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SIMULATION OF MULTI-ASSET OPTION GREEKS UNDER A SPECIAL LÉVY MODEL BY MALLIAVIN CALCULUS

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Abstract

We discuss simulation of sensitivities or Greeks of multi-asset European style options under a special Lévy process model: that is, the subordinated Brownian motion model. The Malliavin calculus method combined with Monte Carlo and quasi-Monte Carlo methods is used in the simulations. Greeks are expressed in terms of the expectations of the option payoff functions multiplied by the weights involving Malliavin derivatives for multi-asset options. Numerical results show that the Malliavin calculus method is usually more efficient than the finite difference method for options with nonsmooth payoffs. The superiority of the former method over the latter is even more significant when both are combined with quasi-Monte Carlo methods.

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1. Introduction

1.1. Subordinated Brownian motions Empirical studies using real financial data from German, Brazilian, North American and Chinese markets [10, 11, 33] show that the Lévy processes based on generalized hyperbolic (GH) distributions can fit real data much better than the well-known Gaussian process for the rate of returns of financial asset prices. Among the GH processes, the normal inverse Gaussian (NIG) and the variance gamma (VG) processes are becoming more and more popular in practice, because they can fit real data better than the Gaussian process. Moreover, their sample paths are easier to obtain than more complex Lévy processes, since both of these processes can be expressed as subordinated (or time changed) Brownian motions.

A positive and increasing almost surely Lévy process $\{Y_t\}_{t\geq 0}$ is called a subordinator [6]. If $\{W_t\}_{t\geq 0}$ is a Brownian motion and $\{Y_t\}_{t\geq 0}$ is a subordinator such

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that $\{W_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ are independent of each other, then the process $\{L_t\}_{t\geq 0}$ given by

$$L_t = \theta Y_t + \sigma W_{Y_t} \tag{1.1}$$

is a subordinated Brownian motion (SBM), where θ and σ are constants. If $\{Y_t\}_{t\geq 0}$ is the inverse Gaussian (IG) process or gamma process, then $\{L_t\}_{t\geq 0}$, given by equation (1.1), is the NIG process or VG process, respectively. Let $\varphi_{Y_t}(u) = \mathbb{E}(e^{iuY_t})$ be the characteristic function of a subordinator (with $i = \sqrt{-1}$), and let $\psi_{Y_t}(u) = \log[\mathbb{E}(e^{iuY_t})]$. Then the characteristic function of the SBM L_t given by (1.1) is

$$\begin{aligned}
\varphi_{L_t}(u) = \mathbb{E}(e^{iuL_t}) &= \mathbb{E}[e^{iu(\theta Y_t+\sigma W_{Y_t})}] = \mathbb{E}\{\mathbb{E}[e^{iu(\theta Y_t+\sigma W_{Y_t})}|Y_t = y]\}\\
&= \mathbb{E}[e^{iu\theta Y_t} E(e^{i\sigma u W_y}|Y_t = y)] = \mathbb{E}[e^{iu\theta Y_t} e^{i^2\sigma^2 u^2 y/2}|Y_t = y]\\
&= \mathbb{E}[e^{i(u\theta+i\sigma^2u^2/2)Y_t}] = \varphi_{Y_t}(u\theta + \tfrac{1}{2} i\sigma^2 u^2).
\end{aligned}$$

A random variable X has the IG distribution with parameters $\delta > 0$ and $\gamma > 0$ (denoted by $X \sim \mathrm{IG}(\delta, \gamma)$), if its probability density function (PDF) is given by

$$f_{\mathrm{IG}}(x; \delta, \gamma) = \sqrt{\frac{\gamma}{2\pi x^3}} \exp\Big[-\frac{\gamma(x-\delta)^2}{2\delta^2 x}\Big] \cdot 1_{\{x\in\mathbb{R}:x>0\}}(x),$$

where $\mathbb{R}$ is the set of all real numbers, the indicator function of the set $\{x \in \mathbb{R} : x > 0\}$ is defined by $1_{\{x\in\mathbb{R}:x>0\}}(x) = 1$ if $x > 0$, and $1_{\{x\in\mathbb{R}:x>0\}}(x) = 0$, otherwise. The characteristic function of $X \sim \mathrm{IG}(\delta, \gamma)$ is given by

$$\varphi_X(u) = \exp\Big[-\delta\Big(\sqrt{\gamma^2 + 2iu} - \gamma\Big)\Big].$$

At time t, the value of an IG process $\{Y_t\}_{t\geq 0}$ with parameters $\delta > 0$ and $\gamma > 0$ has distribution $\mathrm{IG}(\delta t, \gamma)$. Thus, its characteristic function is

$$\varphi_{Y_t}(u) = \exp\Big[-\delta t\Big(\sqrt{\gamma^2 + 2iu} - \gamma\Big)\Big].$$

A random variable X has the gamma distribution with parameters $a > 0$ and $b > 0$ (denoted by $X \sim \Gamma(a, b)$), if its PDF is

$$f_\Gamma(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \cdot 1_{\{x\in\mathbb{R}:x>0\}}.$$

The characteristic function of $X \sim \Gamma(a, b)$ is given by

$$\varphi_X(u) = \Big(1 - \frac{iu}{b}\Big)^{-a}.$$

At time t, the value of a gamma process $\{Y_t\}_{t\geq 0}$ with parameters $a > 0$ and $b > 0$ has distribution $\Gamma(at, b)$. Therefore, its characteristic function is

$$\varphi_{Y_t}(u) = \Big(1 - \frac{iu}{b}\Big)^{-at}.$$

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The density of the one-dimensional NIG distribution is given by

$$f_{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu)=\frac{\alpha\delta}{\pi}\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\exp\Big[\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\Big],$$

for $x\in\mathbb{R}$, where $K_1(x)$ is the modified Bessel function of the second kind with parameter one [1] and α, β, δ and μ are model parameters for the NIG distribution. These four parameters in the model can be estimated by a maximum likelihood method or by matching the theoretical and sample moments. More details about definitions, properties and some applications in finance of subordinated Brownian motions can be found in the literature (see, for example, [6, 10, 29]).

1.2. Methods of option sensitivity estimation Consider a European option whose payoff f depends on p asset prices $S_T^{(1)},\ldots,S_T^{(p)}$ at maturity time T. Define $S_T=(S_T^{(1)},\ldots,S_T^{(p)})$ as the asset price vector at maturity. Under some risk-neutral measure, the option price or value can be expressed as (see, for example, [6] or [14])

$$V=V(S_0,K,\Sigma,r,T)=\mathbb{E}[e^{-rT}f(S_T)],$$

where $S_0=(S_0^{(1)},\ldots,S_0^{(p)})$ is the initial asset price vector, K is the strike price, $\Sigma=(\sigma_{ij})_{p\times p}$ is the covariance matrix of the asset price returns (assumed to be constant in this paper), r is the risk-free interest rate (also constant) and $f(S_T)=f(S_T,K)$ is the option payoff function. A risk-neutral measure, such as the one in the above, is unique under the Black–Scholes–Merton model for the underlying asset prices, but it usually is not unique under a model with jumps for the asset prices.

