J. Appl. Probab. **61,** 1485–1501 (2024)
*doi:*10.1017/jpr.2024.30

BIVARIATE TEMPERED SPACE-FRACTIONAL POISSON PROCESS AND SHOCK MODELS

RITIK SONI,* ** AND
ASHOK KUMAR PATHAK,* *** *Central University of Punjab, Bathinda*
ANTONIO DI CRESCENZO,**** ***** AND
ALESSANDRA MEOLI,**** ****** *Università degli Studi di Salerno, Fisciano, Italy*

Abstract

We introduce a bivariate tempered space-fractional Poisson process (BTSFPP) by time-changing the bivariate Poisson process with an independent tempered α-stable subordinator. We study its distributional properties and its connection to differential equations. The Lévy measure for the BTSFPP is also derived. A bivariate competing risks and shock model based on the BTSFPP for predicting the failure times of items that undergo two random shocks is also explored. The system is supposed to break when the sum of two types of shock reaches a certain random threshold. Various results related to reliability, such as reliability function, hazard rates, failure density, and the probability that failure occurs due to a certain type of shock, are studied. We show that for a general Lévy subordinator, the failure time of the system is exponentially distributed with mean depending on the Laplace exponent of the Lévy subordinator when the threshold has a geometric distribution. Some special cases and several typical examples are also demonstrated.

Keywords: Tempered space-fractional Poisson process; Lévy subordinator; shock model; failure distribution; reliability

2020 Mathematics Subject Classification: Primary 60G22; 60G51; 60G55; 60E05
Secondary 60K10; 62N05

1. Introduction

The Poisson process is one of the most widely used counting processes with nice mathematical properties and applications in diverse disciplines of applied sciences, namely insurance, economics, biology, queuing theory, reliability, and statistical physics. In recent years, the construction and generalization of the counting processes via subordination techniques have received a considerable amount of interest from theoretical and application viewpoints [32, 34]. [26] introduced a space-fractional version of the Poisson process by subordinating the

Received 18 August 2023; accepted 21 February 2024.
* Postal address: Department of Mathematics and Statistics, Central University of Punjab, Bathinda, Punjab 151401, India.
** Email address: ritiksoni2012@gmail.com
*** Email address: ashokiitb09@gmail.com (corresponding author)
**** Postal address: Dipartimento di Matematica, Università degli Studi di Salerno, I-84084 Fisciano (SA), Italy.
***** Email address: adicrescenzo@unisa.it
****** Email address: ameoli@unisa.it

homogeneous Poisson process (HPP) with an independent α-stable subordinator; [25] studied the Poisson process by considering an inverse stable subordinator and established its connection with the fractional Poisson process; [27] proposed a unified approach by time-changing the HPP with an independent general Lévy subordinator. For more recent developments in this direction, see [11, 19, 20, 24, 31] and references therein.

Apart from univariate counting processes, researchers have explored multivariate versions of the counting process in recent years for effectively analyzing complex real-world phenomena arising in daily life. However, we remark that the literature on multivariate fractional Poisson processes is quite limited, since this is a recent topic of interest. A multivariate fractional Poisson counting process was defined in [3] by considering a common random time-change of a finite-dimensional independent Poisson process. Along the same lines, [4] obtained asymptotic results for a different multivariate version of the fractional Poisson process. Moreover, among other topics, [5] studied the time-change of a multidimensional space-fractional Poisson process by a common independent gamma subordinator. A multi-parameter fractional Poisson process was considered in [23] using inverse subordinators and Mittag–Leffler functions, and its main characteristics were studied.

In this paper, we introduce a bivariate tempered space-fractional Poisson process (BTSFPP) by time-changing the bivariate Poisson process with an independent tempered α-stable subordinator (TSS) and study its important characteristics. It should be stressed that the BTSFPP under investigation is a natural multivariate extension of the Poisson process with a relativistic stable subordinator studied in [27]. In particular, we derive its Lévy measure and the governing differential equations of the probability mass function (PMF) and probability-generating function (PGF). As an application in a research area of interest in survival analysis and reliability theory, we also propose a shock model for predicting the failure time of items subject to two external random shocks in a counting pattern governed by the BTSFPP. The system is supposed to break when two types of shock reach their random thresholds. The results related to reliability, such as reliability function, hazard rates, failure density, and the probability that the failure occurs due to a certain type of shock, are studied. Several typical examples based on different random threshold distributions are also presented. Later on, for a general Lévy subordinator, we show that the failure time of the system is exponentially distributed with mean depending on the Laplace exponent of the Lévy subordinator when the threshold is geometrically distributed. Graphs of survival function for different values of tempering parameters θ and stability index α are also shown.

We recall that the classical competing risks model deals with failure times subject to multiple causes of failure. It is suitable, for instance, for describing the failures of organisms or devices in the presence of many types of risk. In the basic setting, this model deals with an observable pair of random variables (T, ζ), where T is the time of failure and ζ describes the cause or type of failure. For a description of the main features of this model we refer, for instance, to [2, 8]. A recent research line in this field focuses on the analysis of competing risk models arising from shock models. Specifically, we study the bivariate counting process $(\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1), \mathcal{N}_2^{\alpha,\theta}(t, \lambda_2))$, whose components describe respectively shocks of type 1 and type 2 occurring in $(0, t]$ to a given observed system. Failure of the system occurs as soon as the total number of shocks reaches an integer-valued random threshold L for the first time, so that the cause of failure is $\zeta = n$ if a shock of type $n = 1, 2$ effectively produces the system's failure.

The recent literature in this area includes the following contributions. Various counting processes in one-dimensional and multidimensional settings, as well as in time-changed versions, have been successfully applied to shock models, deterioration models, and further contexts of interest in reliability theory. For instance, [9] discussed a bivariate Poisson process with applications in shock models, and [10] considered a bivariate space-fractional Poisson process, studying competing risks and shock models associated with it. In reliability theory and survival analysis, system failure is discussed primarily using conventional competing risks and shock models. A class of general shock models in which failure arises as a result of competing causes of trauma-related degradation was presented in [22]. A new class of bivariate counting processes that have marginal regularity property was developed in [7] and utilized in a shock model. For recent development in this area, see [6, 10, 34].

The structure of the paper is as follows. In Section 2, we present some preliminary notation and definitions. In Section 3, we introduce the BTSFPP and discuss its connection to differential equations. A bivariate shock system governed by the BTSFPP and some reliability-related results of the failure time of the system are provided in Section 4. Also, we present a bivariate Poisson time-changed shock model when the underlying process is governed by an independent general Lévy subordinator. Finally, some concluding remarks are discussed in the last section.

2. Preliminaries

In this section, some notation and results are given that will be used in the subsequent sections. Let $\mathbb{N}$ denote the set of natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively.

2.1. Generalized Wright function

The generalized Wright function is defined by [17]

$$ {}_p\Psi_q\left[z \,\middle|\, \begin{matrix} (\alpha_i, \beta_i)_{1,p} \\ (a_j, b_j)_{1,q} \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(a_j + b_j k)}, \qquad z, \alpha_i, a_i \in \mathbb{C}, \ \beta_i, b_i \in \mathbb{R}, \tag{2.1} $$

under the convergence condition $\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} \beta_i > -1$.

