Nagoya Math. J., **230** (2018), 160–179
DOI 10.1017/nmj.2017.7

THE PROJECTIVE DIMENSION OF THE EDGE IDEAL OF A VERY WELL-COVERED GRAPH

KYOUKO KIMURA, NAOKI TERAI AND SIAMAK YASSEMI

Abstract. A very well-covered graph is an unmixed graph whose covering number is half of the number of vertices. We construct an explicit minimal free resolution of the cover ideal of a Cohen–Macaulay very well-covered graph. Using this resolution, we characterize the projective dimension of the edge ideal of a very well-covered graph in terms of a pairwise 3-disjoint set of complete bipartite subgraphs of the graph. We also show nondecreasing property of the projective dimension of symbolic powers of the edge ideal of a very well-covered graph with respect to the exponents.

§1. Introduction

Let G be a finite simple graph. We denote $V = V(G)$ the vertex set of G and $E(G)$ the edge set of G. Let K be a field and $S = K[V]$ the polynomial ring whose variables are identified with the vertices of G. We consider the standard (multi-)grading on $K[V]$. We can associate with G the ideal of S:

$$I(G) := (x_i x_j : \{x_i, x_j\} \in E(G)).$$

The ideal $I(G)$ is called the *edge ideal* of G. Let $J(G)$ be the Alexander dual ideal of $I(G)$. Actually, $J(G)$ is the *cover ideal* of G, the ideal generated by all monomials which are products of the vertices of minimal vertex covers of G. The main theme of the study of these ideals is to investigate the relations between the ring properties of $I(G)$ and $J(G)$ and the combinatorics of G. We are interested in characterizing homological invariants of these ideals. There are some results about this direction; see for example, [3, 5, 7–13, 19, 21, 22] and the references therein.

A subset $C \subset V$ is called a *vertex cover* of G if $C \cap e \neq \emptyset$ for each $e \in E(G)$. A vertex cover is said to be minimal if it has no proper subset which is also a vertex cover. A graph G is called unmixed if all minimal vertex covers of G have the same cardinality. When G is unmixed and has no isolated vertex, it is known that $2\,\mathrm{height}\, I(G) \geqslant \#V$. When the equality holds, G

Received November 12, 2015. Revised December 20, 2016. Accepted March 7, 2017.
2010 Mathematics subject classification. Primary 13D02, 13F55; Secondary 05C99.

is called *very well-covered*. Note that the class of very well-covered graphs contains the class of unmixed bipartite graphs with no isolated vertex. Also the class of unmixed bipartite graphs contains the class of Cohen–Macaulay bipartite graphs. Here we say that a graph G is Cohen–Macaulay if the quotient ring $S/I(G)$ is Cohen–Macaulay.

In general, it is hard to construct an explicit minimal free resolution of an ideal. But Herzog and Hibi [6] succeeded in constructing a resolution of $J(G)$ when G is a Cohen–Macaulay bipartite graph. Also Mohammadi and Moradi [12] investigated a resolution of $J(G)$ when G is an unmixed bipartite graph. The first main result of the present paper is a construction of an explicit minimal free resolution of $J(G)$ when G is a Cohen–Macaulay very well-covered graph (Theorem 3.2).

Using our first main result, we derive the characterization of the projective dimension of $S/I(G)$ over S, denoted by $\operatorname{pd} S/I(G)$ when G is a very well-covered graph.

Two edges $\{x_1, y_1\}, \{x_2, y_2\} \in E(G)$ are said to be 3-disjoint in G if there is no other edge in G between vertices x_1, x_2, y_1, y_2. Let $\mathcal{B} = \{B_1, \ldots, B_r\}$ be a set of complete bipartite subgraphs of G. We set $V(\mathcal{B}) = V(B_1) \cup \cdots \cup V(B_r)$. We say $\mathcal{B}$ is pairwise 3-disjoint if $V(B_k) \cap V(B_\ell) = \emptyset$ for any $k \neq \ell$ and there exists $e_k \in E(B_k)$ for each $k = 1, \ldots, r$ such that $e_1, \ldots, e_r$ are pairwise 3-disjoint.

THEOREM 1.1. *Let G be a very well-covered graph. Then*

$$\operatorname{pd} S/I(G) = \max\left\{\#V(\mathcal{B}) - r : \begin{array}{l} \mathcal{B} = \{B_1, \ldots, B_r\} \textit{ is a pairwise 3-disjoint set} \\ \textit{of complete bipartite subgraphs of } G \end{array}\right\}.$$

Herzog and Hibi [6] characterized Cohen–Macaulay bipartite graphs in terms of the original graph. Crupi *et al.* [2] expanded their result for very well-covered graphs. When G is an unmixed bipartite graph, Kummini [10, Proposition 3.2] gave a combinatorial characterization of $\operatorname{pd} S/I(G)$. Later Kimura[9, Theorem 7.1] translated his result in terms of G and Theorem 1.1 is a generalization of it. The characterization for the regularity of $I(G)$, denoted by $\operatorname{reg} I(G)$ has already been done; Kummini [10] studied when G is an unmixed bipartite graph and later Mahmoudi *et al.* [11] generalized Kummini's result for a very well-covered graph G. Theorem 1.1 stands the Alexander dual version of their generalization since $\operatorname{pd} S/I(G) = \operatorname{reg} J(G)$; see [18].