The option sensitivities or Greeks, such as

$$\Delta_1=\frac{\partial V}{\partial S_0^{(1)}},\quad \Gamma_{11}=\frac{\partial^2 V}{\partial (S_0^{(1)})^2},\quad \Gamma_{12}=\frac{\partial^2 V}{\partial S_0^{(1)}\partial S_0^{(2)}},\quad \mathcal{V}_{11}=\frac{\partial V}{\partial\sigma_{11}},$$
$$\mathcal{V}_{12}=\frac{\partial V}{\partial\sigma_{12}}\quad \rho=\frac{\partial V}{\partial r}\quad\text{and}\quad\theta=\frac{\partial V}{\partial T}\text{, and so on,}$$

are important in financial trading, hedging and risk management, where the indices i and j in Γ_{ij} refer to the sensitivity of the option price with respect to the initial asset prices $S_0^{(i)}$ and $S_0^{(j)}$, respectively, and, in $\mathcal{V}_{ij}$, the indices i and j refer to the sensitivity of the option price with respect to σ_{ij}. The values of Greeks are even harder to obtain than values of options themselves.

Conventionally, the central finite difference method is used to estimate the Greeks, since it is more efficient than the forward or backward finite difference method. For example, with the central finite difference method, Δ_j and Γ_{jk} are approximated to be (see, for example, [14, p. 379] for the case of a single asset)

$$\Delta_j=\frac{\partial V}{\partial S_0^{(j)}}\approx\frac{V(S_{0,1})-V(S_{0,2})}{2dS_0^{(j)}},\tag{1.2}$$

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where $S_0 = (S_0^{(1)}, \ldots, S_0^{(p)})$, $S_{0,q} = (S_{0,q}^{(1)}, \ldots, S_{0,q}^{(p)})$ for $q = 1, 2$,

$$S_{0,1}^{(l)} = S_0^{(j)} + dS_0^{(j)} \quad \text{for } l = j \quad \text{and} \quad S_{0,1}^{(l)} = S_0^{(l)} \quad \text{for } l \neq j,$$
$$S_{0,2}^{(l)} = S_0^{(j)} - dS_0^{(j)} \quad \text{for } l = j \quad \text{and} \quad S_{0,2}^{(l)} = S_0^{(l)} \quad \text{for } l \neq j,$$

where $dS_0^{(j)}$ is the increment in $S_0^{(j)}$. Other Greeks can be approximated similarly.

The plain or crude Monte Carlo simulation method is used to estimate option values appearing in the above expressions of Greeks. The choice of $dS_0^{(j)}$ is critical for applying the finite difference method. The value of $dS_0^{(j)}$ cannot be too small or too large. One criterion is that small changes in $dS_0^{(j)}$ will not lead to large changes in the Greeks. In many situations, the choice of $dS_0^{(j)} \approx S_0^{(j)}/1000$ will be good enough. Another very important consideration is the method for choosing the pseudorandom numbers needed for simulating expectations so as to obtain option values for each Greek by the finite difference method (see, for example, (1.2)). To obtain the best possible convergence in the simulation of the Greeks, it has been shown that one should use common pseudorandom numbers for all expectations appearing in each Greek (see the article by L'Ecuyer and Perron [22] for more details on this issue). However, the central finite difference method is difficult to apply when obtaining Greek values for options with nonsmooth payoff functions. The likelihood ratio method is useful if the density functions exist [14].

A third approach, the Malliavin calculus method, was proposed about fifteen years ago and shows advantages over the other two methods for exotic options, as demonstrated in the literature [12, 13, 24, 34]. The main idea of this method is to express the Greeks in terms of option payoff functions multiplied by weight functions that depend on Malliavin derivatives.

1.3. Elements of Malliavin calculus Here we give a very brief introduction to some basic facts on Malliavin calculus. For details, readers are referred to the book by Nualart [26]. Bally [3] and Øksendal [27] provide two other sources for an elementary introduction to Malliavin calculus. Nunno et al. introduced Malliavin calculus for Lévy processes with some applications in finance in a recent book [9].

Fix $T > 0$, $p \in \mathbb{N}$, $q \in \mathbb{N}$ and let $\{W_t = (W_t^{(1)}, \ldots, W_t^{(p)})\}_{t \geq 0}$ be a p-dimensional standard Brownian motion in $\mathbb{R}^p$. For any $t \in [0, T]$, let $\mathcal{F}_t = \sigma(Y_u : u \in [0, t])$ and $\mathcal{G}_t = \sigma(W_{Y_u} : u \in [0, t])$. Let $F = (F_1, \ldots, F_q)'$ be a random column vector in $\mathbb{R}^q$, measurable with respect to $\mathcal{G}_T = \sigma(W_{Y_t} : t \in [0, T])$. Denote the Malliavin derivative on the Gaussian space with respect to the kth component of the underlying Brownian motion by $D^{(k)}$, and define

$$D_s F_k = (D_s^{(1)} F_k, \ldots, D_s^{(p)} F_k)', \quad k = 1, \ldots, q,$$

where the symbol s refers to the *time* variable for the Malliavin derivative $\{D_s F_k : s \geq 0\}$, which is also a stochastic process (see [12] or [26]).

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Furthermore, denote the Skorohod integral [26] over $(0, Y_T]$ with respect to the kth component of the underlying Brownian motion by δ_k: that is, for a suitable smooth stochastic process $\{G_t\}_{t\geq 0}$ in $\mathbb{R}^q$,

$$\delta_k(G_\cdot) = \int_0^{Y_T} G_s \delta W_s^{(k)},$$

where the multivariate Skorohod integral is taken componentwise. The two operators D and δ are linked by the equations

$$\delta_k\langle F, G_\cdot\rangle = \langle \delta_k(G_\cdot), F\rangle - \int_0^{Y_T} \langle D_t^{(k)} F, G_t\rangle\, dt$$

and

$$\mathbb{E}\bigg[\int_0^{Y_T} (D_t^{(k)} F) G_t \, dt\bigg] = \mathbb{E}[F\delta_k(G_\cdot)]$$

for $F \in D^{1,2}$, $G \in \mathrm{Dom}(\delta)$ and $k = 1, \dots, q$.

So far in the literature, application of Malliavin calculus to Greek estimations has been mainly used for asset prices under (geometric) Brownian motions. Davis and Johansson [7] discuss simulation of Greeks of options under jump-diffusion processes for asset prices, while Kawai and Kohatsu-Higa [18] address simulation of option Greeks under subordinated Brownian motion for up to two assets, as well as density estimation in multiple dimensions. There are almost no results on option Greek simulations with Malliavin calculus combined with quasi-Monte Carlo methods reported in the literature. In this paper, our contributions are twofold. We (1) derive Greek formulas for multi-asset options under subordinated Brownian motions for European style options and (2) apply the derived formulas, combined with Monte Carlo and quasi-Monte Carlo methods, to simulate the Greeks.

The rest of this paper is organized as follows. Formulas of Greeks for multi-asset options are given in Section 2. Numerical test examples are given in Section 3 with a concluding summary in Section 4.