2.2. Lévy subordinator

A Lévy subordinator, denoted by $\{S(t)\}_{t\geq 0}$, is a nondecreasing Lévy process with Laplace transform [1, Section 1.3.2] $\mathbb{E}(\mathrm{e}^{-uS(t)}) = \mathrm{e}^{-t\psi(u)}$, $u \geq 0$, where $\psi(u)$ is the Laplace exponent given by [30, Theorem 3.2]

$$ \psi(u) = \eta u + \int_0^{\infty} (1 - \mathrm{e}^{-ux})\, \nu(\mathrm{d}x), \qquad \eta \geq 0. $$

Here, η is the drift coefficient and ν is a nonnegative Lévy measure on the positive half-line satisfying $\int_0^{\infty} \min\{x, 1\}\, \nu(\mathrm{d}x) < \infty$ and $\nu([0, \infty)) = \infty$, so that $\{S(t)\}_{t\geq 0}$ has strictly increasing sample paths almost surely (a.s.)—for more details, see [29, Theorem 21.3].

For $\alpha \in (0, 1)$ and $\theta > 0$, the *tempered α-stable subordinator* $\{S^{\alpha,\theta}(t)\}_{t\geq 0}$ is defined by the Laplace transform [21]

$$ \mathbb{E}\big[\mathrm{e}^{-uS^{\alpha,\theta}(t)}\big] = \mathrm{e}^{-t((u+\theta)^{\alpha} - \theta^{\alpha})}, \tag{2.2} $$

with Laplace exponent $\psi(u) = (u + \theta)^{\alpha} - \theta^{\alpha}$. Further, the Lévy density $\nu(s)$ associated with ψ is (see [15, (5)] and [28])

$$ \nu(s) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{\mathrm{e}^{-\theta s}}{s^{\alpha+1}}, \qquad s > 0. \tag{2.3} $$

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Let $f_{S^{\alpha,\theta}(t)}(x, t)$ denote the probability density function (PDF) of the TSS. By independent and stationary increments of the Lévy subordinator, the joint density is defined as

$$f_{S^{\alpha,\theta}(t_1),S^{\alpha,\theta}(t_2)}(x_1, t_1; x_2, t_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = f_{S^{\alpha,\theta}(t_2-t_1)}(x_2 - x_1, t_2 - t_1) f_{S^{\alpha,\theta}(t_1)}(x_1, t_1)\,\mathrm{d}x_1\,\mathrm{d}x_2. \quad (2.4)$$

2.3. Tempered space-fractional Poisson process

Let $\{\mathcal{N}(t, \lambda)\}_{t\geq 0}$ be the homogeneous Poisson process with parameter $\lambda > 0$. The tempered space-fractional Poisson process (TSFPP) denoted by $\{\mathcal{N}^{\alpha,\theta}(t, \lambda)\}_{t\geq 0}$ is defined by time-changing the homogeneous Poisson process with an independent TSS as [14] $\mathcal{N}^{\alpha,\theta}(t, \lambda) := \mathcal{N}(S^{\alpha,\theta}(t), \lambda)$. Its PMF $p^{\alpha,\theta}(k, t)$ is given by [13, (26)]

$$p^{\alpha,\theta}(k, t) = \frac{(-1)^k}{k!} \mathrm{e}^{t\theta^\alpha} \sum_{i=0}^{\infty} \frac{\theta^i}{\lambda^i i!}\, {}_1\Psi_1\left[-\lambda^\alpha t \,\middle|\, \begin{matrix} (1, \alpha) \\ (1 - k - i, \alpha) \end{matrix}\right].$$

2.4. Backward shift operators

Let B be the backward shift operator defined by $B[\xi(k)] = \xi(k - 1)$. For the fractional difference operator $(I - B)^\alpha$, we have (see [26] and [33, p. 91])

$$(I - B)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{j} (-1)^i B^i, \qquad \alpha \in (0, 1),$$

where I is an identity operator. Furthermore, let $\{B_i\}$, $i \in \{1, 2, \ldots, m\}$, be the operators defined as

$$B_i[\xi(k_1, k_2, \ldots, k_m)] = \xi(k_1, k_2, \ldots, k_i - 1, \ldots, k_m).$$

For the $m = 1$ case, the B_i act the same as the operator B.

3. Bivariate tempered space-fractional Poisson process

Let $\{\mathcal{N}_i(t, \lambda_i)\}_{t\geq 0}$, $i = 1, 2$, be two independent homogeneous Poisson processes with parameters λ_i, $i = 1, 2$, respectively. Then, for $\alpha \in (0, 1)$, we define the BTSFPP $\{\mathcal{Q}^{\alpha,\theta}(t)\}_{t\geq 0}$ as

$$\mathcal{Q}^{\alpha,\theta}(t) := \big(\mathcal{N}_1(S^{\alpha,\theta}(t), \lambda_1), \mathcal{N}_2(S^{\alpha,\theta}(t), \lambda_2)\big) := \big(\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1), \mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)\big), \quad (3.1)$$

where $S^{\alpha,\theta}$ is the TSS, independent of $\mathcal{N}_1$ and $\mathcal{N}_2$.

Throughout the paper, we work with the bivariate process. Here, we denote any arbitrary bivariate vector of constants by $\mathbf{a} = (a_1, a_2)$, where a_1 and a_2 are nonnegative integers. Let $\mathbf{b} = (b_1, b_2)$, and $\mathbf{0} = (0, 0)$ be the null vector. We write $\mathbf{a} \geq \mathbf{b}$ (or $\mathbf{a} \leq \mathbf{b}$) to mean that $a_i \geq b_i$ (or $a_i \leq b_i$) for $i = 1, 2$. Further, we write $\mathbf{k} = (k_1, k_2)$ and $\mathbf{r} = (r_1, r_2)$.

Next, we derive the PMF, PGF, and associated differential equations for the BTSFPP.

Proposition 3.1. *For $\alpha \in (0, 1)$ and $\mathbf{k} \geq \mathbf{0}$, the PMF $q^{\alpha,\theta}(\mathbf{k}, t) = \mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(t) = \mathbf{k}\}$ is given by*

$$q^{\alpha,\theta}(\mathbf{k}, t) = \left(-\frac{1}{\lambda_1 + \lambda_2}\right)^{k_1+k_2} \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{k_1!k_2!} \mathrm{e}^{t\theta^\alpha} \times \sum_{i=0}^{\infty} \frac{\theta^i}{i!(\lambda_1 + \lambda_2)^i}\, {}_1\Psi_1\left[-(\lambda_1 + \lambda_2)^\alpha t \,\middle|\, \begin{matrix} (1, \alpha) \\ (1 - (k_1 + k_2) - i, \alpha) \end{matrix}\right]. \quad (3.2)$$

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Proof. First, we have

$$\begin{aligned} q^{\alpha,\theta}(\mathbf{k}, t) &= \mathbb{P}\big(\{\mathcal{Q}^{\alpha,\theta}(t)=\mathbf{k}\} \cap \big\{\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1)+\mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)=k_1+k_2\big\}\big) \\ &= \mathbb{P}\big(\mathcal{Q}^{\alpha,\theta}(t)=\mathbf{k} \mid \big\{\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1)+\mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)=k_1+k_2\big\}\big) \\ &\quad \times \mathbb{P}\big(\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1)+\mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)=k_1+k_2\big). \end{aligned} \tag{3.3}$$

