}}$ and $\|f\|_{WH^{\varphi,\infty,s}_{A,\mathrm{at}}} \lesssim \|f\|_{WH^{\varphi}_{A,m}}$. This finishes the proof of Step 2 and hence Lemma 3.9. □

REMARK 3.10.

(i) Lemma 3.9 is an anisotropic extension of Liang *et al.* [41, Theorem 3.5], namely, when $A := 2\mathrm{I}_{n\times n}$ and $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, our result is reduced to Liang *et al.* [41, Theorem 3.5].

(ii) When $\varphi$ is an anisotropic $p$-growth function with $i(\varphi) = I(\varphi) = p$, where $p \in (0, 1]$, Lemma 3.9 gives the $q$-atomic characterization of $WH^{\varphi}_A$ with $q \in (q(\varphi), \infty]$ which includes the $\infty$-atomic characterization of $WH^{\varphi}_A$ in [55, Theorem 1].

(iii) When $q \in (1, \infty]$ and $\varphi$ is as in (2.6) with taking $\omega \equiv 1$ and $\Phi(t) := t^p$ for all $t \in [0, \infty)$, where $p \in (0, 1]$, Lemma 3.9 is also new.

LEMMA 3.11. [13, Lemma 2.3] *Let $A$ be a dilation on $\mathbb{R}^n$. Then there exists a collection*

$$\mathcal{Q} := \{Q^k_\alpha \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in \mathrm{I}_k\}$$

*of open subsets, where $\mathrm{I}_k$ is some index set, such that:*

(i) $|\mathbb{R}^n \setminus \cup_\alpha Q^k_\alpha| = 0$ *for each fixed $k$ and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$;*

(ii) *for any $\alpha, \beta, k, \ell$ with $\ell \geqslant k$, either $Q^k_\alpha \cap Q^\ell_\beta = \emptyset$ or $Q^\ell_\alpha \subset Q^k_\beta$;*

(iii) *for each $(\ell, \beta)$ and each $k < \ell$, there exists a unique $\alpha$ such that $Q^\ell_\beta \subset Q^k_\alpha$;*

(iv) *there exist some negative integer $v$ and positive integer $u$ such that, for any $Q^k_\alpha$ with $k \in \mathbb{Z}$ and $\alpha \in \mathrm{I}_k$, there exists $x_{Q^k_\alpha} \in Q^k_\alpha$ satisfying that, for any $x \in Q^k_\alpha$,*

$$x_{Q^k_\alpha} + B_{vk-u} \subset Q^k_\alpha \subset x + B_{vk+u}.$$

In what follows, for convenience, we call $k$ the *level* of the dyadic cube $Q^k_\alpha$ with $k \in \mathbb{Z}$ and $\alpha \in \mathrm{I}_k$ and denote it by $\ell(Q^k_\alpha)$, and call $\mathcal{Q}$ of Lemma 3.11 *dyadic cubes*.

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

For any $\psi \in L^1$ and $\xi \in \mathbb{R}^n$, let $\widehat{\psi}(\xi) := \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i \xi \cdot x}\, dx$.

LEMMA 3.12. [13, Proposition 2.14] *Let $A$ be a dilation on $\mathbb{R}^n$ and $s \in \mathbb{N}$. Then there exist $\theta, \psi \in \mathcal{S}$ such that:*

(i) $\operatorname{supp} \theta \subset B_0 \in \mathcal{B}$, *$\int_{\mathbb{R}^n} \theta(x) x^\gamma\, dx = 0$ for any $\gamma \in \mathbb{N}^n$ with $|\gamma| \leqslant s$, and $\widehat{\theta}(\xi) \geqslant C > 0$ for any $\xi \in \{x \in \mathbb{R}^n : a \leqslant \rho(x) \leqslant b\}$, where $a, b \in (0,1)$ are constants;*

(ii) $\operatorname{supp} \widehat{\psi}$ *is compact and bounded away from the origin;*

(iii) *$\sum_{j\in\mathbb{Z}} \widehat{\psi}((A^*)^j \xi)\widehat{\theta}((A^*)^j \xi) = 1$ for any $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$, where $A^*$ denotes the transpose of $A$.*

*Moreover, for any $f \in \mathcal{S}'_0$, $f = \sum_{j\in\mathbb{Z}} f * \psi_j * \theta_j$ in $\mathcal{S}'$.*

By Lemmas 3.1 and 3.11, we have the following lemma.

LEMMA 3.13. *Let $q \in [1,\infty)$ and $\varphi \in \mathbb{A}_q(A)$. Then, for any $Q \in \mathcal{Q}$ and $t \in (0,\infty)$, it holds true that*

$$\varphi(c_Q + B_{v\ell(Q)-u}, t) \sim \varphi(Q,t) \sim \varphi(x_Q + B_{v\ell(Q)+u}, t),$$

*where the implicit constants are independent of $Q$ and $t$.*

The proof of the following lemma is a slight modification of the proof of [41, Proposition 4.2], the details being omitted.

LEMMA 3.14. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3. If $f \in WH^\varphi_A$, then $f$ vanishes weakly at infinity.*

LEMMA 3.15. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3, $s \in \mathbb{N} \cap [m(\varphi),\infty)$, $q \in (q(\varphi),\infty)$ and $\widetilde{q} \in (q(\varphi), q)$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.3) and (2.8), respectively. For a sequence of multiples of anisotropic $(\varphi, q, s)$-atoms, $\{a_i\}_i$, associated with dilated balls $\{x_i + B_{l_i}\}_i$, where $\{l_i\}_i \subset \mathbb{Z}$, satisfying that there exists some $k \in \mathbb{Z}$ such that, for each $i$, $\|a_i\|_{L^q_\varphi(x_i+B_{l_i})} \lesssim 2^k$ and, for any $x \in \mathbb{R}^n$, $\sum_i \chi_{x_i+B_{l_i}}(x) \lesssim 1$, then there exists a positive constant $C$, independent of $\{a_i\}_i$, such that, for any $t \in (0,\infty)$,*

$$\int_{\mathbb{R}^n} \left[\sum_i S(a_i)(x)\right]^{\widetilde{q}} \varphi(x,t)\, dx \leqslant C \sum_i 2^{k\widetilde{q}} \varphi(x_i + B_{l_i}, t).$$

*Proof.* For any multiple of anisotropic $(\varphi, q, s)$-atom, $a_i$, associated with some dilated ball $x_i + B_{l_i}$. Since $\varphi \in \mathbb{A}_\infty(A)$ and $q \in (q(\varphi),\infty)$, it follows