2. Formulas of option Greeks

Under a Lévy process model for asset prices, the market is usually no longer complete, and the equivalent martingale measure is not unique. However, such an equivalent martingale measure can be found by methods such as Esscher transform, mean-correcting martingale measure, and so on. Assume that there are $p\,(\geq 1)$ risky assets. Let $\{Y_t\}_{t\geq 0}$ be a subordinator process, and φ_{Y_t}, ψ_{Y_t}, $\mathcal{F}_t$ and $\mathcal{G}_t$ be the same as in Section 1. Using the mean-correcting martingale measure, one can derive equations for the underlying asset prices: that is,

$$\begin{aligned} S_t^{(j)} &= S_0^{(j)} \exp\bigg[rt - \psi_{Y_t}\bigg\{-i\bigg(\eta_j + \frac{1}{2}\sum_{l=1}^{p} c_{jl}^2\bigg)\bigg\} + \eta_j Y_t + \sum_{l=1}^{p} c_{jl} W_{Y_t}^{(l)}\bigg], \\ S_t^{(j)}|_{t=0} &= S_0^{(j)}, \quad j = 1, \dots, p, \end{aligned} \tag{2.1}$$

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where r is the risk-free interest rate, η_j, $j = 1, \ldots, p$ are constants, $C = (c_{jk})_{p\times p}$ is a matrix such that $\Sigma = (\sigma_{jk})_{p\times p} = CC'$ is the covariance matrix of asset price returns (that is, C can be taken as the Cholesky decomposition of Σ [15]), $\{Y_t\}_{t\geq 0}$ is a subordinator process, $\{W_t = (W_t^{(1)}, \ldots, W_t^{(p)})\}_{t\geq 0}$ is a p-dimensional standard Brownian motion (that is, $\{W_t^{(1)}\}_{t\geq 0}, \ldots, \{W_t^{(p)}\}_{t\geq 0}$ are independent one-dimensional standard Brownian motions), and

$$\left|\psi_{Y_t}\left[-i\left(\eta_j + \frac{1}{2}\sum_{l=1}^{p} c_{jl}^2\right)\right]\right| < \infty, \quad t \in [0, T],\ j = 1, 2, \ldots, p.$$

Denote $\sigma_j^2 = \sigma_{jj}$. Then $\sigma_j^2 = \sigma_{jj} = \sum_{l=1}^{p} c_{jl}^2$. If Y_t has an IG or a VG distribution with parameters $a > 0$ and $b > 0$, then equation (2.1) becomes

$$S_t^{(j)} = S_0^{(j)} \exp\left(r_j t + \eta_j Y_t + \sum_{l=1}^{p} c_{jl} W_{Y_t}^{(j)}\right) \quad \text{with } S_t^{(j)}|_{t=0} = S_0^{(j)},\ j = 1, \ldots, p. \tag{2.2}$$

Here,

$$r_j = r + a[\sqrt{b^2 - 2(\eta_j + \sigma_j^2/2)}] - b \quad \text{with } b \geq \sqrt{\max_{1\leq j\leq p} |2\eta_j + \sigma_j^2|} \quad \text{if } Y_t \sim \mathrm{IG}(at, b),$$

or

$$r_j = r + a\log[1 - (\eta_j + \sigma_j^2/2)/b] \quad \text{with } b > \max_{1\leq j\leq p}(\eta_j + \sigma_j^2/2) \quad \text{if } Y_t \sim \Gamma(at, b).$$

We concentrate on simulation of Greeks under NIG and VG models in this paper. Simulation of option Greeks under other GH models can be treated similarly, but with more effort.

Let $\delta_{ij} = 1$ for $i = j$ and zero otherwise, and define e_i to be the ith row of the $p \times p$ identity matrix. If matrix $C = (c_1', \ldots, c_p')'$ and its inverse $C^{-1} = (\beta_{ij})_{p\times p} = (\beta_1, \ldots, \beta_p)$ are partitioned into row and column vectors, respectively, then $c_i\beta_j = \delta_{ij}$ for $i, j = 1, 2, \ldots, p$. We consider the following European style option whose payoff f depends on asset prices at maturity or terminal $S_T = (S_T^{(1)}, \ldots, S_T^{(p)})$ given by

$$V = \mathbb{E}[e^{-rT} f(S_T)].$$

Theorem 2.1. *Assume that the asset prices are given by equation (2.1) and $f : \mathbb{R}^p \to \mathbb{R}$ is a measurable function such that $\mathbb{E}[f^2(S_T)]$ is locally uniformly bounded in S_0.*

(1) *If $\mathbb{E}[Y_T^{-2}] < \infty$, then for $j = 1, 2, \ldots, p$,*

$$\Delta_j = \frac{\partial V}{\partial S_0^{(j)}} = \frac{e^{-rT}}{S_0^{(j)}} \mathbb{E}\left(f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p} \beta_{lj} W_{Y_T}^{(l)}\right) \quad \textit{for } j = 1, 2, \ldots, p. \tag{2.3}$$

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(2) *If* $\mathbb{E}[Y_T^{-4}] < \infty$, *then for* $j, k = 1, 2, \ldots, p$,

$$\Gamma_{jk} = \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[f(S_T)\Big\{-\frac{\delta_{jk}}{S_0^{(j)}Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)} + \frac{1}{S_0^{(k)}Y_T^2}\Big(\Big(\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big)\Big(\sum_{p=1}^{p}\beta_{pk}W_{Y_T}^{(p)}\Big) - Y_T\sum_{l=1}^{p}\beta_{lj}\beta_{lp}\Big)\Big\}\Big]. \tag{2.4}$$

(3) *If* $\mathbb{E}[Y_T^{-2}] < \infty$, *then for* $j, k = 1, 2, \ldots, p$,

$$\widetilde{\mathcal{V}}_{jk} = \frac{\partial V}{\partial c_{jk}} = e^{-rT}\mathbb{E}\Big[f(S_T)\Big(\frac{W_{Y_T}^{(k)}}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)} - \beta_{kj}\Big)\Big]. \tag{2.5}$$

Proof. The proof of this theorem requires the order of differentiation and expectation to be exchanged. The validity of such an exchange can be verified by a similar method to that of Kawai and Kohatsu-Higa [18], and, therefore, it is omitted here. We concentrate on the derivation of formulas (2.3)–(2.5) below. By definition,

$$\Delta_j = \frac{\partial V}{\partial S_0^{(j)}} = e^{-rT}\mathbb{E}\Big[\frac{\partial f(S_T)}{\partial S_0^{(j)}}\Big] = e^{-rT}\mathbb{E}\Big[\sum_{l=1}^{d}\partial_l f(S_T)\frac{\partial S_T^{(l)}}{\partial S_0^{(j)}}\Big] = \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}[\partial_j f(S_T)S_T^{(j)}],$$

since $S_T^{(l)}$ is independent of $S_0^{(j)}$ when $l \neq j$, where $\partial_l f$ is the partial derivative of f with respect to the lth variable. Conditioning on $\mathcal{F}_T$ yields $\Delta_j = (e^{-rT}/S_0^{(j)})\mathbb{E}\{\mathbb{E}[\partial_j f(S_T)S_T^{(j)} \,|_{\mathcal{F}_T}]\}$ and, given $\mathcal{F}_T$, we have $D_u^{(k)}S_t^{(j)} = S_t^{(j)}c_{jk}1_{\{u\le Y_t\}}$ and

$$D_u S_t^{(j)} = (D_t^{(1)}S_t^{(j)}, \ldots, D_t^{(d)}S_t^{(j)}) = (S_t^{(j)}c_{j1}1_{\{u\le Y_t\}}, \ldots, S_t^{(j)}c_{jd}1_{\{u\le Y_t\}}) = S_t^{(j)}1_{\{u\le Y_t\}}c_j.$$

Moreover, for any p-variate differentiable function f,

$$D_u f(S_t) = \sum_{l=1}^{p}\partial_l f(S_t)D_u S_t^{(l)} = \sum_{l=1}^{p}\partial_l f(S_t)S_t^{(l)}1_{\{u\le Y_t\}}c_l \quad \text{for } u, t \in [0, T].$$