Using a conditioning argument along similar lines to [3, Proposition 4], we get

$$\mathbb{P}\big(\mathcal{Q}^{\alpha,\theta}(t)=\mathbf{k} \mid \big\{\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1)+\mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)=k_1+k_2\big\}\big) = \frac{(k_1+k_2)!}{k_1!k_2!}\frac{\lambda_1^{k_1}\lambda_2^{k_2}}{(\lambda_1+\lambda_2)^{k_1+k_2}}.$$

Now, we calculate

$$\begin{aligned} &\mathbb{P}\big(\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1)+\mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)=k_1+k_2\big) \\ &= \mathbb{E}\Big[\mathbb{P}(N_1(r, \lambda_1)+N_2(r, \lambda_2))=k_1+k_2\big|_{r=S^{\alpha,\theta}(t)}\Big] \\ &= \mathbb{E}\left[\frac{((\lambda_1+\lambda_2)r)^{k_1+k_2}}{(k_1+k_2)!}\mathrm{e}^{-r(\lambda_1+\lambda_2)}\bigg|_{r=S^{\alpha,\theta}(t)}\right] \\ &= \frac{(-1)^{k_1+k_2}}{(k_1+k_2)!}\mathrm{e}^{t\theta^\alpha}\sum_{i=0}^{\infty}\frac{\theta^i}{(\lambda_1+\lambda_2)^i i!}\,{}_1\Psi_1\left[-(\lambda_1+\lambda_2)^\alpha t\,\middle|\,\begin{matrix}(1, \alpha)\\ (1-(k_1+k_2)-i, \alpha)\end{matrix}\right]. \end{aligned}$$

With the help of (3.3), we get the PMF. The convergence of ${}_1\Psi_1$ follows from the condition in (2.1) as $\alpha-\alpha=0>-1$. □

Remark 3.1. When $\theta=0$, (3.2) reduces to the PMF of the bivariate space-fractional Poisson process studied in [10].

Theorem 3.1. *For* $\mathbf{u}=(u_1, u_2)\in[0, 1]^2$*, the PGF* $G^{\alpha,\theta}(\mathbf{u}; t)$ *for the BTSFPP is given by*

$$G^{\alpha,\theta}(\mathbf{u}; t)=\exp\{-t([\lambda_1(1-u_1)+\lambda_2(1-u_2)+\theta]^\alpha-\theta^\alpha)\},$$

and it satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}G^{\alpha,\theta}(\mathbf{u}; t)=-([\lambda_1(1-u_1)+\lambda_2(1-u_2)+\theta]^\alpha-\theta^\alpha)G^{\alpha,\theta}(\mathbf{u}; t), \qquad G^{\alpha,\theta}(\mathbf{u}; 0)=1. \tag{3.4}$$

Proof. For $\lambda>0$, the PGF for the TSFPP is given by [13]

$$\mathbb{E}\big[u^{\mathcal{N}^{\alpha,\theta}(t,\lambda)}\big]=\mathbb{E}\big[\mathbb{E}\big[u^{\mathcal{N}(S^{\alpha,\theta}(t),\lambda)} \mid S^{\alpha,\theta}(t)\big]\big]=\mathbb{E}\big[\mathrm{e}^{-\lambda(1-u)S^{\alpha,\theta}(t)}\big]=\mathrm{e}^{-t((\lambda(1-u)+\theta)^\alpha-\theta^\alpha)}.$$

We define the PGF as $G^{\alpha,\theta}(\mathbf{u}; t)=\mathbb{E}\big[\mathbf{u}^{\mathcal{Q}^{\alpha,\theta}(t)}\big]=\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k}, t)$. Hence, we get

$$\begin{aligned} G^{\alpha,\theta}(u; t) &= \mathbb{E}\big[\mathbb{E}\big[u^{\mathcal{Q}^{\alpha,\theta}(t,\lambda)} \mid S^{\alpha,\theta}(t)\big]\big] \\ &= \mathbb{E}\big[e^{(\lambda_1(u_1-1)+\lambda_2(u_2-1))S^{\alpha,\theta}(t)}\big] \\ &= \mathrm{e}^{-t([\lambda_1(1-u_1)+\lambda_2(1-u_2)+\theta]^\alpha-\theta^\alpha)}. \end{aligned}$$

By calculus we obtain (3.4), and the condition trivially holds for $t=0$. □

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Theorem 3.2. *The PMF in (3.2) satisfies the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}q^{\alpha,\theta}(\mathbf{k},t)=-(\lambda_1+\lambda_2)^{\alpha}\left(\left(I-\frac{\lambda_1B_1+\lambda_2B_2-\theta}{\lambda_1+\lambda_2}\right)^{\alpha}-\left(\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\right)q^{\alpha,\theta}(\mathbf{k},t),$$

with $q^{\alpha,\theta}(\mathbf{0},t)=1$.

Proof. From (3.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}G^{\alpha,\theta}(\mathbf{u};t)=-(\lambda_1+\lambda_2)^{\alpha}\left(\left(1-\frac{\lambda_1u_1+\lambda_2u_2-\theta}{\lambda_1+\lambda_2}\right)^{\alpha}-\left(\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\right)G^{\alpha,\theta}(\mathbf{u};t). \quad (3.5)$$

Now, we concentrate our attention on simplifying the following:

$$\begin{aligned}\left(1-\frac{\lambda_1u_1+\lambda_2u_2-\theta}{\lambda_1+\lambda_2}\right)^{\alpha}&=\left(1+\frac{\theta}{\lambda_1+\lambda_2}-\frac{\lambda_1u_1+\lambda_2u_2}{\lambda_1+\lambda_2}\right)^{\alpha}\\&=\sum_{j\geq 0}\binom{\alpha}{j}\left(1+\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha-j}(-1)^j\left(\frac{\lambda_1u_1+\lambda_2u_2}{\lambda_1+\lambda_2}\right)^{j}\\&=\sum_{j\geq 0}\binom{\alpha}{j}\left(1+\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha-j}\frac{(-1)^j}{(\lambda_1+\lambda_2)^j}\sum_{\substack{\mathbf{r}\geq\mathbf{0},\\ r_1+r_2=j}}\frac{j!}{r_1!r_2!}\lambda_1^{r_1}\lambda_2^{r_2}u_1^{r_1}u_2^{r_2}.\end{aligned}$$

Therefore, from (3.5),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}G^{\alpha,\theta}(\mathbf{u};t)\\&=-(\lambda_1+\lambda_2)^{\alpha}\left(\left(1-\frac{\lambda_1u_1+\lambda_2u_2-\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k},t)-\left(\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k},t)\right)\\&=-(\lambda_1+\lambda_2)^{\alpha}\sum_{j\geq 0}\binom{\alpha}{j}\left(1+\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha-j}\frac{(-1)^j}{(\lambda_1+\lambda_2)^j}\sum_{\substack{\mathbf{r}\geq\mathbf{0},\\ r_1+r_2=j}}\frac{j!}{r_1!r_2!}\lambda_1^{r_1}\lambda_2^{r_2}\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1+r_1}u_2^{k_2+r_2}q^{\alpha,\theta}(\mathbf{k},t)\\&\quad+(\lambda_1+\lambda_2)^{\alpha}\left(\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k},t)\\&=-(\lambda_1+\lambda_2)^{\alpha}\sum_{j\geq 0}\binom{\alpha}{j}\left(1+\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha-j}\frac{(-1)^j}{(\lambda_1+\lambda_2)^j}\sum_{\substack{\mathbf{r}\geq\mathbf{0},\\ r_1+r_2=j}}\frac{j!}{r_1!r_2!}\lambda_1^{r_1}\lambda_2^{r_2}\sum_{\mathbf{k}\geq\mathbf{r}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k}-\mathbf{r},t)\\&\quad+(\lambda_1+\lambda_2)^{\alpha}\left(\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k},t)\\&=-(\lambda_1+\lambda_2)^{\alpha}\sum_{\mathbf{k}\geq\mathbf{r}}u_1^{k_1}u_2^{k_2}\sum_{j\geq 0}\binom{\alpha}{j}\left(1+\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha-j}\frac{(-1)^j}{(\lambda_1+\lambda_2)^j}\sum_{\substack{\mathbf{r}\geq\mathbf{0},\\ r_1+r_2=j}}\frac{j!}{r_1!r_2!}\lambda_1^{r_1}\lambda_2^{r_2}q^{\alpha,\theta}(\mathbf{k}-\mathbf{r},t)\\&\quad+(\lambda_1+\lambda_2)^{\alpha}\left(\frac{\theta}{\lambda_1+\lambda_2}\right)^{\alpha}\sum_{\mathbf{k}\geq\mathbf{0}}u_1^{k_1}u_2^{k_2}q^{\alpha,\theta}(\mathbf{k},t).\end{aligned}$$