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

that $\varphi \in \mathbb{A}_q(A)$. Let $U_0(x_i + B_{l_i+2\sigma}) := x_i + B_{l_i+2\sigma}$. By the boundedness on $L^q(\mathbb{R}^n, \varphi(\cdot, t))$, uniformly in $t \in (0, \infty)$, of the anisotropic Lusin-area function $S$ (see [13, Theorem 3.2]), Lemma 3.1 with $\varphi \in \mathbb{A}_q(A)$, and $\|a_i\|_{L^q_\varphi(x_i+B_{l_i})} \lesssim 2^k$, we see that, for any $i$,

$$\begin{aligned}
&\|S(a_i)\|_{L^q_\varphi(U_0(x_i+B_{l_i+2\sigma}))}\\
&\quad\leqslant \sup_{t\in(0,\infty)} \left[\frac{1}{\varphi(U_0(x_i + B_{l_i+2\sigma}), t)} \int_{\mathbb{R}^n} |S(a_i)(x)|^q \varphi(x, t)\, dx\right]^{1/q}\\
&\quad\lesssim \sup_{t\in(0,\infty)} \left[\frac{1}{\varphi(U_0(x_i + B_{l_i+2\sigma}), t)} \int_{\mathbb{R}^n} |a_i(x)|^q \varphi(x, t)\, dx\right]^{1/q}\\
&\quad\lesssim \sup_{t\in(0,\infty)} \left[\frac{1}{\varphi(x_i + B_{l_i}, t)} \int_{x_i+B_{l_i}} |a_i(x)|^q \varphi(x, t)\, dx\right]^{1/q}\\
&\quad\sim \|a_i\|_{L^q_\varphi(x_i+B_{l_i})} \lesssim 2^k. \qquad (3.21)
\end{aligned}$$

Let $U_j(x_i + B_{l_i+2\sigma}) := x_i + (B_{l_i+j+2\sigma} \setminus B_{l_i+j-1+2\sigma})$, where $j \in \mathbb{Z}_+$. By [36, (2.11)], we know that, for any $j \in \mathbb{Z}_+$, $i$ and $x \in U_j(x_i + B_{l_i+2\sigma})$,

$$S(a_i)(x) \lesssim \|a_i\|_{L^q_\varphi(x_i+B_{l_i})}[b(\lambda_-)^{s+1}]^{-j}.$$

From this and $\|a_i\|_{L^q_\varphi(x_i+B_{l_i})} \lesssim 2^k$, we deduce that, for any $j \in \mathbb{Z}_+$ and $i$,

$$\|S(a_i)\|_{L^q_\varphi(U_j(x_i+B_{l_i+2\sigma}))} \lesssim 2^k[b(\lambda_-)^{s+1}]^{-j}. \qquad (3.22)$$

By repeating the proof of [41, pp. 660–661] with [41, (4.1)] and [41, (4.6)] replaced by (3.21) and (3.22), respectively, we know that, for any $t \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \left[\sum_i S(a_i)(x)\right]^{\widetilde{q}} \varphi(x, t)\, dx \leqslant C \sum_i 2^{k\widetilde{q}} \varphi(x_i + B_{l_i}, t).$$

This finishes the proof of Lemma 3.15. ∎

*Proof of Theorem 2.10.* Suppose $(\varphi, q, s)$ is an admissible anisotropic triplet as in Definition 3.7.

**Step 1.** In this step, we show $WH^\varphi_{A,S} \subset WH^\varphi_A$. By Lemma 3.9, it suffices to prove $WH^\varphi_{A,S} \subset WH^{\varphi,q,s}_{A,\mathrm{at}}$.

Assuming that $f \in \mathcal{S}'_0$ and $S(f) \in WL^{\varphi}$, we prove that $f \in WH^{\varphi,q,s}_{A,\mathrm{at}}$ and $\|f\|_{WH^{\varphi,q,s}_{A,\mathrm{at}}} \lesssim \|S(f)\|_{WL^{\varphi}}$. For any $k \in \mathbb{Z}$, let $\Omega_k := \{S(f) > 2^k\}$ and

$$\mathcal{R}_k := \{Q \in \mathcal{Q} : |Q \cap \Omega_k| > |Q|/2, |Q \cap \Omega_{k+1}| \leqslant |Q|/2\}.$$

Then, for each dyadic cube $Q \in \mathcal{Q}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{R}_k$. For any $Q \in \mathcal{Q}$, let

$$\widetilde{Q} := \{(y, m) \in \mathbb{R}^n \times \mathbb{R} : y \in Q, m \sim v\ell(Q) + u\},$$

and, here and hereafter, $m \sim v\ell(Q) + u$ always means

$$v\ell(Q) + u + \sigma \leqslant m < v[\ell(Q) - 1] + u + \sigma,$$

where $\ell(Q)$, $v$ and $u$ are the same as in Lemma 3.11.

Let $\theta$ and $\psi$ be as in Lemma 3.12 and let each $\theta$ be of the vanishing moments up to order $s$ with $s \geqslant m(\varphi)$. We use $\{Q^{\ell}_k\}_{\ell}$ to denote the set of all maximal dyadic cubes in $\mathcal{R}_k$. For any $Q \in \mathcal{R}_k$, by Lemma 3.11(ii), there exists a unique maximal dyadic cube $Q^{\ell}_k$ such that $Q \subset Q^{\ell}_k$.

For any $f \in WH^{\varphi}_{A,S}$, by the Step 1 of the proof of [36, Theorem 2.14], we know that $f = \sum_{k\in\mathbb{Z}} \sum_{\ell} a^{\ell}_k$ in $\mathcal{S}'$, where $a^{\ell}_k := \sum_{Q\subset Q^{\ell}_k, Q\in\mathcal{R}_k} e_Q$, where

$$e_Q(x) := \int_{\widetilde{Q}} f * \psi_{-m}(y)\theta_{-m}(x - y)\, dy\, d\sigma(m)$$

and $\sigma(m)$ is the *counting measure* on $\mathbb{R}$. Notice that

$$Q^{\ell}_k \subset x_{Q^{\ell}_k} + B_{v\ell(Q^{\ell}_k)+u} \subset B^{\ell}_k := x_{Q^{\ell}_k} + B_{v[\ell(Q^{\ell}_k)-1]+u+3\sigma}.$$

Then, from this and $\|a^{\ell}_k\|_{L^q(\mathbb{R}^n,\varphi(\cdot,t))} \lesssim 2^k[\varphi(Q^{\ell}_k, t)]^{1/q}$ (see [36, p. 297]), it follows that, for any $t \in (0, \infty)$,

$$\begin{aligned}&\left\{\frac{1}{\varphi(B^{\ell}_k, t)} \int_{B^{\ell}_k} |a^{\ell}_k(x)|^q \varphi(x, t)\, dx\right\}^{1/q}\\ &\quad\leqslant \left\{\frac{1}{\varphi(Q^{\ell}_k, t)} \int_{\mathbb{R}^n} |a^{\ell}_k(x)|^q \varphi(x, t)\, dx\right\}^{1/q} \lesssim 2^k,\end{aligned}$$

which implies that $\|a^{\ell}_k\|_{L^q_{\varphi}(B^{\ell}_k)} \lesssim 2^k$. By this and the Steps 3–5 of the proof of [36, Theorem 2.14], we know that $a^{\ell}_k$ is a multiple of an anisotropic $(\varphi, q, s)$-atom associated with $B^{\ell}_k$.