In particular, $D_u f(S_T) = \sum_{l=1}^{p}\partial_l f(S_T)S_T^{(l)}1_{\{u\le Y_T\}}c_l$, which yields

$$\begin{aligned}\int_0^{Y_T} D_u f(S_T)(\beta_j)\,du &= \sum_{l=1}^{p}\partial_l f(S_T)S_T^{(l)}(c_l\beta_j)\int_0^{Y_T} 1_{\{u\le Y_T\}}\,du\\ &= \sum_{l=1}^{p}\partial_l f(S_T)S_T^{(l)}\delta_{lj}\int_0^{Y_T} du = \partial_j f(S_T)S_T^{(j)}Y_T.\end{aligned}$$

Hence

$$\partial_j f(S_T)S_T^{(j)} = \int_0^{Y_T} D_u f(S_T)\frac{\beta_j}{Y_T}\,du \tag{2.6}$$

and, therefore,

$$\begin{aligned}\Delta_j &= \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}[\mathbb{E}\{\partial_j f(S_T)S_T^{(j)}\big|_{\mathcal{F}_T}\}] = \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[\mathbb{E}\Big\{\int_0^{Y_T} D_u f(S_T)\frac{\beta_j}{Y_T}\,du\Big|_{\mathcal{F}_T}\Big\}\Big] \\ &= \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[\mathbb{E}\Big\{f(S_T)\delta\Big(\frac{\beta_j}{Y_T}\Big)\Big|_{\mathcal{F}_T}\Big\}\Big] = \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[f(S_T)\delta\Big(\frac{\beta_j}{Y_T}\Big)\Big].\end{aligned}$$

Now, for given $\mathcal{F}_T$,

$$\delta\Big(\frac{\beta_j}{Y_T}\Big) = \frac{1}{Y_T}\sum_{l=1}^{p}\int_0^{Y_T}\beta_{lj}\,dW_t^{(l)} = \frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}$$

and so

$$\begin{aligned}\Delta_j &= \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[\mathbb{E}\Big\{f(S_T)\delta\Big(\frac{\beta_j}{Y_T}\Big)\Big|_{\mathcal{F}_T}\Big\}\Big] \\ &= \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[\mathbb{E}\Big\{f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big|_{\mathcal{F}_T}\Big\}\Big] = \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big(f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big),\end{aligned}$$

which proves (2.3). To prove (2.4), notice that $(1/Y_T)\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}$ is independent of $S_0^{(k)}$, so

$$\begin{aligned}\Gamma_{jk} &= \frac{\partial^2 V}{\partial S_0^{(j)}\partial S_0^{(k)}} = \frac{\partial\Delta_j}{\partial S_0^{(k)}} \\ &= -\frac{e^{-rT}\delta_{jk}}{(S_0^{(j)})^2}\mathbb{E}\Big[f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big] + \frac{e^{-rT}}{S_0^{(j)}}\mathbb{E}\Big[\frac{\partial f(S_T)}{\partial S_0^{(k)}}\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big] \\ &= -\frac{e^{-rT}\delta_{jk}}{(S_0^{(j)})^2}\mathbb{E}\Big[f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big] + \frac{e^{-rT}}{S_0^{(j)}S_0^{(k)}}\mathbb{E}\Big[\mathbb{E}\Big\{\partial_k f(S_T)S_{kT}\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big|_{\mathcal{F}_T}\Big\}\Big] \\ &= -\frac{e^{-rT}\delta_{jk}}{(S_0^{(j)})^2}\mathbb{E}\Big[f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big] \\ &\quad + \frac{e^{-rT}}{S_0^{(j)}S_0^{(k)}}\mathbb{E}\Big[\mathbb{E}\Big\{\int_0^{Y_T} D_u f(S_T)\frac{\beta_k}{Y_T}\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\,du\,|_{\mathcal{F}_T}\Big\}\Big] \\ &= -\frac{e^{-rT}\delta_{jk}}{(S_0^{(j)})^2}\mathbb{E}\Big[f(S_T)\frac{1}{Y_T}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big] + \frac{e^{-rT}}{S_0^{(j)}S_0^{(k)}}\mathbb{E}\Big[\mathbb{E}\Big\{f(S_T)\delta\Big(\frac{\beta_k}{{Y_T}^2}\sum_{l=1}^{p}\beta_{lj}W_{Y_T}^{(l)}\Big)\Big|_{\mathcal{F}_T}\Big\}\Big].\end{aligned} \tag{2.7}$$

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For given $\mathcal{F}_T$,

$$\begin{aligned}
\delta\Big[\frac{\beta_k}{Y_T}\Big(\frac{1}{Y_T}\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\Big] &= \frac{1}{Y_T^2}\Big[\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\delta(\beta_k) - \int_0^{Y_T} D_u\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\beta_k\,du\Big] \\
&= \frac{1}{Y_T^2}\Big[\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\Big(\sum_{p=1}^{d}\beta_{pk}W_{pY_T}\Big) \\
&\quad - \int_0^{Y_T}\Big\{D_u^{(1)}\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big),\ldots,D_u^{(d)}\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\Big\}\beta_k\,du\Big] \\
&= \frac{1}{Y_T^2}\Big[\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\Big(\sum_{p=1}^{d}\beta_{pk}W_{pY_T}\Big) \\
&\quad - \int_0^{Y_T}(\beta_{1j}1_{\{u\le Y_T\}},\ldots,\beta_{dj}1_{\{u\le Y_T\}})\beta_k\,du\Big] \\
&= \frac{1}{Y_T^2}\Big[\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\Big(\sum_{p=1}^{d}\beta_{pk}W_{pY_T}\Big) - \int_0^{Y_T}\beta_j'\beta_k 1_{\{u\le Y_T\}}\,du\Big] \\
&= \frac{1}{Y_T^2}\Big[\Big(\sum_{l=1}^{d}\beta_{lj}W_{lY_T}\Big)\Big(\sum_{p=1}^{d}\beta_{pk}W_{pY_T}\Big) - Y_T\sum_{l=1}^{d}\beta_{lj}\beta_{lp}\Big]. \qquad (2.8)
\end{aligned}$$

When equation (2.8) is substituted into (2.7), we obtain the required expression for Γ_{jk} in (2.4). For equation (2.5), since $S_T^{(l)} = S_0^{(l)}\exp(\mu_l Y_T + \sum_{q=1}^{p} c_{lq}W_{Y_T}^{(q)})$, for given $\mathcal{F}_T$, we obtain $\partial S_T^{(l)}/\partial c_{jk} = S_T^{(l)}W_{Y_T}^{(k)}\delta_{lj}$. Thus,

$$\widetilde{\mathcal{V}}_{jk} = \frac{\partial V}{\partial c_{jk}} = e^{-rT}\mathbb{E}\Big[\sum_{l=1}^{p}\partial_l f(S_T)\frac{\partial S_T^{(l)}}{\partial c_{jk}}\Big] = e^{-rT}\mathbb{E}[\mathbb{E}\{\partial_j f(S_T)S_T^{(j)}W_{Y_T}^{(k)}|_{\mathcal{F}_T}\}]. \qquad (2.9)$$