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Since

$$\sum_{\substack{\mathbf{r}\geq\mathbf{0},\\ r_1+r_2=j}} \frac{j!}{r_1!r_2!}\lambda_1^{r_1}\lambda_2^{r_2}q^{\alpha,\theta}(\mathbf{k}-\mathbf{r}, t) = (\lambda_1 B_1 + \lambda_2 B_2)^j q^{\alpha,\theta}(\mathbf{k}, t),$$

we obtain the desired differential equation. □

Next, we derive the Lévy measure for the BTSFPP.

Theorem 3.3. *The discrete Lévy measure $\mathcal{V}_{\alpha,\theta}$ for the BTSFPP is given by*

$$\mathcal{V}_{\alpha,\theta}(\,\cdot\,) = \sum_{k_1,k_2>0} \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{k_1!k_2!}\frac{\alpha\Gamma(k_1+k_2-\alpha)}{\Gamma(1-\alpha)}\delta_{\{\mathbf{k}\}}(\,\cdot\,)(\theta+\lambda_1+\lambda_2)^{\alpha-k_1-k_2},$$

where $\delta_{\{\mathbf{k}\}}(\,\cdot\,)$ is the Dirac measure concentrated at $\mathbf{k}$.

Proof. The PMF for the bivariate Poisson process $\mathcal{N}(t) = (\mathcal{N}_1(t, \lambda_1), \mathcal{N}_2(t, \lambda_2))$ is [3]

$$\mathbb{P}\{\mathcal{N}_1(t, \lambda_1) = k_1, \mathcal{N}_2(t, \lambda_2) = k_2\} = \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{k_1!k_2!}t^{k_1+k_2}\mathrm{e}^{-(\lambda_1+\lambda_2)t}.$$

Using (2.3) and applying the formula from [29, p. 197] to calculate the Lévy measure, we get

$$\begin{aligned}\mathcal{V}_{\alpha,\theta}(\,\cdot\,) &= \int_0^\infty \sum_{k_1,k_2>0} \mathbb{P}\{\mathcal{N}_1(s, \lambda_1) = k_1, \mathcal{N}_2(s, \lambda_2) = k_2\}\delta_{\{\mathbf{k}\}}(\,\cdot\,)\nu(s)\,\mathrm{d}s \\ &= \sum_{k_1,k_2>0} \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{k_1!k_2!}\delta_{\{\mathbf{k}\}}(\,\cdot\,)\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \mathrm{e}^{s(\theta+\lambda_1+\lambda_2)}s^{k_1+k_2-\alpha-1}\,\mathrm{d}s.\end{aligned}$$

Using the integral formula [12, (3.351.3)], we can simplify this as

$$\mathcal{V}_{\alpha,\theta}(\,\cdot\,) = \sum_{k_1,k_2>0} \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{k_1!k_2!}\frac{\alpha(k_1+k_2-\alpha-1)!}{\Gamma(1-\alpha)}\delta_{\{\mathbf{k}\}}(\,\cdot\,)(\theta+\lambda_1+\lambda_2)^{\alpha-k_1-k_2}.$$

Hence, the theorem is proved. □

With the aim of calculating hazard rates, we establish the following lemma.

Lemma 3.1. *For $h \in \mathbb{N}_0$,*

$$\begin{aligned}\frac{\mathrm{d}^h}{\mathrm{d}u^h}\big[\mathrm{e}^{-t((u+\theta)^\alpha-\theta^\alpha)}\big] &= \sum_{k=0}^{h}\frac{1}{k!}\mathrm{e}^{-t((u+\theta)^\alpha-\theta^\alpha)}\sum_{j=0}^{k}\binom{k}{j}t^k(-1)^j(((u+\theta)^\alpha-\theta^\alpha))^{k-j} \\ &\quad\times\sum_{i=0}^{j}\binom{j}{i}(\alpha i)_h(u+\theta)^{\alpha i-h}(-\theta^\alpha)^{j-i},\end{aligned}$$

where $(x)_h = x(x-1)\cdots(x-h+1)$ denotes the falling factorial.

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Proof. Let $V(u) = -t((u+\theta)^\alpha - \theta^\alpha)$ and $W(V(u)) = \mathrm{e}^{-t((u+\theta)^\alpha - \theta^\alpha)}$. Then, applying Hoppe's formula [16] to the function $W(V(u))$, we get

$$\frac{\mathrm{d}^h}{\mathrm{d}u^h} W(V(u)) = \sum_{k=0}^{h} \frac{1}{k!} \mathrm{e}^{-t((u+\theta)^\alpha - \theta^\alpha)} T_{h,k}(V(u)), \tag{3.6}$$

where $T_{h,k}(V(u))$ is computed as

$$\begin{aligned}
T_{h,k}(V(u)) &= \sum_{j=0}^{k} \binom{k}{j} (-V(u))^{k-j} \frac{\mathrm{d}^h}{\mathrm{d}u^h} (V(u))^j \\
&= \sum_{j=0}^{k} \binom{k}{j} (t((u+\theta)^\alpha - \theta^\alpha))^{k-j} \frac{\mathrm{d}^h}{\mathrm{d}u^h} (-t((u+\theta)^\alpha - \theta^\alpha))^j \\
&= \sum_{j=0}^{k} \binom{k}{j} (t((u+\theta)^\alpha - \theta^\alpha))^{k-j} (-t)^j \sum_{i=0}^{j} \binom{j}{i} (-\theta^\alpha)^{j-i} \frac{\mathrm{d}^h}{\mathrm{d}u^h} (u+\theta)^{\alpha i} \\
&= \sum_{j=0}^{k} \binom{k}{j} t^k (-1)^j (((u+\theta)^\alpha - \theta^\alpha))^{k-j} \sum_{i=0}^{j} \binom{j}{i} (\alpha i)_h (u+\theta)^{\alpha i - h} (-\theta^\alpha)^{j-i}.
\end{aligned}$$

Hence, through (3.6), we have proved the lemma. □

4. Bivariate shock models

We design a shock model that is subjected to two shocks of types 1 and 2. Let T be a nonnegative absolutely continuous random variable that represents the failure time of a system subject to two possible causes of failure. Set $\zeta = n$, which represents the failure of the system occurring due to a shock of type n for $n = 1, 2$. We define the total number of shocks $\{\mathcal{Z}(t)\}_{t\geq 0}$ during the time interval $[0, t]$ as

$$\mathcal{Z}(t) = \mathcal{N}_1^{\alpha,\theta}(t, \lambda_1) + \mathcal{N}_2^{\alpha,\theta}(t, \lambda_2),$$

where $\mathcal{N}_1^{\alpha,\theta}(t, \lambda_1)$ and $\mathcal{N}_2^{\alpha,\theta}(t, \lambda_2)$ are processes counting the number of shocks of type n for $n = 1, 2$, respectively, during the time interval $[0, t]$.