In this article we also treat the projective dimension of symbolic powers of the edge ideal of a very well-covered graph. The following problem is widely open:

PROBLEM 1.2. Let G be a graph. Then is it true that

$$\operatorname{pd} S/I(G)^{(i)} \geqslant \operatorname{pd} S/I(G)^{(i-1)}$$

for $i \geqslant 2$?

We give a partial affirmative answer for the case that G is a very well-covered graph. We also point out that the corresponding result also holds for Stanley depth.

Now we explain the organization of the paper. In Section 2, we recall the structure of a very well-covered graph which was given by Crupi *et al.* [2] and the association a very well-covered graph with semidirected graph by Mahmoudi *et al.* [11]. In Section 3, we construct an explicit minimal free resolution of $J(G)$ for a Cohen–Macaulay very well-covered graph G and in Section 4, we prove Theorem 1.1. In Section 5, we show that the projective dimension of symbolic powers of the edge ideal of a very well-covered graph is nondecreasing.

§2. The structure of very well-covered graphs

In this section, we recall the structure of very well-covered graphs. Almost all of the results in this section are proved by Crupi *et al.* [2] and Mahmoudi *et al.* [11].

Let G be a finite simple graph with no isolated vertex. If G is unmixed, then $2 \operatorname{height} I(G) \geqslant \# V(G)$ is known. A graph G is called *very well-covered* if G is unmixed with $2 \operatorname{height} I(G) = \# V(G)$.

A subset $W \subset V(G)$ is called an independent set if there is no edge of G between any two vertices in W. An independent set is said to be maximal if it is maximal among independent sets of G.

A very well-covered graph has the following structure.

THEOREM 2.1. [2, Proposition 2.3] *Let G be a very well-covered graph with height h. Then there is a relabeling of vertices $V(G) = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ such that the following conditions are satisfied:*

(i) $X = \{x_1, \ldots, x_h\}$ *is a minimal vertex cover of G and* $Y = \{y_1, \ldots, y_h\}$ *is a maximal independent set of G;*
(ii) $\{x_k, y_k\} \in E(G)$ *for* $k = 1, \ldots, h$*;*

(iii) *if* $\{z_i, x_j\}, \{y_j, x_k\} \in E(G)$, *then* $\{z_i, x_k\} \in E(G)$ *for distinct* i, j, k *and for* $z_i \in \{x_i, y_i\}$;
(iv) *if* $\{x_i, y_j\} \in E(G)$, *then* $\{x_i, x_j\} \notin E(G)$.

On the other hand, the graph G on $V = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ with (i), (ii), (iii), (iv) in Theorem 2.1 is a very well-covered graph.

Cohen–Macaulay very well-covered graphs have the following additional property.

THEOREM 2.2. [2, Theorem 3.6] *Let* G *be a very well-covered graph with height* h. *Then* G *is Cohen–Macaulay if and only if there is a relabeling of vertices* $V(G) = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ *with (i), (ii), (iii), (iv) of Theorem 2.1 and the following property:*

(v) *if* $\{x_i, y_j\} \in E(G)$, *then* $i \leqslant j$.

Now we associate a very well-covered graph G with a semidirected graph $\mathfrak{d}_G$ as in [11]. We recall the notion of a semidirected graph. A semidirected graph $\mathfrak{d}$ consists of the vertex set $V(\mathfrak{d}) = \{p_1, \ldots, p_h\}$, the set of undirected edges $E_u(\mathfrak{d})$, and the set of directed edges $E_d(\mathfrak{d})$, where if $p_ip_j \in E_d(\mathfrak{d})$, then $\{p_i, p_j\} \notin E_u(\mathfrak{d})$. We say $A \subset V(\mathfrak{d})$ is *independent* if $\{p_i, p_j\} \notin E_u(\mathfrak{d})$ and $p_ip_j, p_jp_i \notin E_d(\mathfrak{d})$ for any $p_i, p_j \in A$, $i \neq j$. Let $\Delta_{\mathfrak{d}}$ denote the set of all independent sets in $\mathfrak{d}$. Then $\Delta_{\mathfrak{d}}$ is a simplicial complex on $V(\mathfrak{d})$ and is called the independence complex of $\mathfrak{d}$.