Similarly to equation (2.6), for given $\mathcal{F}_T$,

$$\partial_j f(S_T)S_T^{(j)}W_{Y_T}^{(k)} = \int_0^{Y_T} D_u f(S_T)\frac{W_{Y_T}^{(k)}\beta_j}{Y_T}\,du,$$

and hence,

$$\mathbb{E}[\partial_j f(S_T)S_T^{(j)}W_{Y_T}^{(k)}] = \mathbb{E}\Big[\int_0^{Y_T} D_u f(S_T)\frac{W_{Y_T}^{(k)}\beta_j}{Y_T}\,du\Big] = \mathbb{E}\Big[f(S_T)\delta\Big(\frac{W_{Y_T}^{(k)}\beta_j}{Y_T}\Big)\Big], \qquad (2.10)$$

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and

$$\delta\Big(\frac{W^{(k)}_{Y_T}\beta_j}{Y_T}\Big) = \frac{1}{Y_T}\delta(W^{(k)}_{Y_T}\beta_j) = \frac{1}{Y_T}\Big[W^{(k)}_{Y_T}\delta(\beta_j) - \int_0^{Y_T}(D_u W^{(k)}_{Y_T})\beta_j\,du\Big]$$
$$= \frac{1}{Y_T}\Big[W^{(k)}_{Y_T}\sum_{l=1}^{p}\beta_{lj}W^{(l)}_{Y_T} - \int_0^{Y_T}(D^{(1)}_u W^{(k)}_{Y_T},\ldots,D^{(p)}_u W^{(k)}_{Y_T})\beta_j\,du\Big]$$
$$= \frac{1}{Y_T}\Big[W^{(k)}_{Y_T}\sum_{l=1}^{p}\beta_{lj}W^{(l)}_{Y_T} - \int_0^{Y_T}(\delta_{1k}1_{\{u\le Y_T\}},\ldots,\delta_{pk}1_{\{u\le Y_T\}})\beta_j\,du\Big]$$
$$= \frac{1}{Y_T}\Big[W^{(k)}_{Y_T}\sum_{l=1}^{p}\beta_{lj}W^{(l)}_{Y_T} - \Big(\int_0^{Y_T}1_{\{u\le Y_T\}}\,du\Big)e'_k\beta_j\Big]$$
$$= \frac{1}{Y_T}\Big[W^{(k)}_{Y_T}\sum_{l=1}^{p}\beta_{lj}W^{(l)}_{Y_T} - Y_T\beta_{kj}\Big] = \frac{W^{(k)}_{Y_T}}{Y_T}\sum_{l=1}^{p}\beta_{lj}W^{(l)}_{Y_T} - \beta_{kj}. \tag{2.11}$$

Thus equation (2.5) follows from the equations (2.9)–(2.11), which completes the proof of the theorem. □

REMARK 2.1. In the rest of the paper, we assume that Σ has the Cholesky decomposition, $\Sigma = CC'$, where $C = (c_{ij})_{p\times p}$ is a lower triangular matrix, and

$$c_{11} = \sqrt{\sigma_{11}}, \quad c_{i1} = \frac{\sigma_{i1}}{\sqrt{\sigma_{11}}}, \quad 2 \le i \le p;$$
$$c_{ii} = \sqrt{\sigma_{ii} - \sum_{k=1}^{i-1} c_{ik}^2}, \quad c_{ij} = \frac{\sigma_{11} - \sum_{k=1}^{i-1} c_{ik}c_{jk}}{c_{ii}}, \quad 2 \le j \le p,\ j+1 \le i \le p;$$
$$c_{ij} = 0,\ i < j.$$

Thus, from the above expressions, $\mathcal{V}_{j_0k_0}$ of the option is given by

$$\mathcal{V}_{j_0k_0} = \frac{\partial V}{\partial \sigma_{j_0k_0}} = \frac{\partial V}{\partial c_{j_0k_0}}\frac{\partial c_{j_0k_0}}{\partial \sigma_{j_0k_0}} = \widetilde{\mathcal{V}}_{j_0k_0}\frac{\partial c_{j_0k_0}}{\partial \sigma_{j_0k_0}}.$$

3. Numerical tests

In this section, we simulate the Greeks by both a finite difference method and the Malliavin calculus method combined with the Monte Carlo and the quasi-Monte Carlo methods, and compare their performances.

The Monte Carlo (MC) simulation method has been widely used in financial applications since Boyle [4] first proposed it in 1977. It is the main method used to handle problems with high dimensions (>3). Its main drawback is that computation is slow, sometimes even after using variance reduction techniques. To overcome this, people have applied the quasi-Monte Carlo (QMC) methods (also called low-discrepancy sequence (LDS) methods), where deterministic or quasirandom sequences are used, instead of pseudorandom ones. There are, basically, has two classes of LDS [8, 21, 25]: digital net sequences, such as Halton's sequence, Sobol's sequence, Faure's sequence, Niederreiter's (t, m, s)-nets and (t, s)-sequences; and integration lattice sequences, such as good lattice points (GLP).

The asymptotic convergence rate of the error of a p-dimensional integration is $O((\log N)^p/N)$ when a digital net LDS is applied, whereas it is $O(1/\sqrt{N})$ when a pseudorandom number sequence is used, where N is the number of points or sample size or replications, and p is the dimension of the problem. The asymptotic convergence rate of the error is even better when a GLP is applied to estimate an integral with smooth and periodic integrand: that is, it is $O((\log N)^{\alpha p}/N^{\alpha})$, where $\alpha\,(> 1)$ is a parameter related to the smoothness of the integrand. Niederreiter's book [25] is an excellent reference for QMC methods. Dick and Pillichshammer [8] provide more updated information on QMC methods, whereas Hua and Wang [16] and Sloan and Joe [30] discuss GLP and lattice rules with rank $r\ (\geq 1)$. Applications of GLP and lattice rules with rank $r\ (\geq 1)$ to option pricing and sensitivity can be found in the work of Boyle et al. [5] and Lai [19].

To compare the efficiencies of different methods, we need a benchmark for fair comparisons. If the exact value of the quantity to be estimated is known, then we could use the absolute error or relative error for comparison. Otherwise, we use $\sigma_N^2 \cdot t = N\delta_N^2 \cdot t$ for comparison, where σ_N^2 is the unbiased sample variance, $\delta_N = \sigma_N/\sqrt{N}$ is the standard error and t is the simulation time.

For a given LDS sequence, following Tufflin [31], a randomized LDS sequence produced by introducing random shift (probably the simplest randomized QMC (RQMC) method), can be defined as follows. Assume that we estimate $\mu = E[f(X)]$, where X is a p-dimensional random vector. Let $\{X_i\}_{i=1}^m \subset [0, 1]^p$ be a finite LDS sequence and $\{R_j\}_{j=1}^n \subset [0, 1]^p$ be a finite sequence of random vectors. For each fixed j, we form a sequence $\{Y_{i,j}\}_{i=1}^m$ with $Y_{i,j} = X_i + R_j \pmod 1$ componentwise. The operation of $x = y \pmod 1$ is to take the fractional part of the number $y \geq 0$, so that $x \in [0, 1]$. When such a randomized LDS is applied, the integration error still has the same asymptotic convergence rate as the error does when the original LDS is applied (see [31]). Define

$$\mu_j = \frac{1}{m}\sum_{i=1}^{m} f(Y_{i,j}) \quad \text{and} \quad \overline{\mu} = \frac{1}{n}\sum_{j=1}^{n}\mu_j.$$

Then the unbiased sample variance is

$$\overline{\sigma}^2 = \frac{\sum_{j=1}^n(\mu_j - \overline{\mu})^2}{n-1} = \frac{n\sum_{j=1}^n \mu_j^2 - (\sum_{j=1}^n \mu_j)^2}{(n-1)n}$$

and the standard error of the randomized LDS is defined by $\delta = \overline{\sigma}/\sqrt{n}$. The variance of the MC method is calculated as usual with mn points. When two methods are used to simulate the same quantity with the same sample size, one way to measure the

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efficiency of the second method over the first one is defined as in [28] to be

$$\text{efficiency} = \frac{\sigma_1^2 \cdot t_1}{\sigma_2^2 \cdot t_2},$$

where σ_1^2 and t_1 are the variance and time used for the first method, respectively, and, similarly, for σ_2^2 and t_2 of the second method. Other definitions of efficiencies can be found in the literature [2], and the efficiency here is the ratio of the efficiencies, based on other definitions. The variance reduction ratio (VRR) is σ_1^2/σ_2^2. If the efficiency is greater than one, then the second method is regarded as more efficient than the first. If the times used by the two methods are close, then the efficiency is close to the variance reduction ratio.