We introduce a random threshold L that takes values in the set of natural numbers. Hence, at the first time when $\mathcal{Q}(t) = L$, the failure occurs. The probability distribution and the reliability function of L are respectively defined by

$$\begin{aligned}
q_k &= \mathbb{P}(L = k), \qquad k \in \mathbb{N}, \\
\overline{q}_k &= \mathbb{P}(L > k), \qquad k \in \mathbb{N}_0.
\end{aligned} \tag{4.1}$$

Let $g_T(t)$ be the PDF of T, defined as $T = \inf\{t \geq 0 \colon \mathcal{Z}(t) = L\}$. Then, we have $g_T(t) = g_1(t) + g_2(t)$, $t \geq 0$, where the sub-densities $g_n(t)$ are defined by

$$g_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}\{T \leq t, \zeta = n\}, \qquad n = 1, 2.$$

Also, the probability that the failure occurs due to a shock of type n is given by

$$\mathbb{P}(\zeta = n) = \int_0^{\infty} g_n(t)\,\mathrm{d}t, \qquad n = 1, 2. \tag{4.2}$$

Furthermore, in terms of the joint PMF, the hazard rates are given by

$$\begin{aligned} h_1(k_1, k_2; t) &= \lim_{\tau\to 0^+} \frac{\mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(t+\tau) = (k_1+1, k_2) \mid \mathcal{Q}^{\alpha,\theta}(t) = (k_1, k_2)\}}{\tau}, \\ h_2(k_1, k_2; t) &= \lim_{\tau\to 0^+} \frac{\mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(t+\tau) = (k_1, k_2+1) \mid \mathcal{Q}^{\alpha,\theta}(t) = (k_1, k_2)\}}{\tau}, \end{aligned} \tag{4.3}$$

with $(k_1, k_2) \in \mathbb{N}_0^2$. Hence, conditioning on L and with the help of (4.1), the failure densities take the form

$$g_n(t) = \sum_{k=1}^{\infty} q_k \sum_{k_1+k_2=k-1} \mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(t) = (k_1, k_2)\} h_n(k_1, k_2; t), \qquad n = 1, 2. \tag{4.4}$$

The reliability function of T, denoted by $\overline{R}_T(t) = \mathbb{P}\{T > t\}$, is given by

$$\overline{R}_T(t) = \sum_{k=0}^{\infty} \overline{q}_k \sum_{k_1+k_2=k} \mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(t) = (k_1, k_2)\}, \qquad \overline{q}_0 = 1. \tag{4.5}$$

Proposition 4.1. *Under the assumptions of the model in (3.1) and for $n = 1, 2$, the hazard rates $h_n(k_1, k_2; t)$, $t \geq 0$, are given by*

$$\begin{aligned} h_n(k_1, k_2; t) &= \alpha\lambda_n(\Lambda+\theta)^{\alpha-1}\mathrm{e}^{-t(\Lambda+\theta)^{\alpha}} \left(\sum_{l=0}^{\infty} \frac{\theta^l}{\Lambda^l l!}\, {}_1\Psi_1\left[-\Lambda^{\alpha} t \,\middle|\, \begin{matrix} (1, \alpha) \\ (1-h-l, \alpha) \end{matrix}\right]\right)^{-1} \\ &\quad \times \sum_{k=0}^{h} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j t^k (((\Lambda+\theta)^{\alpha} - \theta^{\alpha}))^{k-j} \sum_{i=0}^{j} \binom{j}{i} (\alpha i)_h (\Lambda+\theta)^{\alpha i} (-\theta^{\alpha})^{j-i}, \end{aligned} \tag{4.6}$$

where $\Lambda = \lambda_1 + \lambda_2$ and $h = k_1 + k_2$.

Proof. We fix $n = 1$. With the help of (2.4) and considering the BTSFPP as bivariate HPP with tempered α-stable stopping time, we have

$$\begin{aligned} &\mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(\tau) = (k_1+1, k_2),\ \mathcal{Q}^{\alpha,\theta}(t) = (k_1, k_2)\} \\ &= \int_0^{\infty}\int_0^{y} \mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(y) = (k_1+1, k_2), \mathcal{Q}^{\alpha,\theta}(x) = (k_1, k_2)\} f_{S^{\alpha,\theta}(\tau-t)}(y-x, \tau-t) f_{S^{\alpha,\theta}(t)}(x, t)\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty}\int_0^{y} \mathbb{P}\{\mathcal{N}_1(y-x, \lambda_1) = 1, \mathcal{N}_2(y-x, \lambda_2) = 0\} \\ &\qquad \times \mathbb{P}\{\mathcal{N}_1(x, \lambda_1) = k_1, \mathcal{N}_2(x, \lambda_2) = k_2\} f_{S^{\alpha,\theta}(\tau-t)}(y-x, \tau-t) f_{S^{\alpha,\theta}(t)}(x, t)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty}\int_0^{y} \frac{\lambda_1^{k_1+1}\lambda_2^{k_2}}{k_1!k_2!} \mathrm{e}^{-(\lambda_1+\lambda_2)y} x^{k_1+k_2}(y-x) f_{S^{\alpha,\theta}(\tau-t)}(y-x, \tau-t) f_{S^{\alpha,\theta}(t)}(x, t)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

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By using Tonelli's theorem, we get

$$\begin{aligned}
&\mathbb{P}\{\mathcal{Q}^{\alpha,\theta}(\tau)=(k_1+1,k_2),\ \mathcal{Q}^{\alpha,\theta}(t)=(k_1,k_2)\}\\
&=\frac{\lambda_1^{k_1+1}\lambda_2^{k_2}}{k_1!k_2!}\int_0^\infty\int_x^\infty x^h f_{S^{\alpha,\theta}(t)}(x,t) f_{S^{\alpha,\theta}(\tau-t)}(y-x,\tau-t)\mathrm{e}^{-(\lambda_1+\lambda_2)y}(y-x)\,\mathrm{d}y\,\mathrm{d}x\\
&=\frac{\lambda_1^{k_1+1}\lambda_2^{k_2}}{k_1!k_2!}\int_0^\infty \mathrm{e}^{-(\lambda_1+\lambda_2)x}x^h f_{S^{\alpha,\theta}(t)}(x,t)\,\mathrm{d}x\int_0^\infty y\mathrm{e}^{-(\lambda_1+\lambda_2)y} f_{S^{\alpha,\theta}(\tau-t)}(y,\tau-t)\,\mathrm{d}y\\
&=\frac{\lambda_1^{k_1+1}\lambda_2^{k_2}}{k_1!k_2!}(-1)^h\frac{\mathrm{d}^h}{\mathrm{d}x^h}\mathbb{E}\big[\mathrm{e}^{-xS^{\alpha,\theta}(t)}\big]\Big|_{x=\Lambda}\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}\big[\mathrm{e}^{-yS^{\alpha,\theta}(\tau-t)}\big]\Big|_{y=\Lambda}.
\end{aligned}$$

Hence, using the definition of conditional density in (4.3), the required form is obtained with the help of (3.2) and Lemma 3.1. For the $n=2$ case the proof follows along the same lines. □

In the next propositions, we derive the failure densities and the reliability function of the system and obtain the distribution (4.2) of failure due to the nth type of shock.