A semidirected graph $\mathfrak{d}$ is called *acyclic* if $\mathfrak{d}$ has no directed cycle and is called *transitively closed* if the following two properties are satisfied for any distinct i, j, k:

(TC1) if $p_ip_j \in E_d(\mathfrak{d})$ and $p_jp_k \in E_d(\mathfrak{d})$, then $p_ip_k \in E_d(\mathfrak{d})$;
(TC2) if $p_ip_j \in E_d(\mathfrak{d})$ and $\{p_j, p_k\} \in E_u(\mathfrak{d})$, then $\{p_i, p_k\} \in E_u(\mathfrak{d})$.

Let $\mathfrak{d}$ be an acyclic and transitively closed semidirected graph. We define $p_j \succ p_i$ if $p_ip_j \in E_d(\mathfrak{d})$, and $p_j \succeq p_i$ if $p_j = p_i$ or $p_j \succ p_i$. Then $\prec$ is a partial order on $V(\mathfrak{d})$. For a subset $A \subset V(\mathfrak{d})$, we define $p_j \succeq A$ if $p_j \succeq p_i$ for some $p_i \in A$.

Let $\mathfrak{d}$ be a transitively closed semidirected graph. Two vertices $p_i, p_j \in V(\mathfrak{d})$ are called *strongly connected* if $p_ip_j, p_jp_i \in E_d(\mathfrak{d})$. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_t$ be the strongly connected components of $\mathfrak{d}$. Note that $V(\mathfrak{d})$ can be decomposed as $\mathcal{Z}_1 \sqcup \cdots \sqcup \mathcal{Z}_t$. Then we define a new semidirected graph $\widehat{\mathfrak{d}}$, called the *acyclic reduction* of $\mathfrak{d}$, as follows:

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

$$V(\widehat{\mathfrak{d}}) = \{q_1, \ldots, q_t\};$$
$$E_u(\widehat{\mathfrak{d}}) = \{\{q_a, q_b\} : \{p_i, p_j\} \in E_u(\mathfrak{d}) \text{ for some } p_i \in \mathcal{Z}_a \text{ and for some } p_j \in \mathcal{Z}_b\};$$
$$E_d(\widehat{\mathfrak{d}}) = \{q_a q_b : p_i p_j \in E_d(\mathfrak{d}) \text{ for some } p_i \in \mathcal{Z}_a \text{ and for some } p_j \in \mathcal{Z}_b\}.$$

Note that $\widehat{\mathfrak{d}}$ is acyclic. Also $\widehat{\mathfrak{d}}$ is transitively closed since $\mathfrak{d}$ is transitively closed.

Let G be a very well-covered graph with (i), (ii), (iii), (iv) of Theorem 2.1. Then we define the semidirected graph $\mathfrak{d}_G$ as follows:

$$V(\mathfrak{d}_G) = \{p_1, \ldots, p_h\};$$
$$E_u(\mathfrak{d}_G) = \{\{p_i, p_j\} : \{x_i, x_j\} \in E(G)\};$$
$$E_d(\mathfrak{d}_G) = \{p_i p_j : \{x_i, y_j\} \in E(G)\}.$$

Note that $\mathfrak{d}_G$ is transitively closed. Now let us consider the acyclic reduction $\widehat{\mathfrak{d}_G}$. We define a new graph $\widehat{G}$, called the *acyclic reduction* of G as follows:

$$\begin{aligned} V(\widehat{G}) &= \{u_1, \ldots, u_t\} \cup \{v_1, \ldots, v_t\}; \\ E(\widehat{G}) &= \{\{u_a, v_a\} : a = 1, \ldots, t\} \\ &\quad \cup \{\{u_a, u_b\} : \{q_a, q_b\} \in E_u(\widehat{\mathfrak{d}_G})\} \cup \{\{u_a, v_b\} : q_a q_b \in E_d(\widehat{\mathfrak{d}_G})\}. \end{aligned}$$

Then $\widehat{G}$ is a Cohen–Macaulay very well-covered graph [11, Lemma 4.5]. Note that $\mathfrak{d}_{\widehat{G}} = \widehat{\mathfrak{d}_G}$ and we denote it by $\widehat{\mathfrak{d}}_G$.

For $\emptyset \neq \widehat{A} \in \Delta_{\widehat{\mathfrak{d}}_G}$, we set $\Omega_{\widehat{A}} := \bigcup_{q_b \succeq A} \mathcal{Z}_b$ and $\Omega_\emptyset := \emptyset$.

Lemma 2.3. [11, Lemma 4.10] *Let G be a very well-covered graph satisfying (i), (ii), (iii), (iv) of Theorem 2.1. Then*

$$\operatorname{Ass}(S/I(G)) = \{(x_i : p_i \notin \Omega_{\widehat{A}}) + (y_i : p_i \in \Omega_{\widehat{A}}) : \widehat{A} \in \Delta_{\widehat{\mathfrak{d}}_G}\}.$$

Finally, we note the following lemma, which will be a key to the construction of a minimal free resolution of $J(G)$ when G is a Cohen–Macaulay very well-covered graph; see the next section.