In this paper, all the programs are run under Windows XP using MATLAB (version 7.0.4), the computer is a TOSHIBA Satellite Pro laptop with Intel Core 2 Duo CPU @ 2.10 GHz, 2.00 GB of RAM. The pseudorandom number generator used is the built-in generator (that is, *rand*) provided by MATLAB. Inverse Gaussian random variates can be generated by the method proposed by Michael et al. [23] (denote this as the MSH method). For low-discrepancy sequences, Sobol's sequence is used, since it usually gives the best performance out of the available digital net low-discrepancy sequences that can be implemented. We use the direction numbers of Joe and Kuo [17], so that our implementation of Sobol's sequence can generate points of dimension as high as 1111. We use the approximation method of Wichurat [32] to generate the standard normal random variate. This method achieves 16-digit accuracy, as claimed.

Note that we do not simulate Greeks by a finite difference (FD) method combined with low-discrepancy sequences in order to save time. A fairer comparison would be between two different methods combined with the same type of sequence with the same number of points or sample size. We conjecture that, although the FD method combined with an LDS will be more efficient than the central finite difference method combined with MC, it is still less efficient than the Malliavin calculus method combined with the same type of LDS.

We consider two examples: basket type down-and-out (DNO) and corridor options under *multivariate NIG and VG* models. That is, the asset prices are given by (2.2), where

$$r_j = r + a\sqrt{b^2 - 2(\eta_j + \sigma_j^2/2)} - b \quad \text{with } b \geq \sqrt{\max_{1\leq j\leq d}(|2\eta_j + \sigma_j^2|)} \quad \text{if } Y_t \sim \mathrm{IG}(at, b),$$

or

$$r_j = r + a\log[1 - (\eta_j + \sigma_j^2/2)/b] \quad \text{with } b > \max_{1\leq j\leq d}(\eta_j + \sigma_j^2/2) \quad \text{if } Y_t \sim \Gamma(at, b).$$

We have taken the parameter values as follows: $r = 0.1$, $T = 1$, $d = 6$, $S_{j0} = 50 + 5 \times (j-1)$, $dS_{j0} = 0.003 \times S_{j0}$ for $1 \leq j \leq d$; $\sigma_j = 0.2$ for $j \leq 3$ and $\sigma_j = 0.3$ for $4 \leq j \leq d$, $\rho_{jk} = 0.6$ for $1 \leq j, k \leq 3$ or $4 \leq j, k \leq d$, $\rho_{jk} = -0.4$, otherwise, $\sigma_{jk} = \rho_{jk}\sigma_j\sigma_k$ for $1 \leq j, k \leq d$; the number of random shifts $n = 10$. The choice of $n = 10$ may be a little

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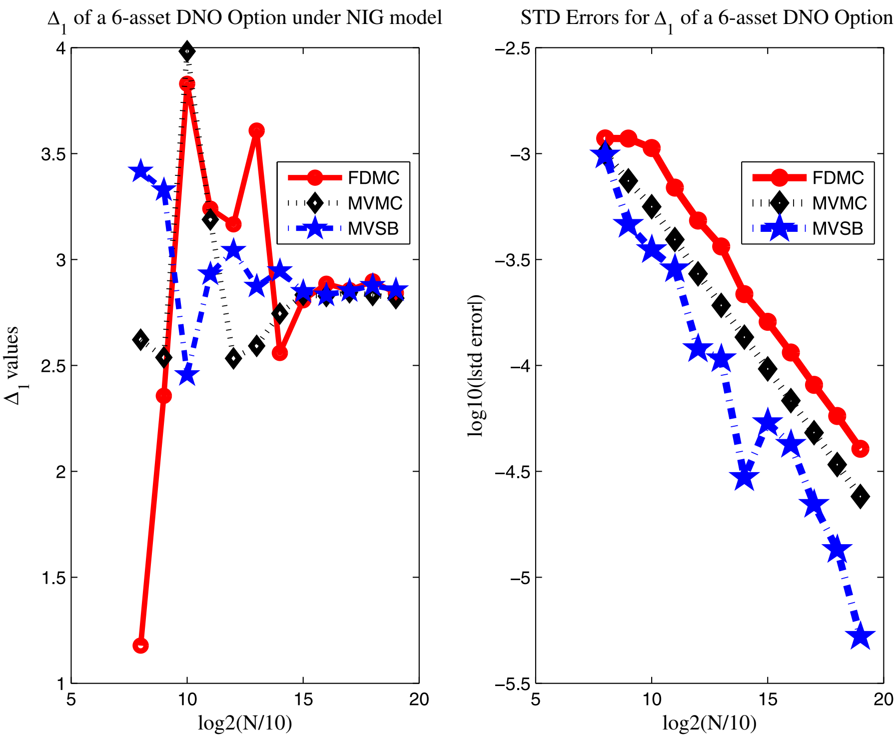

FIGURE 1. Comparison of simulations of Δ_1 of a 6-dim. basket type DNO option under NIG model with $K = \min(S_0)$ and $\eta_0 = 0.3$.

TABLE 1. Comparison of simulation efficiencies and VRRs of Δ_1 and Γ_{11} for a basket type DNO option with 6 assets under NIG model by finite difference and Malliavin calculus combined with MC and QMC. Notice that to shorten the width of the table, the following notations are used: FD for FDMC, MC for MVMC and SB for MVSB. The same comment applies to Table 2.