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

By Lemmas 3.13 and 3.1 with $\varphi\in\mathbb{A}_q(A)$, $|Q_k^\ell\cap\Omega_k|>|Q_k^\ell|/2$ and the disjointness of $\{Q_k^\ell\}_\ell$, we conclude that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}\sup_{k\in\mathbb{Z}}\left\{\sum_\ell\varphi\left(B_k^\ell,\frac{2^k}{\lambda}\right)\right\}&\lesssim\sup_{k\in\mathbb{Z}}\left\{\sum_\ell\varphi\left(Q_k^\ell,\frac{2^k}{\lambda}\right)\right\}\\&\lesssim\sup_{k\in\mathbb{Z}}\left\{\sum_\ell\left(\frac{|Q_k^\ell|}{|Q_k^\ell\cap\Omega_k|}\right)^q\varphi\left(Q_k^\ell\cap\Omega_k,\frac{2^k}{\lambda}\right)\right\}\\&\lesssim\sup_{k\in\mathbb{Z}}\varphi\left(\Omega_k,\frac{2^k}{\lambda}\right),\end{aligned}$$

which implies that $\|f\|_{WH^{\varphi,q,s}_{A,\mathrm{at}}}\lesssim\|S(f)\|_{WL^\varphi}$. This finishes the proof of Step 1.

**Step 2.** In this step, we show $WH^\varphi_A\subset WH^\varphi_{A,S}$. Suppose $f\in WH^\varphi_A$. By Lemma 3.14, we see that $f\in\mathcal{S}_0'$. It remains to show $\|S(f)\|_{WL^\varphi}\lesssim\|f\|_{WH^\varphi_A}$.

By Lemma 3.9, we know that, for any $f\in WH^\varphi_A=WH^{\varphi,q,s}_{A,\mathrm{at}}$ with $q\in(q(\varphi),\infty)$ and $s\in[m(\varphi),\infty)\cap\mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.3) and (2.8), respectively, there exist a sequence of multiples of anisotropic $(\varphi,q,s)$-atoms, $\{f_i^k\}_{k\in\mathbb{Z},i}$, associated with dilated balls $\{B_i^k\}_{k\in\mathbb{Z},i}$, such that $f=\sum_{k\in\mathbb{Z}}\sum_i f_i^k$ in $\mathcal{S}'$, $\sum_i\chi_{B_i^k}(x)\lesssim1$ for any $x\in\mathbb{R}^n$ and $k\in\mathbb{Z}$, $\|f_i^k\|_{L^q_\varphi(B_i^k)}\lesssim2^k$ for any $k\in\mathbb{Z}$ and $i$, and

$$\|f\|_{WH^{\varphi,q,s}_{A,\mathrm{at}}}\sim\inf\left\{\lambda\in(0,\infty):\sup_{k\in\mathbb{Z}}\left\{\sum_i\varphi\left(B_i^k,\frac{2^k}{\lambda}\right)\right\}\leqslant1\right\}.$$

Thus, it suffices to prove that, for any $\alpha,\lambda\in(0,\infty)$,

$$\varphi\left(\{S(f)>\alpha\},\frac{\alpha}{\lambda}\right)\lesssim\sup_{k\in\mathbb{Z}}\left\{\sum_i\varphi\left(B_i^k,\frac{2^k}{\lambda}\right)\right\}.\tag{3.23}$$

To show (3.23), we may assume that there exists $k_0\in\mathbb{Z}$ such that $\alpha=2^{k_0}$ without loss of generality. Write

$$f=\sum_{k=-\infty}^{k_0-1}\sum_i f_i^k+\sum_{k=k_0}^{\infty}\sum_i f_i^k=:F_1+F_2.$$

For $F_1$, by repeating the estimate of $F_1$ in the proof of [41, Theorem 4.5] with [41, Lemma 4.4] replaced by Lemma 3.15, we know that, for any

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

$\lambda \in (0, \infty)$,

$$\varphi\left(\{S(F_1) > 2^{k_0}\}, \frac{2^{k_0}}{\lambda}\right) \lesssim \sup_{k\in\mathbb{Z}} \left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\}. \tag{3.24}$$

Let us now deal with $F_2$. By repeating the estimate of (3.11) in the proof of Lemma 3.9 with $(F_2)_m^*$ replaced by $S(F_2)$, we know that, for any $\lambda \in (0, \infty)$,

$$\varphi\left(\{S(F_2) > 2^{k_0}\}, \frac{2^{k_0}}{\lambda}\right) \lesssim \sup_{k\in\mathbb{Z}} \left\{\sum_i \varphi\left(B_i^k, \frac{2^k}{\lambda}\right)\right\}. \tag{3.25}$$

Combining (3.24) and (3.25), we finally obtain (3.23), which implies that $\|S(f)\|_{WL^\varphi} \lesssim \|f\|_{WH^{\varphi,q,s}_{A,\mathrm{at}}}$. This finishes the proof of Step 2 and hence Theorem 2.10. ∎

## §4. Proofs of Theorems 2.11 and 2.12

To obtain the anisotropic $g$-function characterization of $WH^\varphi_A$, we begin with recalling some notation and establishing several technical lemmas.

For any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{j,k} := A^{-j}([0,1)^n + k)$ be the dilated cube, $x_{Q_{j,k}} := A^{-j}k$ be its "lower-left" corner, $\mathcal{Q}_j := \{Q_{j,k} : k \in \mathbb{Z}^n\}$ and $\widetilde{\mathcal{Q}} := \bigcup_{j\in\mathbb{Z}} \mathcal{Q}_j$. Obviously, for any $k_1, k_2 \in \mathbb{Z}^n$ with $k_1 \neq k_2$, $Q_{j,k_1} \cap Q_{j,k_2} = \emptyset$.