Lemma 2.4. [11, Lemma 4.9] *Let G be a very well-covered graph satisfying (i), (ii), (iii), (iv) of Theorem 2.1. Then $\Omega_{\widehat{A}}$ does not contain any undirected edge in $\mathfrak{d}_G$ for any $\widehat{A} \in \Delta_{\widehat{\mathfrak{d}}_G}$.*

§3. A minimal free resolution of $J(G)$ where G is Cohen–Macaulay

The proof of Theorem 1.1 is inspired of that for the case of Cohen–Macaulay bipartite graphs in [9]. This proof is based on an explicit minimal

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

free resolution of the cover ideal $J(G)$ given by Herzog and Hibi [6]. In this section, we construct an explicit minimal free resolution of $J(G)$ for a Cohen–Macaulay very well-covered graph G. This is done by a similar construction to the one by Herzog and Hibi [6].

Let G be a Cohen–Macaulay very well-covered graph on $V = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ with the properties (i), (ii), (iii), (iv) in Theorem 2.1 and (v) in Theorem 2.2. For $A \in \Delta_{\mathfrak{d}_G}$, we denote

$$u_A = \prod_{p_i \notin \Omega_A} x_i \prod_{p_i \in \Omega_A} y_i.$$

By Lemma 2.3, we know

$$J(G) = (u_A : A \in \Delta_{\mathfrak{d}_G}).$$

REMARK 3.1. Our notation of x_i, y_j are converse to the one in Herzog and Hibi [6]. Also "minimal" in our construction corresponds to "maximal" in that of [6].

We set $\mathcal{L} = \{\Omega_A : A \in \Delta_{\mathfrak{d}_G}\}$. For $\Omega \in \mathcal{L}$, let $A(\Omega)$ denote the minimal elements in Ω with respect to $\prec$. Note that by Lemma 2.4, no two vertices in Ω form an undirected edge in $\mathfrak{d}_G$. It then follows that $A(\Omega)$ is independent in $\mathfrak{d}_G$ and the set $A(\Omega)$ is the unique face A in $\Delta_{\mathfrak{d}_G}$ with $\Omega = \Omega_A$. Hence there is a one-to-one correspondence between $\Omega \in \mathcal{L}$ and $A \in \Delta_{\mathfrak{d}_G}$.

Now we construct a ($\mathbb{N}^{2h}$-graded) minimal free resolution $\mathcal{F}_\bullet$ of $J(G)$ as follows. For all $i \geqslant 0$, let $\mathcal{F}_i$ denote the free S-module with basis $e(\Omega, T)$, where $\Omega \in \mathcal{L}$ and $T \subset V(\mathfrak{d}_G)$ satisfying

$$\Omega \cap T \subset A(\Omega), \qquad \#(\Omega \cap T) = i, \qquad \Omega \cup T = V(\mathfrak{d}_G).$$

The degree of $e(\Omega, T)$ is defined by $\deg u_{A(\Omega),T}$ where we define

$$u_{A,T} := u_A \prod_{p_i \in \Omega_A \cap T} x_i = \prod_{p_i \in \Omega_A} y_i \prod_{p_i \notin \Omega_A} x_i \prod_{p_i \in \Omega_A \cap T} x_i$$

for $A \in \Delta_{\mathfrak{d}_G}$. Also for all $i \geqslant 1$, we define the differential $\partial_i \colon \mathcal{F}_i \longrightarrow \mathcal{F}_{i-1}$ by

$$\partial_i(e(\Omega, T)) := \sum_{p_\ell \in \Omega \cap T} (-1)^{\sigma(\Omega \cap T, p_\ell)} (y_\ell e(\Omega \setminus \{p_\ell\}, T) - x_\ell e(\Omega, T \setminus \{p_\ell\})),$$

where for $Q \subset V(\mathfrak{d}_G)$ and $p_\ell \in Q$, we set $\sigma(Q, p_\ell) := \#\{p_k \in Q : k < \ell\}$. Since Ω contains no undirected edge in $\mathfrak{d}_G$, we have that $\Omega \setminus \{p_\ell\} = \Omega_{A'} \in \mathcal{L}$ where

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

A' is the set of minimal elements of $\Omega \setminus \{p_\ell\}$ with respect to $\prec$ and it is an independent set in $\mathfrak{d}_G$. Then it is easy to see that $e(\Omega \setminus \{p_\ell\}, T), e(\Omega, T \setminus \{p_\ell\})$ are free bases of $\mathcal{F}_{i-1}$ and ∂_i possesses the multidegree.

The following is the main result of this section.

THEOREM 3.2. *$(\mathcal{F}_\bullet, \partial_\bullet)$ is a $\mathbb{N}^{2h}$-graded minimal free resolution of $J(G)$.*

Before proving the theorem, we recall some notion on graphs. Let G be a simple graph on the vertex set V. Take $W \subset V$. The induced subgraph of G on W is the graph whose vertex set is W and whose edge set is the set of all edges of G whose two end vertices are in W. Also, $G \setminus W$ denotes the induced subgraph $G_{V \setminus W}$. For $x \in V$, we denote by $N_G(x)$ the set of neighbors of x in G: $N_G(x) := \{y \in V : \{x, y\} \in E(G)\}$.