Proposition 4.2. *Under the assumptions of the model in (3.1), for $n=1,2$ and $t\geq 0$, the failure density is of the form*

$$g_n(t)=\alpha\lambda_n(\Lambda+\theta)^{\alpha-1}\mathrm{e}^{-t((\Lambda+\theta)^\alpha-\theta^\alpha)}\times$$
$$\sum_{k=1}^\infty q_k\frac{(-1)^{k-1}}{(k-1)!}\sum_{l=0}^{k-1}\frac{t^l}{l!}\sum_{j=0}^{l}\binom{l}{j}(-1)^j(((\Lambda+\theta)^\alpha-\theta^\alpha))^{l-j}\sum_{i=0}^{j}\binom{j}{i}(\alpha i)_{k-1}(\Lambda+\theta)^{\alpha i}(-\theta^\alpha)^{j-i}.$$

Proof. On substituting the PMF (3.2) and (4.6) into (4.4), we get

$$\begin{aligned}
g_n(t)&=\sum_{k=1}^\infty q_k\sum_{k_1+k_2=k-1}q^{\alpha,\theta}(k,t)(\Lambda+\theta)^{\alpha-1}\mathrm{e}^{-t(\Lambda+\theta)^\alpha}\left(\sum_{l=0}^\infty\frac{\theta^l}{\Lambda^l l!}{}_1\Psi_1\left[-\Lambda^\alpha t\,\middle|\,\begin{matrix}(1,\alpha)\\(1-h-l,\alpha)\end{matrix}\right]\right)^{-1}\\
&\quad\times\alpha\lambda_n\sum_{l=0}^{k_1+k_2}\frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}(-1)^j t^l(((\Lambda+\theta)^\alpha-\theta^\alpha))^{l-j}\sum_{i=0}^{j}\binom{j}{i}(\alpha i)_{k_1+k_2}(\Lambda+\theta)^{\alpha i}(-\theta^\alpha)^{j-i}\\
&=\alpha\lambda_n(\Lambda+\theta)^{\alpha-1}\mathrm{e}^{-t((\Lambda+\theta)^\alpha-\theta^\alpha)}\sum_{k=1}^\infty q_k\frac{(-1)^{k-1}}{(k-1)!\Lambda^{k-1}}\sum_{k_1=0}^{k-1}\frac{(k-1)!\lambda_1^{k_1}\lambda_2^{k-1-k_1}}{k_1!(k-1-k_1)!}\\
&\quad\times\sum_{l=0}^{k-1}\frac{t^l}{l!}\sum_{j=0}^{l}\binom{l}{j}(-1)^j(((\Lambda+\theta)^\alpha-\theta^\alpha))^{l-j}\sum_{i=0}^{j}\binom{j}{i}(\alpha i)_{k-1}(\Lambda+\theta)^{\alpha i}(-\theta^\alpha)^{j-i}.
\end{aligned}$$

Using the binomial theorem, the failure density is obtained. □

Proposition 4.3. *Under the assumptions of the model in (3.1), the reliability function of T is given by*

$$\overline{R}_T(t)=\sum_{k=0}^\infty\overline{q}_k\frac{(-1)^k}{k!}\mathrm{e}^{t\theta^\alpha}\sum_{i=0}^\infty\frac{\theta^i}{\Lambda^i i!}{}_1\Psi_1\left[-\Lambda^\alpha t\,\middle|\,\begin{matrix}(1,\alpha)\\(1-k-i,\alpha)\end{matrix}\right],\qquad t\geq 0.$$

Proof. The reliability function can be obtained by substituting (3.2) into (4.5) and simplifying it using the binomial theorem as carried out in the previous proof. □

Proposition 4.4. *Under the assumptions of the model in (3.1), for $n=1, 2$ we also have*

$$\mathbb{P}(\zeta = n) = \alpha\lambda_n \frac{(\Lambda+\theta)^{\alpha-1}}{(\Lambda+\theta)^{\alpha}-\theta^{\alpha}} \times$$

$$\sum_{k=1}^{\infty} q_k \frac{(-1)^{k-1}}{(k-1)!} \sum_{l=0}^{k-1} \sum_{j=0}^{l} \binom{l}{j} (-1)^j ((\Lambda+\theta)^{\alpha}-\theta^{\alpha})^{-j} \sum_{i=0}^{j} \binom{j}{i} (\alpha i)_{k-1} (\Lambda+\theta)^{\alpha i} (-\theta^{\alpha})^{j-i}.$$

Proof. With the help of Proposition 4.2, the probability (4.2) gives

$$\mathbb{P}(\zeta = n) = \alpha\lambda_n (\Lambda+\theta)^{\alpha-1} \times$$

$$\sum_{k=1}^{\infty} q_k \frac{(-1)^{k-1}}{(k-1)!} \sum_{l=0}^{k-1} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} (-1)^j (((\Lambda+\theta)^{\alpha}-\theta^{\alpha}))^{l-j} \sum_{i=0}^{j} \binom{j}{i} (\alpha i)_{k-1} (\Lambda+\theta)^{\alpha i} (-\theta^{\alpha})^{j-i}$$

$$\times \int_0^{\infty} \mathrm{e}^{-t((\Lambda+\theta)^{\alpha}-\theta^{\alpha})} t^l \, \mathrm{d}t.$$

Using the integral formula of [12, (3.351.3)], we get the proposition. □

4.1. Generalized shock models

Let $S := \{S(t)\}_{t\geq 0}$ be a Lévy subordinator. In the next theorem, we evaluate the reliability function of T when the threshold L has geometric distribution with parameter $p \in (0, 1]$, i.e.

$$\overline{q}_k = (1-p)^k, \qquad k = 0, 1, 2, \ldots, \tag{4.7}$$

and when the shocks arrive according to a process $N := \{N(t)\}_{t\geq 0}$, where

$$N(t) := (\mathcal{N}_1(S(t), \lambda_1), \mathcal{N}_2(S(t), \lambda_2)), \tag{4.8}$$

and the components of N are two time-changed independent homogeneous Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. The time change is represented by an independent generic subordinator S.

Recalling that $T = \inf\{t \geq 0 \colon \mathcal{N}_1(S(t), \lambda_1) + \mathcal{N}_2(S(t), \lambda_2) = L\}$, the distribution (4.7) stems from the customary assumption that the failure happens at the occurrence of the first critical event of a sequence of Bernoulli trials having parameter p, where each trial is performed as soon as the sum of shocks reaches any integer level.

Theorem 4.1. *For $(x_1, x_2) \in \mathbb{N}_0^2$ and under the assumptions of the model in (4.7) and (4.8), the reliability function of T is*

$$\overline{F}_T(t) = \mathrm{e}^{-t\psi((\lambda_1+\lambda_2)p)}, \tag{4.9}$$

where $\psi(\,\cdot\,)$ is the Laplace exponent of the subordinator S.