Definition 4.1. Let $r, \lambda \in (0, \infty)$. For any sequence $\mathbf{s} := \{s_Q\}_{Q\in\widetilde{\mathcal{Q}}} \subset \mathbb{C}$, its *majorant sequence* $s^*_{r,\lambda} := \{(s^*_{r,\lambda})_Q\}_{Q\in\widetilde{\mathcal{Q}}}$ is defined by setting, for all $Q \in \widetilde{\mathcal{Q}}$,

$$(s^*_{r,\lambda})_Q := \left[\sum_{P\in\widetilde{\mathcal{Q}}, |P|=|Q|} \frac{|s_P|^r}{[1 + |Q|^{-1}\rho(x_Q - x_P)]^\lambda}\right]^{1/r}.$$

Recall that the *anisotropic Hardy–Littlewood maximal function* $\mathcal{M}_A(f)$ of any locally integrable function $f$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_A f(x) := \sup_{x\in B, B\in\mathcal{B}} \frac{1}{|B|} \int_B |f(y)|\, dy. \tag{4.1}$$

Lemma 4.2. [9, Lemma 6.2] *Let $j \in \mathbb{Z}$, $a, r \in (0, \infty)$ with $a \leqslant r$ and $\lambda \in (r/a, \infty)$. Then there exists a positive constant $C$ such that, for any sequence*

$\mathbf{s} := \{s_Q\}_{Q\in\widetilde{\mathcal{Q}}} \subset \mathbb{C}$,

$$\sum_{|Q|=b^{-j}} (s^*_{r,\lambda})_Q \chi_Q \leqslant C \left[\mathcal{M}_A\left(\sum_{|Q|=b^{-j}} |s_Q|^a \chi_Q\right)\right]^{1/a}.$$

The following lemma is an anisotropic version of the weak Musielak–Orlicz Fefferman–Stein vector-valued inequality, whose proof is also an obvious modification of the proof of [41, Theorem 2.8], the details being omitted.

LEMMA 4.3. *Let $r \in (1,\infty]$, $\varphi$ be a Musielak–Orlicz function of uniformly lower type $p^-_\varphi$ and of uniformly upper type $p^+_\varphi$ and let $q(\varphi)$ be as in (2.3). If $q(\varphi) < p^-_\varphi \leqslant p^+_\varphi < \infty$, then there exists a positive constant $C$ such that, for any $\{f_j\}_{j\in\mathbb{Z}} \in WL^\varphi(\ell^r,\mathbb{R}^n)$,*

$$\sup_{t\in(0,\infty)} \varphi\left(\left\{\left(\sum_{j\in\mathbb{Z}}[\mathcal{M}_A(f_j)]^r\right)^{1/r} > t\right\}, t\right)$$
$$\leqslant C \sup_{t\in(0,\infty)} \varphi\left(\left\{\left(\sum_{j\in\mathbb{Z}}|f_j|^r\right)^{1/r} > t\right\}, t\right).$$

LEMMA 4.4. [10, p. 423] *For any $f \in \mathcal{S}'$ and $\Phi \in \mathcal{S}$ satisfying* supp $\widehat{\Phi}$ *is compact and bounded away from the origin, the sequences* $\sup(f) := \{\sup(f)_Q\}_{Q\in\widetilde{\mathcal{Q}}}$ *and* $\inf(f) := \{\inf(f)_Q\}_{Q\in\widetilde{\mathcal{Q}}}$ *are defined by setting, respectively, for any $Q \in \widetilde{\mathcal{Q}}$ with $|Q| = b^{-j}$,*

$$\sup(f)_Q := \sup_{y\in Q} |f * \widetilde{\Phi}_j(y)|$$

*and*

$$\inf(f)_Q := \sup\left\{\inf_{y\in P} |f * \widetilde{\Phi}_j(y)| : |P\cap Q| \neq 0, |Q|/|P| = b^\gamma\right\},$$

*where $\widetilde{\Phi}(\cdot) := \overline{\Phi(-\cdot)}$ and $\gamma \in \mathbb{Z}_+$. Then, for any $\lambda, r \in (0,\infty)$ and sufficient large $\gamma \in \mathbb{Z}_+$, there exists a positive constant $C$ such that, for any $Q \in \widetilde{\mathcal{Q}}$,*

$$(\sup(f)^*_{r,\lambda})_Q \leqslant C(\inf(f)^*_{r,\lambda})_Q.$$

LEMMA 4.5. *Let $\varphi$ be an anisotropic growth function as in Definition 2.3. For any $r \in (0,\infty)$ and $\lambda \in (\max\{1, r/2, rq(\varphi)/i(\varphi)\},\infty)$, where $q(\varphi)$ and $i(\varphi)$ are as in (2.3) and (2.4), respectively, then there exists a*

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

*positive constant $C$ such that, for any* $\mathbf{s} := \{s_Q\}_{Q\in\widetilde{\mathcal{Q}}}$,

$$\sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{Q\in\widetilde{\mathcal{Q}}} [(s^*_{r,\lambda})_Q]^2\chi_Q\right]^{1/2} > t\right\}, t\right) \leqslant C \sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{Q\in\widetilde{\mathcal{Q}}} |s_Q|^2\chi_Q\right]^{1/2} > t\right\}, t\right).$$

*Proof.* We show this lemma by borrowing some ideas from the proof of [36, Lemma 3.7]. Let $r\in(0,\infty)$ and $\lambda\in(\max\{1, r/2, rq(\varphi)/i(\varphi)\},\infty)$. If $r<\min\{2, i(\varphi)/q(\varphi)\}$, we choose $a := r$. Otherwise, we choose $a$ such that $r/\lambda < a < \min\{r, 2, i(\varphi)/q(\varphi)\}$. It is possible to choose such an $a$, since $\lambda > \max\{1, r/2, rq(\varphi)/i(\varphi)\}$ implies $r/\lambda < \min\{r, 2, i(\varphi)/q(\varphi)\}$. In both cases, we find that

$$0 < a \leqslant r < \infty, \qquad \lambda > \frac{r}{a}, \qquad \frac{2}{a} > 1 \qquad \text{and} \qquad \frac{i(\varphi)}{a} > q(\varphi).$$

For the above last inequality, by choosing $p\in(0, i(\varphi))$ close to $i(\varphi)$, we further obtain $p/a > q(\varphi)$. Next, let $\widetilde{\varphi}(x,t) := \varphi(x, t^{1/a})$ for all $x\in\mathbb{R}^n$ and $t\in(0,\infty)$. From the fact that $\varphi$ is of uniformly lower type $p$ and of uniformly upper type 1, it follows that $\widetilde{\varphi}$ is of uniformly lower type $p/a$ and of uniformly upper type $1/a$. Therefore, since $1/a > p/a > q(\varphi)$, Lemmas 4.2 and 4.3 yield

$$\begin{aligned}
&\sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{Q\in\widetilde{\mathcal{Q}}} [(s^*_{r,\lambda})_Q]^2\chi_Q\right]^{1/2} > t\right\}, t\right)\\
&\sim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{j\in\mathbb{Z}} \left(\sum_{|Q|=b^{-j}} (s^*_{r,\lambda})_Q\chi_Q\right)^2\right]^{1/2} > t\right\}, t\right)\\
&\lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left(\sum_{j\in\mathbb{Z}} \left[\mathcal{M}_A\left(\sum_{|Q|=b^{-j}} |s_Q|^a\chi_Q\right)\right]^{2/a}\right)^{1/2} > t\right\}, t\right)\\
&\sim \sup_{t\in(0,\infty)} \widetilde{\varphi}\left(\left\{\left(\sum_{j\in\mathbb{Z}} \left[\mathcal{M}_A\left(\sum_{|Q|=b^{-j}} |s_Q|^a\chi_Q\right)\right]^{2/a}\right)^{a/2} > t\right\}, t\right)
\end{aligned}$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