Proof of Theorem 3.2. We use arguments similar to those of Herzog and Hibi [6, Theorem 2.1].

Set $V' := V(\mathfrak{d}_G)$. The free basis $e(\Omega, T)$ of $\mathcal{F}_0$ satisfies $\Omega \cap T = \emptyset$ and $\Omega \cup T = V'$. Thus T is uniquely determined by $\Omega \in \mathcal{L}$: $T = V' \setminus \Omega$. Also $\deg e(\Omega, T) = \deg u_{\Omega,T} = \deg u_{A(\Omega)}$. Define the augmentation $\varepsilon \colon \mathcal{F}_0 \longrightarrow J(G)$ by $e(\Omega, T) \mapsto u_{A(\Omega)}$.

Claim 1. $\mathcal{F}_1 \xrightarrow{\partial_1} \mathcal{F}_0 \xrightarrow{\varepsilon} J(G) \longrightarrow 0$ is a complex.

Take a free basis $e(\Omega, T)$ of $\mathcal{F}_1$. Note that $\#(\Omega \cap T) = 1$ and set $\Omega \cap T = \{p_\ell\}$. Then

$$\begin{aligned}\varepsilon \circ \partial_1(e(\Omega, T)) &= \varepsilon(y_\ell e(\Omega \setminus \{p_\ell\}, T) - x_\ell e(\Omega, T \setminus \{p_\ell\})) \\ &= y_\ell u_{A(\Omega \setminus \{p_\ell\})} - x_\ell u_{A(\Omega)} = 0,\end{aligned}$$

as desired.

Claim 2. $\partial_i \circ \partial_{i-1} = 0$.

Note that

$$\begin{aligned}&\partial_i \circ \partial_{i-1}(e(\Omega, T)) \\ &= \partial_i \left(\sum_{p_\ell \in \Omega \cap T} (-1)^{\sigma(\Omega \cap T, p_\ell)} (y_\ell e(\Omega \setminus \{p_\ell\}, T) - x_\ell e(\Omega, T \setminus \{p_\ell\})) \right)\end{aligned}$$

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

$$
\begin{aligned}
= \sum_{p_\ell \in \Omega \cap T} & (-1)^{\sigma(\Omega \cap T, p_\ell)} \\
\times \Bigg[& y_\ell \sum_{\substack{p_k \in \Omega \cap T \\ k \neq \ell}} (-1)^{\sigma((\Omega \setminus \{p_\ell\}) \cap T, p_k)} \\
& \times (y_k e(\Omega \setminus \{p_\ell, p_k\}, T) - x_k e(\Omega \setminus \{p_\ell\}, T \setminus \{p_k\})) \\
& - x_\ell \sum_{\substack{p_k \in \Omega \cap T \\ k \neq \ell}} (-1)^{\sigma(\Omega \cap (T \setminus \{p_\ell\}), p_k)} \\
& \times (y_k e(\Omega \setminus \{p_k\}, T \setminus \{p_\ell\}) - x_k e(\Omega, T \setminus \{p_\ell, p_k\})) \Bigg].
\end{aligned}
$$

Then the coefficient of $e(\Omega \setminus \{p_\ell, p_k\}, T)$, $k \neq \ell$ vanishes since the differences of $\sigma((\Omega \setminus \{p_\ell\}) \cap T, p_k)$ and $\sigma((\Omega \setminus \{p_k\}) \cap T, p_\ell)$ is just 1. Similarly, the coefficient of $e(\Omega, T \setminus \{p_\ell, p_k\})$, $k \neq \ell$ vanishes. Finally we check the coefficient of $e(\Omega \setminus \{p_\ell\}, T \setminus \{p_k\})$. In this case, $\sigma((\Omega \setminus \{p_\ell\}) \cap T, p_k)$ and $\sigma(\Omega \cap (T \setminus \{p_k\}), p_\ell)$ also differ just by 1. Therefore, we have $\partial_i \circ \partial_{i-1}(e(\Omega, T)) = 0$.

Claim 3. $\mathcal{F}_1 \xrightarrow{\partial_1} \mathcal{F}_0 \xrightarrow{\varepsilon} J(G) \longrightarrow 0$ is exact.