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Proof. Consider the reliability function of T as

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} (1-p)^k \sum_{x_1=0}^{k} \mathbb{P}(\mathcal{N}_1(S(t), \lambda_1) = x_1, \mathcal{N}_2(S(t), \lambda_2) = k - x_1)$$

$$= \sum_{k=0}^{+\infty} (1-p)^k \sum_{x_1=0}^{k} \frac{\lambda_1^{x_1}}{x_1!} \frac{\lambda_2^{k-x_1}}{(k-x_1)!} \int_0^{+\infty} \mathrm{e}^{-(\lambda_1+\lambda_2)s} s^k \mathbb{P}(S(t) \in \mathrm{d}s).$$

We exchange the order of summation and rearrange all the terms to get

$$\overline{F}_T(t) = \sum_{x_1=0}^{+\infty} \frac{\lambda_1^{x_1}(1-p)^{x_1}}{x_1!} \sum_{h=0}^{+\infty} \frac{[\lambda_2(1-p)]^h}{h!} \int_0^{+\infty} s^{x_1+h} \mathrm{e}^{-(\lambda_1+\lambda_2)s} \mathbb{P}(S(t) \in \mathrm{d}s)$$

$$= \sum_{x_1=0}^{+\infty} \frac{\lambda_1^{x_1}(1-p)^{x_1}}{x_1!} \int_0^{+\infty} s^{x_1} \mathrm{e}^{-(\lambda_1+\lambda_2)s+\lambda_2(1-p)s} \mathbb{P}(S(t) \in \mathrm{d}s)$$

$$= \int_0^{+\infty} \mathrm{e}^{-(\lambda_1+\lambda_2)s+\lambda_2(1-p)s+\lambda_1(1-p)s} \mathbb{P}(S(t) \in \mathrm{d}s)$$

$$= \int_0^{+\infty} \mathrm{e}^{-(\lambda_1+\lambda_2)ps} \mathbb{P}(S(t) \in \mathrm{d}s) = \mathrm{e}^{-t\psi((\lambda_1+\lambda_2)p)}. \tag{4.10}$$

Hence, the theorem is proved. □

Remark 4.1. In (4.9), observe that the random failure time T is exponentially distributed with mean depending on the Laplace exponent of the subordinator.

Remark 4.2. As a corollary of Theorem 4.1, it is straightforward to show that if the distribution of L is a mixture of the geometric distribution (4.7), then the distribution of T is a mixture of the exponential distribution (4.10). That is,

$$\overline{F}_T(t) = \int_0^1 \mathrm{e}^{-t\psi((\lambda_1+\lambda_2)p)} \,\mathrm{d}G(p),$$

where G is a distribution on (0, 1).

Next, we discuss examples of some special random thresholds under the assumptions of the model in (3.1).

4.2. Some examples

First, we reproduce the following identity from [14]:

$$\exp[-t((\Lambda(1-u)+\theta)^\alpha - \theta^\alpha)] = \mathrm{e}^{t\theta^\alpha} \sum_{k=0}^{\infty} \frac{(-u)^k}{k!} \sum_{i=0}^{\infty} \frac{\theta^i}{\Lambda^i i!} {}_1\Psi_1\left[-\Lambda^\alpha t \,\middle|\, \begin{matrix} (1, \alpha) \\ (1-k-i, \alpha) \end{matrix}\right]. \tag{4.11}$$

Now, we derive the reliability function of T for two particular cases of the random threshold L.

(i) Let L follow the discrete exponential distribution with reliability function $\overline{q}_k = \mathrm{e}^{-k}$, $k = 0, 1, 2, \ldots$ From Proposition 4.3, and with help of (4.11), we get

$$\overline{R}_T(t) = \exp[-t((\Lambda(1-\mathrm{e}^{-1})+\theta)^\alpha - \theta^\alpha)].$$

Also, the density is given by

$$g_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\overline{R}_T(t) = ((\Lambda(1-\mathrm{e}^{-1})+\theta)^\alpha - \theta^\alpha)\exp[-t((\Lambda(1-{}'e^{-1})+\theta)^\alpha - \theta^\alpha)].$$

Therefore, the hazard rate function denoted by $H_T(t)$ for the random variable T is given by

$$H_T(t) = \frac{g_T(t)}{\overline{R}_T(t)} = ((\Lambda(1-\mathrm{e}^{-1})+\theta)^\alpha - \theta^\alpha), \qquad t \geq 0.$$

(ii) Let L follow the Yule–Simon distribution with parameter p and the reliability function $\overline{q}_k = kB(k, p+1)$, $k = 1, 2, \ldots$, where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the beta function. Then, the reliability function $\overline{R}_T(t)$ takes the form

$$\begin{aligned}
\overline{R}_T(t) &= \sum_{k=0}^{\infty} kB(k, p+1)\frac{(-1)^k}{k!}\mathrm{e}^{t\theta^\alpha}\sum_{i=0}^{\infty}\frac{\theta^i}{\Lambda^i i!}\,{}_1\Psi_1\left[-\Lambda^\alpha t\,\middle|\,\begin{matrix}(1,\alpha)\\(1-k-i,\alpha)\end{matrix}\right]\\
&= \sum_{k=0}^{\infty} k\left(\int_0^1 z^{k-1}(1-z)^p\,\mathrm{d}z\right)\frac{(-1)^k}{k!}\mathrm{e}^{t\theta^\alpha}\sum_{i=0}^{\infty}\frac{\theta^i}{\Lambda^i i!}\,{}_1\psi_1\left[-\Lambda^\alpha t\,\middle|\,\begin{matrix}(1,\alpha)\\(1-k-i,\alpha)\end{matrix}\right]\\
&= \mathrm{e}^{t\theta^\alpha}\int_0^1 (1-z)^p\sum_{k=1}^{\infty} kz^{k-1}\frac{(-1)^k}{k!}\sum_{i=0}^{\infty}\frac{\theta^i}{\Lambda^i i!}\,{}_1\Psi_1\left[-\Lambda^\alpha t\,\middle|\,\begin{matrix}(1,\alpha)\\(1-k-i,\alpha)\end{matrix}\right]\mathrm{d}z\\
&= \int_0^1 (1-z)^p\left(\frac{\mathrm{d}}{\mathrm{d}z}\exp[-t((\Lambda(1-z)+\theta)^\alpha-\theta^\alpha)]\right)\mathrm{d}z.
\end{aligned}$$

Hence, the density function is given by

$$g_T(t) = \int_0^1 (1-z)^p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}z}\exp\left[-t((\Lambda(1-z)+\theta)^\alpha-\theta^\alpha)\right]\right)\mathrm{d}z.$$

Considering these $g_T(t)$ and $\overline{R}_T(t)$, we get the hazard rate function in the case of a Yule–Simon threshold.

Now, we discuss some special cases of the mixing distribution from Remark 4.2 under the assumptions of the model in (4.8).

4.3. Special cases

We now analyze three special cases by specifying the mixing distribution, under the assumption that S is the tempered α-stable subordinator as in (2.2). Evaluation of the reliability functions can be performed using Mathematica.

(i) $\mathrm{d}G(p) = \mathrm{d}p$ (uniform distribution). It is

$$\overline{F}_T(t) = \frac{\mathrm{e}^{t\theta^\alpha}}{\alpha(\lambda_1+\lambda_2)}[\theta E_{(\alpha-1)/\alpha}(t\theta^\alpha) - (\lambda_1+\lambda_2+\theta)E_{(\alpha-1)/\alpha}(t(\lambda_1+\lambda_2+\theta)^\alpha)], \tag{4.12}$$

where $E_l(z) = \int_1^{+\infty}\mathrm{e}^{-uz}/u^l\,\mathrm{d}u$ is a generalized exponential integral.