$$
\begin{aligned}
&\lesssim \sup_{t\in(0,\infty)} \widetilde{\varphi}\left(\left\{\left[\sum_{j\in\mathbb{Z}}\left(\sum_{|Q|=b^{-j}}|s_Q|^a\chi_Q\right)^{2/a}\right]^{a/2} > t\right\}, t\right)\\
&\sim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left(\sum_{j\in\mathbb{Z}}\sum_{|Q|=b^{-j}}|s_Q|^2\chi_Q\right)^{1/2} > t\right\}, t\right)\\
&\sim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left(\sum_{Q\in\widetilde{\mathcal{Q}}}|s_Q|^2\chi_Q\right)^{1/2} > t\right\}, t\right).
\end{aligned}
$$

This finishes the proof of Lemma 4.5. □

*Proof of Theorem 2.11.* Suppose $(\varphi, q, s)$ is an admissible anisotropic triplet as in Definition 3.7. By repeating the proof of Theorem 2.10 with a slight modification, we easily obtain $WH^{\varphi}_{A} \subset WH^{\varphi}_{A,g}$ with continuous inclusion, the details beings omitted. Conversely, to prove $WH^{\varphi}_{A,g} \subset WH^{\varphi}_{A}$, by $\|\cdot\|_{WH^{\varphi}_{A}} \sim \|\cdot\|_{WH^{\varphi}_{A,S}}$, it is sufficient to prove that, for any $f \in WH^{\varphi}_{A,g}$, $\|S(f)\|_{WL^{\varphi}} \lesssim \|g(f)\|_{WL^{\varphi}}$.

Since the proof of $WH^{\varphi}_{A,g} \subset WH^{\varphi}_{A}$ is similar to that of [36, Theorem 3.1], we use the same notation as in the proof of [36, Theorem 3.1] and here we just give out the necessary modifications.

Let $q(\varphi)$ be as in (2.3). Choose $M \in \mathbb{Z}_+$ large enough and $r \in (0, 1]$ such that $r \in (1/M, p/q(\varphi))$. Let $\widetilde{\varphi}(x, t) := \varphi(x, t^{1/r})$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. From the fact that $\varphi$ is of uniformly lower type $p$ and of uniformly upper type 1, it follows that $\widetilde{\varphi}$ is of uniformly lower type $p/r$ and of uniformly upper type $1/r$. Then, since $1/r > p/r > q(\varphi)$, Lemma 4.3 gives

$$
\begin{aligned}
&\sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{j\in\mathbb{Z}}\{\mathcal{M}_A(|f_j|^r)\}^{2/r}\right]^{1/2} > t\right\}, t\right)\\
&\quad\lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{j\in\mathbb{Z}}|f_j|^2\right]^{1/2} > t\right\}, t\right),
\end{aligned}
$$

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

which, together with

$$S(f)(x) \lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}_A\left(\left[\sum_{Q\in\mathcal{Q}_j}|f*\widetilde{\Phi}_j(x_Q)|\chi_Q\right]^r\right)(x)\right]^{2/r}\right\}^{1/2}$$

(see [36, (3.7)]),

further implies that

$$\begin{aligned}
&\sup_{t\in(0,\infty)}\varphi(\{S(f)>t\},t)\\
&\lesssim \sup_{t\in(0,\infty)}\varphi\left(\left\{\left(\sum_{j\in\mathbb{Z}}\left[\mathcal{M}_A\left(\left[\sum_{Q\in\mathcal{Q}_j}|f*\widetilde{\Phi}_j(x_Q)|\chi_Q\right]^r\right)\right]^{2/r}\right)^{1/2}>t\right\},t\right)\\
&\lesssim \sup_{t\in(0,\infty)}\varphi\left(\left\{\left(\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_j}[|f*\widetilde{\Phi}_j(x_Q)|\chi_Q]^2\right)^{1/2}>t\right\},t\right).
\end{aligned}\tag{4.2}$$

Notice that $s_Q\leqslant(s^*_{r,\lambda})_Q$ for any $r,\lambda\in(0,\infty)$ and $Q\in\widetilde{\mathcal{Q}}$. Then, from this, (4.2), Lemmas 4.4 and 4.5 with $r\in(0,\infty)$ and $\lambda\in(\max\{1,r/2,rq(\varphi)/i(\varphi)\},\infty)$, it follows that, for some $\gamma\in\mathbb{Z}_+$ large enough,

$$\begin{aligned}
&\sup_{t\in(0,\infty)}\varphi(\{S(f)>t\},\ t)\\
&\quad\lesssim \sup_{t\in(0,\infty)}\varphi\left(\left\{\left[\sum_{Q\in\widetilde{\mathcal{Q}}}|(\sup(f)^*_{r,\lambda})_Q|^2\chi_Q\right]^{1/2}>t\right\},t\right)\\
&\quad\lesssim \sup_{t\in(0,\infty)}\varphi\left(\left\{\left[\sum_{Q\in\widetilde{\mathcal{Q}}}|(\inf(f)^*_{r,\lambda})_Q|^2\chi_Q\right]^{1/2}>t\right\},t\right)\\
&\quad\lesssim \sup_{t\in(0,\infty)}\varphi\left(\left\{\left[\sum_{Q\in\widetilde{\mathcal{Q}}}|\inf(f)_Q|^2\chi_Q\right]^{1/2}>t\right\},t\right).
\end{aligned}\tag{4.3}$$

Moreover, for any $P\in\widetilde{\mathcal{Q}}$ with $|P|=b^{-i}$ and $s_P:=\inf_{y\in P}|f*\widetilde{\Phi}_{i-\gamma}(y)|$, by checking the proof of [10, Lemma 8.4], we find that $\inf(f)_Q=\sup\{s_P:$

$|Q|/|P| = b^{\gamma}$, $P \in \widetilde{\mathcal{Q}}\}$ and, for any $x \in \mathbb{R}^n$,

$$\sum_{|Q|=b^{-j}} \inf(f)_Q \chi_Q(x) \lesssim b^{\gamma\lambda/r} \sum_{|P|=b^{-j-\gamma}} (s^*_{r,\lambda})_P \chi_P(x).$$