We first note that the first syzygy module of $J(G)$ is generated by

$$
r_{\Omega,\Theta} := y_{\Theta \setminus \Omega} x_{\Omega \setminus \Theta} e(\Omega, V' \setminus \Omega) - y_{\Omega \setminus \Theta} x_{\Theta \setminus \Omega} e(\Theta, V' \setminus \Theta), \quad \Omega, \Theta \in \mathcal{L},
$$

where for $\Omega \subset V'$, we denote

$$
x_\Omega := \prod_{p_\ell \in \Omega} x_\ell \qquad \text{and} \qquad y_\Omega := \prod_{p_\ell \in \Omega} y_\ell.
$$

For $\Omega, \Theta \in \mathcal{L}$, the intersection $\Omega \cap \Theta$ is in $\mathcal{L}$ because $\Omega \cap \Theta = \Omega_{A'}$ where A' is the set of minimal elements of $\Omega \cap \Theta$ with respect to $\prec$. Also we can easily check that

$$
r_{\Omega,\Theta} = y_{\Theta \setminus \Omega} r_{\Omega, \Omega \cap \Theta} - y_{\Omega \setminus \Theta} r_{\Theta, \Omega \cap \Theta}.
$$

Therefore, in order to prove the claim, it is sufficient to prove that $r_{\Omega,\Theta} \in \partial_1(\mathcal{F}_1)$ for $\Omega, \Theta \in \mathcal{L}$ with $\Theta \subset \Omega$; let $\Omega, \Theta \in \mathcal{L}$ be such a pair. Set

$\Omega \setminus \Theta = \{p_{k_1}, \ldots, p_{k_m}\}$, where $k_1 > \cdots > k_m$, and

$$\Omega_0 = \Theta,$$
$$\Omega_j = \Omega_{j-1} \cup \{p_{k_j}\}, \quad j = 1, \ldots, m.$$

We show that $\Omega_j \in \mathcal{L}$ for each j.

Let A_j denote the set of minimal elements of Ω_j. We prove $\Omega_j = \Omega_{A_j}$. It is enough to show that $\Omega_j \supset \Omega_{A_j}$. Take $p_\ell \in \Omega_{A_j}$. Then there exists $p_k \in A_j$ with $p_k \preceq p_\ell$. We note that $k \leqslant \ell$ by the property (v) of Theorem 2.2. We also note that $p_k \in A_j \subset \Omega_j \subset \Omega$, and combining this with $\Omega \in \mathcal{L}$, we have $p_\ell \in \Omega$. By the same reason, when $p_k \in \Theta$, we have $p_\ell \in \Theta$. Since $\Theta \subset \Omega_j$, we have done for the case $p_k \in \Theta$. Thus we assume $p_k \notin \Theta$. If $p_\ell \in \Theta$, then $p_\ell \in \Omega_j$ is obvious. If $p_\ell \notin \Theta$, it follows that $p_k, p_\ell \in \Omega \setminus \Theta$. Since $\ell \geqslant k$ and $p_k \in \Omega_j$, we conclude that $p_\ell \in \Omega_j$ by the construction of Ω_j.

Note that p_{k_j} must be a minimal element of Ω_j and

$$\begin{aligned}\Omega_j \cap (V' \setminus \Omega_{j-1}) = {} & (\Theta \cup \{p_{k_1}, \ldots, p_{k_j}\}) \\ & \cap\ (V' \setminus (\Theta \cup \{p_{k_1}, \ldots, p_{k_{j-1}}\})) = \{p_{k_j}\}.\end{aligned}$$

By the easy calculation, we have

$$r_{\Omega,\Theta} = \sum_{j=1}^{m} \left(\prod_{s=j+1}^{m} y_{k_s} \prod_{s=1}^{j-1} x_{k_s} \right) r_{\Omega_j,\Omega_{j-1}},$$
$$r_{\Omega_j,\Omega_{j-1}} = -\partial(e(\Omega_j, V' \setminus \Omega_{j-1})), \quad j = 1, \ldots, m.$$

Therefore, the claim follows.

Claim 4. $(\mathcal{F}_\bullet, \partial_\bullet)$ is acyclic.

We proceed by induction on h.

If $h = 1$, the cover ideal $J(G) = (x_1, y_1)$. Then $(\mathcal{F}_\bullet, \partial_\bullet)$ is just the Koszul complex associated with $\{x_1, y_1\}$ and thus it is acyclic.

Now we assume $h > 1$. The induced subgraph $G' := G_{V \setminus \{x_1, y_1\}}$ is also a Cohen–Macaulay very well-covered graph. Note that G' satisfies the properties (i), (ii), (iii), (iv) of Theorem 2.1 and (v) of Theorem 2.2 with respect to the induced labeling from G. Let $(\mathcal{F}', \partial')$ denote the complex corresponding to G'. Then by inductive hypothesis, $(\mathcal{F}', \partial')$ is acyclic. Since the inclusion $K[V(G')] \longrightarrow K[V(G)]$ is a flat extension, by tensoring $K[V(G)]$ to $(\mathcal{F}', \partial')$, we obtain the acyclic complex over $K[V(G)]$. We define the map $\phi\colon \mathcal{F}' \longrightarrow \mathcal{F}$ by $e(\Omega, T) \mapsto e(\Omega, T \cup \{p_1\})$. Then ϕ is an injective map of complexes. Also the induced map $J(G') = H_0(\mathcal{F}') \longrightarrow H_0(\mathcal{F}) = J(G)$ is a multiplication by x_1. Let $\mathcal{G}_\bullet$ be the quotient complex $\mathcal{F}/\mathcal{F}'$. Then the short exact sequence of complexes