Efficiencies				VRR			
Δ		Γ_{11}		Δ		Γ_{11}	
FD/MC	FD/SB	FD/MC	FD/SB	FD/MC	FD/SB	FD/MC	FD/SB
$K = \min(S_0)$							
11.7	16.2	7972.2	5006.5	2.8	17.6	1810.8	5721.1
$K = \text{mean}(S_0)$							
16.1	13.9	16 732.6	7661.2	3.7	15.0	4100.1	9201.4
$K = \max(S_0)$							
22.7	15.5	23 154.5	11 431.3	5.2	17.2	5882.4	14 830.8

noisy for other cases, but replacing larger values of n will not change the conclusion that the RQMC method outperforms the MC method, in general, when both methods are applied to derivative pricing and Greek estimations. In our simulations, the sample

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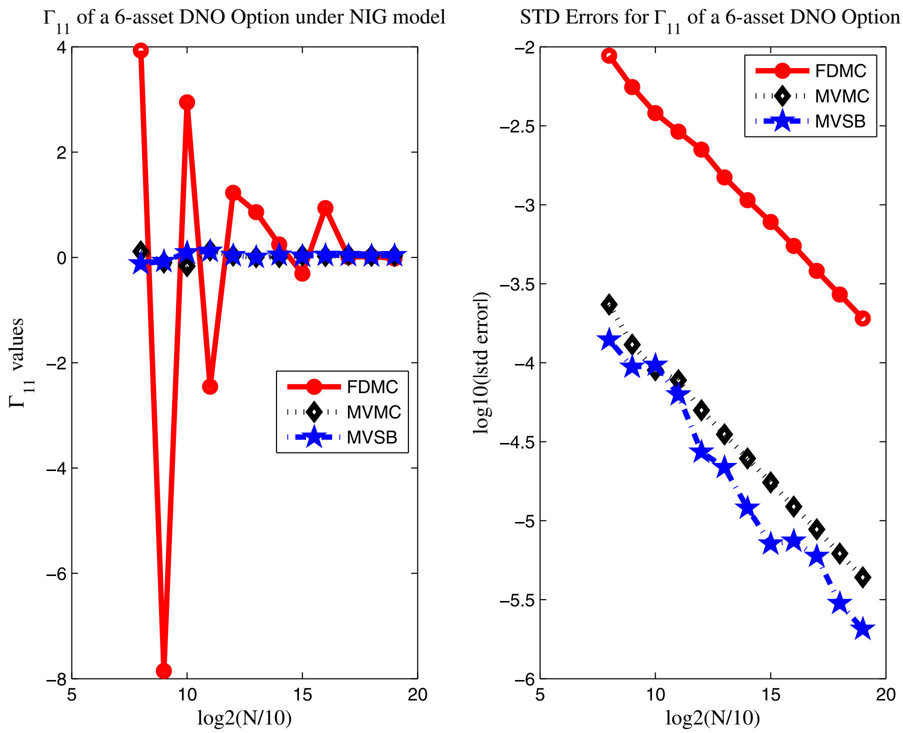

FIGURE 2. Comparison of simulations of Γ_{11} of a 6-dim. basket type DNO option under NIG model with $K = \min(S_0)$ and $\eta_0 = 0.3$.

sizes for the MC method are 2560, 5120, 10 240, 20 480, 40 960, 81 920, 163 840, 327 680, 655 360, 1 310 720, 2 621 440, and 5 242 880; and those for Sobol's sequence are 256, 512, 1024, 2048, 4096, 8192, 16 384, 32 768, 65 536, 131 072, 262 144, and 524 288, each with 10 random shifts. Values of other parameters are specified in the examples. Symbols used in the following tables are explained as follows. The central finite difference method with pseudorandom sequence is denoted by FDMC, the Malliavin calculus method with plain MC is denoted by MVMC and the Malliavin calculus method with Sobol's sequence is denoted by MVSB, using the formulas given in Theorem 2.1. Due to limited space, only some of the numerical results for a basket type DNO option under both NIG and VG models are presented below. Results for a basket type corridor option under NIG and VG are totally omitted here for the same reason. The detailed numerical results are available on request.

EXAMPLE 3.1. Simulation of Greeks of a basket type DNO option under the *multivariate NIG model*. The option has payoff

$$f(S_T, K) = 1_{\{S_T : \sum_{j=1}^{d} \alpha_j S_{jT} \geq K\}}(S_T).$$

The asset prices follow (2.2), where $Y_t \sim \mathrm{IG}(at, b)$. Some parameter values are taken as follows: $\alpha_j = 1/d$, $a = 1$, $b = \max_{1\leq j\leq d}(\sqrt{|2\eta_j + \sigma_j^2|})$, $\eta_j = \eta_0 + 0.005(j-1)$,

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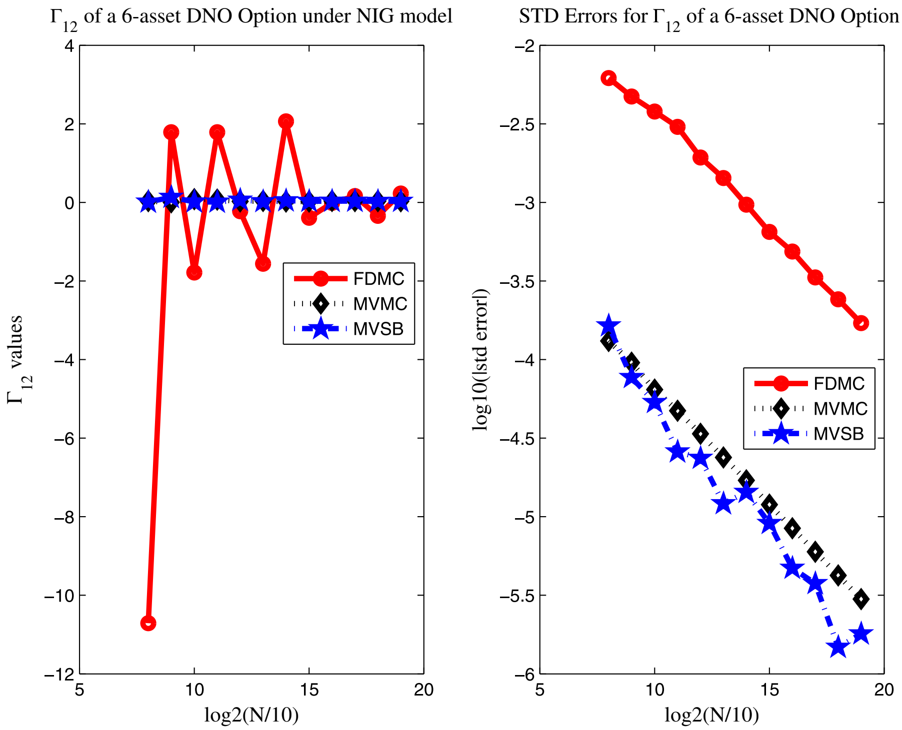

FIGURE 3. Comparison of simulations of Γ_{12} of a 6-dim. basket type DNO option under NIG model with $K = \min(S_0)$ and $\eta_0 = 0.3$.

TABLE 2. Comparison of simulation efficiencies and VRRs of Δ_1 and Γ_{11} for a basket type DNO option with 6 assets under VG model by finite difference and Malliavin calculus combined with MC and QMC.

Efficiencies				VRR			
Δ		Γ_{11}		Δ		Γ_{11}	
FD/MC	FD/SB	FD/MC	FD/SB	FD/MC	FD/SB	FD/MC	FD/SB
$K = \min(S_0)$							
1.9	4.2	711.6	488.0	0.4	4.6	156.9	596.1
$K = \text{mean}(S_0)$							
9.2	4.9	4548.9	2311.8	2.0	5.7	1082.5	2852.7
$K = \max(S_0)$							
14.1	5.6	10 432.1	3826.0	3.0	6.2	2279.2	4667.1

$\eta_0 \in \{-0.2, 0, 0.3\}$, $K \in \{\min(S_0), \text{mean}(S_0), \max(S_0)\}$, where $\min(S_0)$, $\text{mean}(S_0)$ and $\max(S_0)$ are the minimum, mean and maximum of the coordinates of S_0, respectively. Comparisons of simulations for Δ_1, Γ_{11} and Γ_{12} with $\eta_0 = 0.3$ are displayed in Figures 1–3 and Table 1. Each efficiency (or VRR) in Table 1 is the average of efficiencies (or VRRs) over 12 sample sizes listed above (the same comment applies to Table 2).