Combining this, (4.3) and Lemma 4.5, we find that

$$\begin{aligned}
&\sup_{t\in(0,\infty)} \varphi(\{S(f) > t\}, t) \\
&\quad \lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{j\in\mathbb{Z}} \sum_{|P|=b^{-j-\gamma}} |(s^*_{r,\lambda})_P|^2 \chi_P\right]^{1/2} > t\right\}, t\right) \\
&\quad \lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{i\in\mathbb{Z}} \sum_{|P|=b^{-i}} |s_P|^2 \chi_P\right]^{1/2} > t\right\}, t\right) \\
&\quad \lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{\left[\sum_{i\in\mathbb{Z}} \sum_{|Q|=b^{-i}} |f * \widetilde{\Phi}_i|^2 \chi_Q\right]^{1/2} > t\right\}, t\right) \\
&\quad \lesssim \sup_{t\in(0,\infty)} \varphi(\{g(f) > t\}, t),
\end{aligned}$$

which implies that $\|S(f)\|_{WL^{\varphi}} \lesssim \|g(f)\|_{WL^{\varphi}}$. This finishes the proof of Theorem 2.11. □

Next, we consider the anisotropic $g^*_{\lambda}$-function characterization of $WH^{\varphi}_A$. To this end, we need to introduce the following variant of the anisotropic Lusin-area function $S$. Let $\psi \in \mathcal{S}$ such that, for any $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leqslant m(\varphi)$, $\int_{\mathbb{R}^n} \psi(x) x^{\alpha}\, dx = 0$, where $m(\varphi)$ is as in (2.8). For any $k_0 \in \mathbb{N}$, $f \in \mathcal{S}'$ and $x \in \mathbb{R}^n$, let

$$S_{k_0}(f)(x) := \left\{\sum_{k\in\mathbb{Z}} b^{-(k_0+k)} \int_{x+B_{k_0+k}} |f * \psi_{-k}(y)|^2\, dy\right\}^{1/2}.$$

The following technical lemma plays a key role in establishing the anisotropic $g^*_{\lambda}$-function characterization of $WH^{\varphi}_A$, whose proof was motivated by Folland and Stein [20, p. 218, Theorem 7.1], Aguilera and Segovia [1, Theorem 1] and Liang *et al.* [41, Lemma 4.11].

https://doi.org/10.1017/nmj.2018.10 Published online by Cambridge University Press

LEMMA 4.6. *Let $q \in [1, \infty)$, $\varphi$ be an anisotropic growth function as in Definition 2.3 and $\varphi \in \mathbb{A}_q(A)$. Then there exists a positive constant $C$ such that, for any $k_0 \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$,*

$$\sup_{\lambda \in (0,\infty)} \varphi(\{S_{k_0}(f) > \lambda\}, \lambda) \leqslant C b^{(q-p/2)k_0} \sup_{\lambda \in (0,\infty)} \varphi(\{S(f) > \lambda\}, \lambda).$$

*Proof.* For any $\lambda \in (0, \infty)$, $k_0 \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$, let

$$E_{\lambda,k_0} := \{S(f) > \lambda b^{k_0/2}\}$$

and

$$U_{\lambda,k_0} := \{\mathcal{M}_A(\chi_{E_{\lambda,k_0}}) > b^{-k_0-2\sigma}\},$$

where $\mathcal{M}_A$ is as in (4.1). Since $\varphi \in \mathbb{A}_q(A)$, from the boundedness on $L^q(\mathbb{R}^n, \varphi(\cdot, \lambda))$, uniformly in $\lambda \in (0, \infty)$, of $\mathcal{M}_A$ (see, for example, [11, Theorem 2.4]), it follows that, for any $\lambda \in (0, \infty)$, $k_0 \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$,

$$\varphi(U_{\lambda,k_0}, \lambda) \lesssim b^{qk_0} \|\chi_{E_{\lambda,k_0}}\|^q_{L^q(\mathbb{R}^n,\varphi(\cdot,\lambda))} \sim b^{qk_0} \varphi(E_{\lambda,k_0}, \lambda) \tag{4.4}$$

and, by [36, Lemma 3.12], we know that, for any $\lambda \in (0, \infty)$, $k_0 \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$,

$$\begin{aligned} &b^{k_0(1-q)} \int_{(U_{\lambda,k_0})^\complement} [S_{k_0}(f)(x)]^2 \varphi(x, \lambda)\, dx \\ &\quad \lesssim \int_{(E_{\lambda,k_0})^\complement} [S(f)(x)]^2 \varphi(x, \lambda)\, dx. \end{aligned} \tag{4.5}$$

Thus, from $q \in [1, \infty)$, the uniformly lower type $p$ and the uniformly upper type 1 properties of $\varphi$, (4.4) and (4.5), it follows that, for any $\lambda \in (0, \infty)$, $k_0 \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$,

$$\begin{aligned} &\varphi(\{S_{k_0}(f) > \lambda\}, \lambda) \\ &\quad \leqslant \varphi(U_{\lambda,k_0}, \lambda) + \varphi((U_{\lambda,k_0})^\complement \cap \{S_{k_0}(f) > \lambda\}, \lambda) \\ &\quad \lesssim b^{qk_0} \varphi(E_{\lambda,k_0}, \lambda) + \lambda^{-2} \int_{(U_{\lambda,k_0})^\complement} [S_{k_0}(f)(x)]^2 \varphi(x, \lambda)\, dx \\ &\quad \lesssim b^{qk_0} \varphi(\{S(f) > \lambda b^{k_0/2}\}, \lambda) \\ &\qquad + b^{(q-1)k_0} \lambda^{-2} \int_{(E_{\lambda,k_0})^\complement} [S(f)(x)]^2 \varphi(x, \lambda)\, dx \end{aligned}$$

$$\begin{aligned}
&\lesssim b^{(q-p/2)k_0}\varphi(\{S(f)>\lambda b^{k_0/2}\},\lambda b^{k_0/2})\\
&\quad + b^{(q-1)k_0}\lambda^{-2}\int_0^{\lambda b^{k_0/2}} t\varphi(\{S(f)>t\},\lambda)\,dt\\
&\lesssim \left\{b^{(q-p/2)k_0}+b^{(q-1)k_0}\lambda^{-2}\left[\int_0^{\lambda}\lambda\,dt+\int_{\lambda}^{\lambda b^{k_0/2}} t\left(\frac{\lambda}{t}\right)^p dt\right]\right\}\\
&\quad\times \sup_{t\in(0,\infty)}\varphi(\{S(f)>t\},t)\\
&\lesssim b^{(q-p/2)k_0}\sup_{t\in(0,\infty)}\varphi(\{S(f)>t\},t).
\end{aligned}$$

This finishes the proof of Lemma 4.6. □

*Proof of Theorem 2.12.* The proof of Theorem 2.12 is similar to that of [41, Proposition 4.12]. To prove Theorem 2.12, by $\|\cdot\|_{WH^{\varphi}_A}\sim\|\cdot\|_{WH^{\varphi}_{A,S}}$, it suffices to prove $\|\cdot\|_{WH^{\varphi}_{A,S}}\sim\|\cdot\|_{WH^{\varphi}_{A,g^*_\lambda}}$. For any $f\in\mathcal{S}'$ and $x\in\mathbb{R}^n$, by $S(f)(x)\leqslant g^*_\lambda(f)(x)$, the inequality $\|S(f)\|_{WL^\varphi}\leqslant\|g^*_\lambda(f)\|_{WL^\varphi}$ is obvious. It remains to show that, for any $f\in\mathcal{S}'$, $\|g^*_\lambda(f)\|_{WL^\varphi}\lesssim\|S(f)\|_{WL^\varphi}$.