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow H_2(\mathcal{G}) \longrightarrow H_1(\mathcal{F}') \longrightarrow H_1(\mathcal{F}) \longrightarrow H_1(\mathcal{G}) \longrightarrow H_0(\mathcal{F}') \xrightarrow{\cdot x_1} H_0(\mathcal{F}).$$

By inductive hypothesis, $\mathcal{F}'$ is acyclic, that is, $H_i(\mathcal{F}') = 0$ for all $i > 0$. Also the multiplication map is injective. It then follows that $H_i(\mathcal{F}) \cong H_i(\mathcal{G})$ for all $i > 0$. We prove $H_i(\mathcal{G}) = 0$ for all $i > 0$.

We consider the induced subgraph $G'' := G_{V \setminus (\{x_1\} \cup N_G(x_1))}$. Let $G''_0 := G'' \setminus \{\text{isolated vertices of } G''\}$. It was proved by Mahmoudi *et al.* in [11, Proof of Theorem 3.2] that G''_0 is also a Cohen–Macaulay very well-covered graph with respect to the induced labeling of the vertices from G. Let $(\mathcal{F}'', \partial'')$ denote the complex corresponding to G''_0. Similarly to the case of $(\mathcal{F}', \partial')$, we do not distinguish $(\mathcal{F}'', \partial'')$ with $(\mathcal{F}'' \otimes K[V(G)], \partial'' \otimes 1)$. Let $\mathcal{C}_\bullet$ denote the mapping cone of the complex homomorphism

$$\mathcal{F}'' \xrightarrow{\cdot - x_1} \mathcal{F}''.$$

Then we have a short exact sequence of complexes

$$0 \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{C} \longrightarrow \mathcal{F}''[-1] \longrightarrow 0,$$

where $(\mathcal{F}''[-1])_i := \mathcal{F}''_{i-1}$. This induces the long exact sequence

$$\begin{aligned} \cdots \longrightarrow H_2(\mathcal{C}) \longrightarrow H_1(\mathcal{F}'') \\ \xrightarrow{\cdot - x_1} H_1(\mathcal{F}'') \longrightarrow H_1(\mathcal{C}) \longrightarrow H_0(\mathcal{F}'') \\ \xrightarrow{\cdot - x_1} H_0(\mathcal{F}'') \longrightarrow H_0(\mathcal{C}) \longrightarrow 0. \end{aligned}$$

By inductive hypothesis, the complex $\mathcal{F}''$ is also acyclic, and the multiplication map $J(G'') = H_0(\mathcal{F}'') \xrightarrow{\cdot - x_1} H_0(\mathcal{F}'') = J(G'')$ is injective, we have $H_i(\mathcal{C}) = 0$ for $i \geqslant 1$.

Finally we prove that $\mathcal{C} \cong \mathcal{G}$. Note that

$$\mathcal{C}_\ell = \mathcal{F}''_{\ell-1} \oplus \mathcal{F}''_\ell, \quad \ell \geqslant 0,$$

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

where $\mathcal{F}''_{-1} = 0$. Set $\mathcal{L}'' = \{\Omega''_{A''} : A'' \in \mathfrak{d}_{G''_0}\}$. Then the free basis of $\mathcal{C}_\ell$ is $\mathfrak{C}_\ell = \mathfrak{B}_{\ell-1} \cup \mathfrak{B}_\ell$ where

$$\mathfrak{B}_\ell = \left\{ e(\Omega'', T'') : \begin{array}{l} \Omega'' \in \mathcal{L}'', T'' \subset V(\mathfrak{d}_{G''_0}), \\ \Omega'' \cap T'' \subset A(\Omega''), \#(\Omega'' \cap T'') = \ell, \Omega'' \cup T'' = V(\mathfrak{d}_{G''_0}) \end{array} \right\}.$$

On the other hand, the free basis of $\mathcal{G}_\ell$ is

$$\mathfrak{G}_\ell := \left\{ e(\Omega, T) : \begin{array}{l} \Omega \in \mathcal{L}, p_1 \in \Omega, T \subset V(\mathfrak{d}_G), \\ \Omega \cap T \subset A(\Omega), \#(\Omega \cap T) = \ell, \Omega \cup T = V(\mathfrak{d}_G) \end{array} \right\}.$$

We define $K[V(G)]$-linear homomorphism $\psi_\ell \colon \mathcal{C}_\ell \longrightarrow \mathcal{G}_\ell$ by

$$\psi_\ell(e(\Omega'', T'')) = \begin{cases} e(\Omega'' \cup \Omega_{\{p_1\}}, T'' \cup \{p_1\} \cup \{p_k : \{p_1, p_k\} \in E_u(\mathfrak{d}_G)\}), & e(\Omega'', T'') \in \mathfrak{B}_{\ell-1}, \\ e(\Omega'' \cup \Omega_{\{p_1\}}, T'' \cup \{p_k : \{p_1, p_k\} \in E_u(\mathfrak{d}_G)\}), & e(\Omega'', T'') \in \mathfrak{B}_\ell. \end{cases}$$