(ii)

$$\mathrm{d}G(p) = \frac{ab(1+ap)^{-(b+1)}}{1-(1+a)^{-b}}\,\mathrm{d}p,$$

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with $a > 0$ and $(b > -1) \wedge (b \neq 0)$ (truncated Lomax). Set $a := (\lambda_1 + \lambda_2)/\theta$ and $b + 1 := \alpha$. It is

$$\overline{F}_T(t) = \frac{e^{t\theta^\alpha}(\alpha - 1)}{\alpha[1 - (1 + (\lambda_1 + \lambda_2)/\theta)^{1-\alpha}]} \times \left[E_{2-(1/\alpha)}(t\theta^\alpha) - \left(1 + \frac{\lambda_1 + \lambda_2}{\theta}\right)^{1-\alpha} E_{2-(1/\alpha)}\left(t\theta^\alpha \left(1 + \frac{\lambda_1 + \lambda_2}{\theta}\right)^\alpha\right)\right]. \quad (4.13)$$

(iii)

$$dG(p) = \frac{b}{a}\left(\frac{p-c}{a}\right)^{b-1} e^{-((p-c)/a)^b}\, dp,$$

where a and b are positive values, and c is a real value (truncated three-parameter Weibull). Set $a = 1/(\lambda_1 + \lambda_2)$, $b = \alpha$, and $c = -\theta/(\lambda_1 + \lambda_2)$. It is

$$\overline{F}_T(t) = \frac{1 - e^{-(t+1)[(\lambda_1+\lambda_2+\theta)^\alpha - \theta^\alpha]}}{(t+1)[1 - e^{-[(\lambda_1+\lambda_2+\theta)^\alpha - \theta^\alpha]}]}. \quad (4.14)$$

The graphs in Figures 1, 2, and 3 illustrate the special cases for some particular values of the parameters.

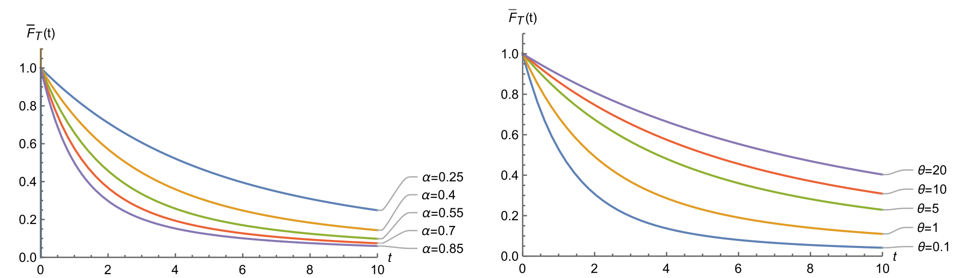

FIGURE 1. Plots of the reliability function (4.12) with $\lambda_1 = \lambda_2 = 1$ and $\theta = 1$ on the left, $\alpha = 0.5$ on the right.

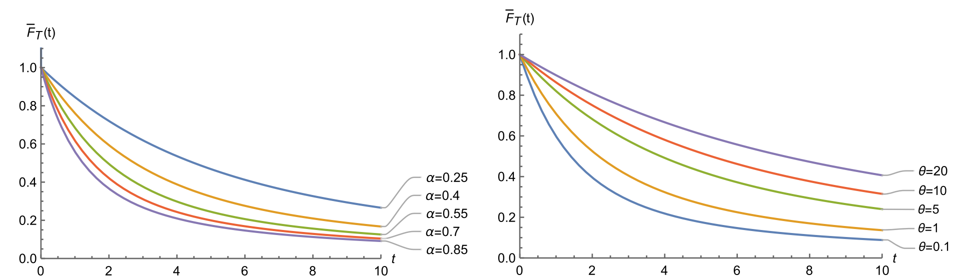

FIGURE 2. Plots of the reliability function (4.13) with $\lambda_1 = \lambda_2 = 1$ and $\theta = 1$ on the left, $\alpha = 0.5$ on the right.

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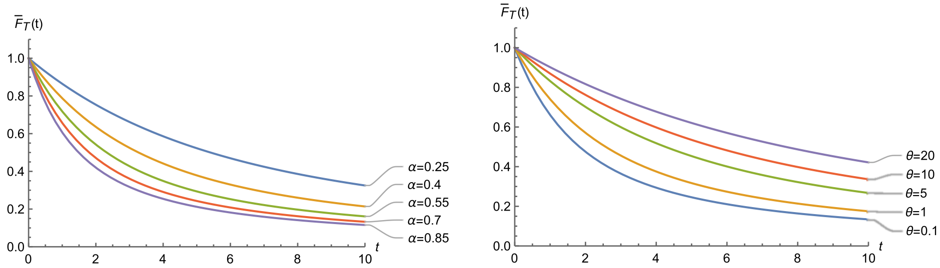

FIGURE 3. Plots of the reliability function (4.14) with $\lambda_1 = \lambda_2 = 1$ and $\theta = 1$ on the left, $\alpha = 0.5$ on the right.

5. Concluding remarks

In this paper, we have proposed a bivariate tempered space-fractional Poisson process by time-changing the bivariate Poisson process with an independent tempered α-stable subordinator. First, we derived the expression for the probability mass function and expressed it in terms of the generalized Wright function, then we obtained the governing differential equations for the PMF and the PGF. We also derived the Lévy measure density for the BTSFPP. Tempering the distribution of an α-stable subordinator by a decreasing exponential gives the new process the property of behaving like a stable subordinator at small times, but with lighter tails at large times. As a consequence, all moments are finite and its density is also infinitely divisible, although it is no more self-similar. Parameter estimation procedures are also well known (see, for instance, [18]). Moreover, as outlined in [27], time-changing a Poisson process with a tempered stable subordinator, rather than with a stable subordinator, has its advantages since it results in high jumps occurring with smaller probability. This might be of interest when it comes to modeling real phenomena. Many financial applications, like option pricing, rely on tempered stable distributions, but, to our knowledge, the use of such processes in reliability is totally unexplored. Therefore, we presented a bivariate competing risk and shock model based on the BTSFPP and derived various reliability quantities to predict the life of the system. Finally, we discussed a generalized shock model and various typical examples. We have focused our study here on the bivariate case, but our work can be explored in multivariate cases also. We could possibly develop the model in subsequent studies by taking into account nonhomogeneous, multistable, and multifractional counting processes. The next step in the research might potentially include consideration of the ageing properties of the random failure time and some additional reliability notions.

Acknowledgements

The authors are grateful to the editor and anonymous referees for their constructive and helpful comments which have significantly improved the article. A.D.C. and A.M. are members of the GNCS group of INdAM (Istituto Nazionale di Alta Matematica).

Funding information

This work is partially supported by CSIR India (File No: 09/1051(11349)/2021-EMR-I), DST-SERB Government of India (MATRICS Grant: MTR/2022/000796) and European Union – Next Generation EU through MUR-PRIN 2022 PNRR, project P2022XSF5H Stochastic

Models in Biomathematics and Applications, and MUR–PRIN 2022, Project Anomalous Phenomena on Regular and Irregular Domains: Approximating Complexity for the Applied Sciences (no. 2022XZSAFN).

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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