For any $f\in\mathcal{S}'$ and $x\in\mathbb{R}^n$, we have

$$\begin{aligned}
[g^*_\lambda(f)(x)]^2 &= \sum_{k\in\mathbb{Z}} b^{-k}\int_{x+B_k}|f*\varphi_{-k}(y)|^2\left[\frac{b^k}{b^k+\rho(x-y)}\right]^{\lambda}dy\\
&\quad+\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} b^{-k}\int_{x+(B_{k+j}\setminus B_{k+j-1})}\cdots\\
&\lesssim \sum_{j=0}^{\infty} b^{-j(\lambda-1)}[S_j(f)(x)]^2. \qquad (4.6)
\end{aligned}$$

Since $\lambda\in(2q/p,\infty)$, it follows that there exists $\varepsilon\in(0,1)$ such that $\lambda-2\varepsilon\in(2q/p,\infty)$. Let $C_{(\varepsilon)}:=1/(1-b^{-\varepsilon})$. Then, from (4.6), the uniformly lower type $p$ and the uniformly upper type 1 properties of $\varphi$, Lemma 4.6, and $\lambda-2\varepsilon\in(2q/p,\infty)$, we deduce that, for any $\alpha\in(0,\infty)$,

$$\begin{aligned}
\varphi(\{g^*_\lambda(f)>\alpha\},\alpha) &\lesssim \varphi\left(\left\{\sum_{j=0}^{\infty} b^{-j(\lambda-1)/2}S_j(f)>\frac{1}{C_{(\varepsilon)}}\sum_{j=0}^{\infty} b^{-j\varepsilon}\alpha\right\},\alpha\right)\\
&\lesssim \sum_{j=0}^{\infty}\varphi\left(\left\{S_j(f)>\frac{1}{C_{(\varepsilon)}}b^{j(\lambda-1)/2-j\varepsilon}\alpha\right\},\alpha\right)
\end{aligned}$$

$$\begin{aligned}
&\sim \sum_{j=0}^{\infty} \varphi(\{S_j(f) > \beta\}, C_{(\varepsilon)} b^{j\varepsilon - j(\lambda-1)/2}\beta) \\
&\lesssim \sum_{j=0}^{\infty} b^{jp(\varepsilon-\lambda/2+1/2)} \varphi(\{S_j(f) > \beta\}, \beta) \\
&\lesssim \sum_{j=0}^{\infty} b^{jp(\varepsilon-\lambda/2+1/2)} b^{j(q-p/2)} \sup_{\beta\in(0,\infty)} \varphi(\{S(f)(x) > \beta\}, \beta) \\
&\lesssim \sup_{\beta\in(0,\infty)} \varphi(\{S(f)(x) > \beta\}, \beta),
\end{aligned}$$

which implies that $\|g_\lambda^*(f)\|_{WL^\varphi} \lesssim \|S(f)\|_{WL^\varphi}$. This finishes the proof of (2.10) and hence Theorem 2.12. □

**Acknowledgments.** The authors would like to thank the referees for their very carefully reading and so many useful remarks which did make this article more readable.

## References


[1] N. Aguilera and C. Segovia, *Weighted norm inequalities relating the $g_\lambda^*$ and the area functions*, Studia Math. **61** (1977), 293–303.

[2] J. Álvarez and M. Milman, *$H^p$ continuity properties of Calderón–Zygmund-type operators*, J. Math. Anal. Appl. **118** (1986), 63–79.

[3] K. Astala, T. Iwaniec, P. Koskela and G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. **317** (2000), 703–726.

[4] Z. Birnbaum and W. Orlicz, *Über die verallgemeinerung des begriffes der zueinander konjugierten potenzen*, Studia Math. **3** (1931), 1–67.

[5] A. Bonami, J. Feuto and S. Grellier, *Endpoint for the DIV-CURL lemma in Hardy spaces*, Publ. Mat. **54** (2010), 341–358.

[6] A. Bonami, S. Grellier and L. D. Ky, *Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets*, J. Math. Pures Appl. (9) **97** (2012), 230–241.

[7] A. Bonami, T. Iwaniec, P. Jones and M. Zinsmeister, *On the product of functions in BMO and $H^1$*, Ann. Inst. Fourier (Grenoble) **57** (2007), 1405–1439.

[8] M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. **164** (2003), vi + 122 pp.

[9] M. Bownik, *Boundedness of operators on Hardy spaces via atomic decompositions*, Proc. Amer. Math. Soc. **133** (2005), 3535–3542.

[10] M. Bownik, *Anisotropic Triebel–Lizorkin spaces with doubling measures*, J. Geom. Anal. **17** (2007), 387–424.

[11] M. Bownik and K.-P. Ho, *Atomic and molecular decompositions of anisotropic Triebel–Lizorkin spaces*, Trans. Amer. Math. Soc. **358** (2006), 1469–1510.

[12] M. Bownik, B. Li, D. Yang and Y. Zhou, *Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators*, Indiana Univ. Math. J. **57** (2008), 3065–3100.