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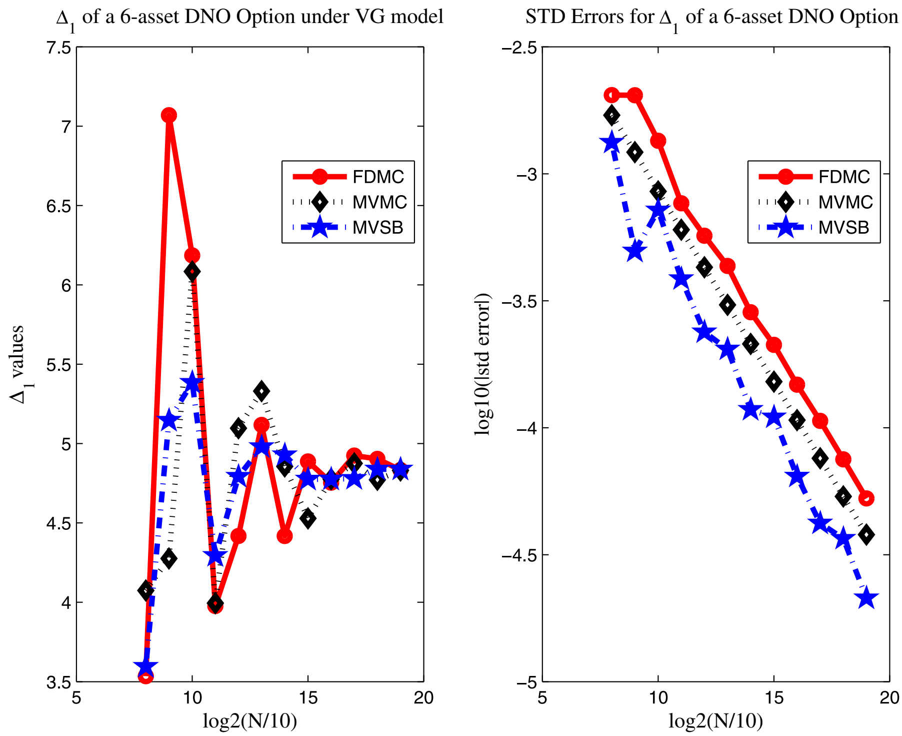

FIGURE 4. Comparison of simulations of Δ_1 of a 6-dim. basket type DNO option under VG model with $K = \text{mean}(S_0)$ and $\mu_0 = 0.3$.

Figures with $K = \text{mean}(S_0)$ and $\max(S_0)$, $\eta_0 = -0.2$ and 0 are quite similar, and so they are omitted here to save space.

When simulating Δ_1, from Figure 1 and Table 1, we observe that the Malliavin calculus (MVC) method, no matter what kind of sequence is used, is more efficient than the FDMC method. The MVSB method is the most efficient out of these three methods if we only compare variances or standard errors without considering CPU times, whereas the MVMC method is more efficient than the MVSB method most of the time if CPU times are also considered. This is because MVMC is faster than MVSB. Notice that this may change depending on how the Sobol's sequence is implemented. The efficiencies or VRRs are no more than 20 in this case. When simulating Γ_{11} and Γ_{12} for this option under the *multivariate NIG model*, from Figures 2, 3 and Table 1, we obtain very similar conclusions as in the case of simulation of Δ_1, except that the efficiencies or VRRs are larger (both up to more than ten thousand).

EXAMPLE 3.2. Simulation of Greeks of a basket type DNO option under the *multivariate VG model*. The option has the same payoff as in Example 3.1. The

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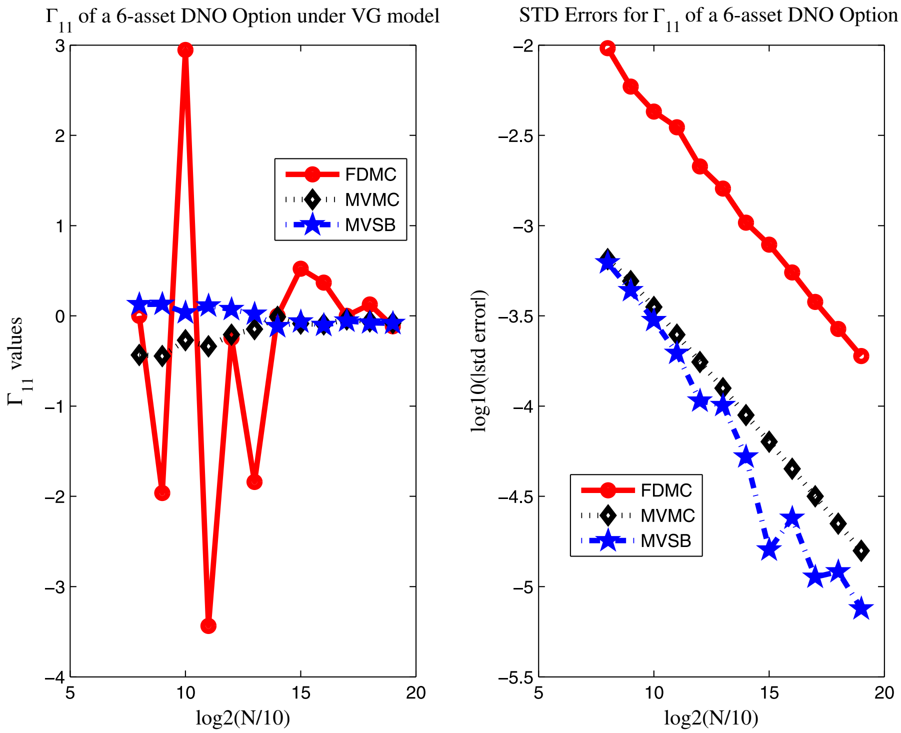

FIGURE 5. Comparison of simulations of Γ_{11} of a 6-dim. basket type DNO option under VG model with $K = \min(S_0)$ and $\mu_0 = 0.3$.

asset prices follow the multivariate VG model (2.2), where $Y_t \sim \Gamma(at, b)$ with $a, b > 0$. Some parameters are taken as follows: $\alpha_j = 1/d$, $r_j = r - \omega_j$, $\omega_j = -(1/\nu)\ln(1 - \eta_j\nu - \sigma_j^2\nu/2)$, $\eta_j = \eta_0 + 0.01 * j$, $\eta_0 \in \{-0.2, 0, 0.3\}$, $\nu = 0.3$. Comparisons of simulations for Δ_1 and Γ_{11} are displayed in Figures 4, 5, and Table 2, from which we reach a similar conclusion to that of Example 3.1, except that the efficiencies or VRRs are smaller.

4. Summary

We have derived Greek formulas of multi-asset European style options under subordinated Brownian motions. The numerical results show that the Malliavin calculus method is more efficient than the finite difference method for exotic options with nonsmooth payoffs. The superiority of the first method over the second is even more significant when the quasi-Monte Carlo method based on the Sobol's sequence is used. The efficiencies reach up to more than ten thousand times in six-asset option cases when simulating the Γ. Further improvements in efficiency would be expected, if variance reduction methods, localization techniques (see for example, [12]) and GPU (graphical processing unit) computational skills were to be used. These ideas may motivate some research topics in near future. It would also be interesting to compare

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the results obtained by the Malliavin calculus method with the method proposed by L' Ecuyer [20], if the latter is applicable to the problems discussed in this paper.

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