[13] M. Bownik, B. Li, D. Yang and Y. Zhou, *Weighted anisotropic product Hardy spaces and boundedness of sublinear operators*, Math. Nachr. **283** (2010), 392–442.
[14] T. A. Bui, J. Cao, L. D. Ky, D. Yang and S. Yang, *Musielak–Orlicz Hardy spaces associated with operators satisfying reinforced off-diagonal estimates*, Anal. Geom. Metr. Spaces **1** (2013), 69–129.
[15] D.-C. Chang, D. Yang and S. Yang, "*Real-variable theory of Orlicz-type function spaces associated with operators — a survey*", in *Some Topics in Harmonic Analysis and Applications*, Adv. Lect. Math. **34**, Int. Press, Somerville, MA, 2016, 27–70.
[16] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogènes*, Lect. Notes Math. **242**, Springer, Berlin, New York, 1971, v + 160 pp.
[17] L. Diening, *Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces*, Bull. Sci. Math. **129** (2005), 657–700.
[18] Y. Ding and S. Lan, *Anisotropic Hardy space estimates for multilinear operators*, Adv. Math. (China) **38** (2009), 168–184.
[19] R. Fefferman and F. Soria, *The space weak $H^1$*, Studia Math. **85** (1986), 1–16.
[20] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes **28**, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982, xii + 285 pp.
[21] M. Frazier, B. Jawerth and G. Weiss, *Littlewood–Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics **79**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, American Mathematical Society, Providence, RI, 1991, viii + 132 pp.
[22] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies **116**, North-Holland, Amsterdam, 1985, x + 604 pp.
[23] L. Grafakos, *Modern Fourier Analysis*, 2nd ed., Graduate Texts in Mathematics **250**, Springer, New York, 2009, xvi + 504 pp.
[24] S. E. Graversen, G. Peškir and M. Weber, *The continuity principle in exponential type Orlicz spaces*, Nagoya Math. J. **137** (1995), 55–75.
[25] S. Hou, D. Yang and S. Yang, *Lusin area function and molecular characterizations of Musielak–Orlicz Hardy spaces and their applications*, Commun. Contemp. Math. **15** (2013), 423–434.
[26] T. Iwaniec and J. Onninen, *$H^1$-estimates of Jacobians by subdeterminants*, Math. Ann. **324** (2002), 341–358.
[27] S. Janson, *Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation*, Duke Math. J. **47** (1980), 959–982.
[28] R. Jiang and D. Yang, *New Orlicz–Hardy spaces associated with divergence form elliptic operators*, J. Funct. Anal. **258** (2010), 1167–1224.
[29] R. Johnson and C. J. Neugebauer, *Homeomorphisms preserving $A_p$*, Rev. Mat. Iberoam. **3** (1987), 249–273.
[30] N. J. Kalton, *Linear operators on $L_p$ for $0 < p < 1$*, Trans. Amer. Math. Soc. **259** (1980), 319–355.
[31] T. Kilpeläinen, P. Koskela and H. Masaoka, *Harmonic Hardy–Orlicz spaces*, Ann. Acad. Sci. Fenn. Math. **38** (2013), 309–325.
[32] L. D. Ky, *Bilinear decompositions and commutators of singular integral operators*, Trans. Amer. Math. Soc. **365** (2013), 2931–2958.
[33] L. D. Ky, *New Hardy spaces of Musielak–Orlicz type and boundedness of sublinear operators*, Integral Equations Operator Theory **78** (2014), 115–150.

[34] L. D. Ky, *Bilinear decompositions for the product space* $H^1_L \times BMO_L$, Math. Nachr. **287** (2014), 1288–1297.

[35] A. K. Lerner, *Sharp weighted norm inequalities for Littlewood–Paley operators and singular integrals*, Adv. Math. **226** (2011), 3912–3926.

[36] B. Li, X. Fan and D. Yang, *Littlewood–Paley characterizations of anisotropic Hardy spaces of Musielak–Orlicz type*, Taiwanese J. Math. **19** (2015), 279–314.

[37] B. Li, D. Yang and W. Yuan, *Anisotropic Hardy spaces of Musielak–Orlicz type with applications to boundedness of sublinear operators*, The Scientific World Journal **2014** (2014), 19 pp.

[38] Y. Liang, J. Huang and D. Yang, *New real-variable characterizations of Musielak–Orlicz Hardy spaces*, J. Math. Anal. Appl. **395** (2012), 413–428.

[39] Y. Liang and D. Yang, *Musielak–Orlicz Campanato spaces and applications*, J. Math. Anal. Appl. **406** (2013), 307–322.

[40] Y. Liang and D. Yang, *Intrinsic Littlewood–Paley function characterizations of Musielak–Orlicz Hardy spaces*, Trans. Amer. Math. Soc. **367** (2015), 3225–3256.

[41] Y. Liang, D. Yang and R. Jiang, *Weak Musielak–Orlicz Hardy spaces and applications*, Math Nachr. **289** (2016), 637–677.

[42] H. Liu, "*The weak* $H^p$ *spaces on homogenous groups*", in *Harmonic Analysis (Tianjin, 1988)*, Lecture Notes in Mathematics **1984**, Springer, Berlin, 1991, 113–118.

[43] S. Martínez and N. Wolanski, *A minimum problem with free boundary in Orlicz spaces*, Adv. Math. **218** (2008), 1914–1971.

[44] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics **1034**, Springer, Berlin, 1983, iii + 222 pp.

[45] E. Nakai and Y. Sawano, *Orlicz–Hardy spaces and their duals*, Sci. China Math. **57** (2014), 903–962.

[46] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Int. Acad. Polon. Ser. A **8** (1932), 207–220.

[47] C. Qi, H. Zhang and B. Li, *Boundedness of Littlewood–Paley functions on anisotropic weak Hardy spaces of Musielak–Orlicz type*, J. Xinjiang Univ. Natur. Sci. **33** (2016), 287–295.

[48] T. Quek and D. Yang, *Calderón–Zygmund-type operators on weighted weak Hardy spaces over* $\mathbb{R}^n$, Acta Math. Sin. (Engl. Ser.) **16** (2000), 141–160.

[49] E. M. Stein, M. H. Taibleson and G. Weiss, *Weak type estimates for maximal operators on certain* $H^p$ *classes*, Rend. Circ. Mat. Palermo **1** (1981), 81–97.

[50] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. **28** (1979), 511–544.

[51] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics **1381**, Springer, Berlin, 1989, vi + 193 pp.

[52] T. D. Tran, *Musielak–Orlicz Hardy spaces associated with divergence form elliptic operators without weight assumptions*, Nagoya Math. J. **216** (2014), 71–110.

[53] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics **78**, Birkhäuser, Basel, 1983, 284 pp.

[54] D. Yang, Y. Liang and L. D. Ky, *Real-Variable Theory of Musielak–Orlicz Hardy Spaces*, Lecture Notes in Mathematics **2182**, Springer, Cham, 2017, xiii + 466 pp.

[55] H. Zhang, C. Qi and B. Li, *Anisotropic weak Hardy spaces of Musielak–Orlicz type and their applications*, Front. Math. China **12** (2017), 993–1022.

Bo Li
*College of Mathematics and System Science*
*Xinjiang University*
*Urumqi 830046*
*PR China*
bli.math@outlook.com

Ruirui Sun
*College of Mathematics and System Science*
*Xinjiang University*
*Urumqi 830046*
*PR China*
1565343765@qq.com

Minfeng Liao
*College of Mathematics and System Science*
*Xinjiang University*
*Urumqi 830046*
*PR China*
2434042446@qq.com

Baode Li
*College of Mathematics and System Science*
*Xinjiang University*
*Urumqi 830046*
*PR China*
1246530557@qq.com