Then ψ_ℓ is well-defined. Note that $A(\Omega'' \cup \Omega_{\{p_1\}}) = A(\Omega'') \cup \{p_1\}$ and $T'' \cap \{p_1\} = \emptyset$. It is easy to see that all ψ_ℓ are bijective and (ψ_ℓ) induces an isomorphism of complexes. $\square$

REMARK 3.3. In [11, Lemma 3.4], Mahmoudi *et al.* characterized the regularity of a Cohen–Macaulay very well-covered graph by using G' and G'' (the notations are G_2, G_1, respectively).

By the above explicit minimal free resolution of $J(G)$, we obtain the nonzero extremal Betti numbers of $J(G)$. Recall that $\beta_{i,\sigma}(J(G))$ is *extremal* if $\beta_{j,\tau}(J(G)) = 0$ for all $j \geqslant i$ and $\tau \succ \sigma$ with $|\tau| - |\sigma| \geqslant j - i$, where $|\sigma| = \sigma_1 + \cdots + \sigma_n$ for $\sigma \in \mathbb{N}^n$.

COROLLARY 3.4. *The free basis $e(\Omega, T)$ corresponds to an extremal Betti number of $J(G)$ if and only if the following two conditions are satisfied:*

(i) *$A(\Omega)$ is a facet of $\Delta_{\mathfrak{d}_G}$;*
(ii) *$\Omega \cap T = A(\Omega)$.*

Proof. We use the same notation as above.

We first assume that $e(\Omega, T)$ corresponds to an extremal Betti number of $J(G)$. Suppose $\Omega \cap T \neq A(\Omega)$. We set $A' := A(\Omega) \setminus (\Omega \cap T)$ and $S := T \cup A'$. Then the degree of $e(\Omega, S)$ is

$$\begin{aligned}\deg e(\Omega, S) &= \deg \prod_{p_i \in \Omega} y_i \prod_{p_i \notin \Omega} x_i \prod_{p_i \in \Omega \cap S} x_i \\ &= \deg \prod_{p_i \in \Omega} y_i \prod_{p_i \notin \Omega} x_i \prod_{p_i \in \Omega \cap T} x_i \prod_{p_i \in A'} x_i \\ &= \deg e(\Omega, T) + \deg \prod_{p_i \in A'} x_i.\end{aligned}$$

Hence $e(\Omega, T)$ does not correspond to an extremal Betti number, a contradiction. Therefore, we assume $\Omega \cap T = A(\Omega)$. Suppose that $A(\Omega)$ is not a facet. Then there exists $p_i \in V(\mathfrak{d}_G)$ with $A(\Omega) \cup \{p_i\} \in \Delta_{\mathfrak{d}_G}$. Let p_{i_0} be such a vertex with maximum index. Set $\Theta := \Omega_{A(\Omega) \cup \{p_{i_0}\}}$ and $S = T \cup \{p_{i_0}\}$. We claim $\Theta = \Omega \cup \{p_{i_0}\}$. Indeed, $\Theta \supset \Omega \cup \{p_{i_0}\}$ is obvious. Conversely, assume $p_k \in \Theta$. If $p_k \succeq p_\ell$ for some $p_\ell \in A(\Omega)$, then $p_k \in \Omega$. Otherwise, by applying Lemma 2.4 to $\Theta = \Omega_{A(\Omega) \cup \{p_{i_0}\}}$, we have $\{p_k\} \cup A(\Omega) \in \Delta_{\mathfrak{d}_G}$. On the other hand, $p_k \succeq p_{i_0}$ also satisfied. In particular $i_0 \leqslant k$. Then the maximality of i_0 implies that $p_{i_0} = p_k$, as desired. Hence

$$\begin{aligned}\deg e(\Theta, S) &= \deg \prod_{p_i \in \Theta} y_i \prod_{p_i \notin \Theta} x_i \prod_{p_i \in \Theta \cap S} x_i \\ &= \deg y_{i_0} \prod_{p_i \in \Omega} y_i \prod_{p_i \notin \Omega} x_i \prod_{p_i \in \Omega \cap T} x_i \\ &= \deg e(\Omega, T) + \deg y_{i_0}.\end{aligned}$$

This is also a contradiction.

Next, let $e(\Omega, T)$ be a free basis of $\mathcal{F}_\ell$ of degree σ satisfying the conditions (i), (ii). Suppose that there is a free basis $e(\Theta, S)$ of $\mathcal{F}_k$ of degree τ with $k \geqslant \ell$, $\tau \succ \sigma$, and $|\tau| - |\sigma| \geqslant k - \ell$. Note that $J(G)$ has a linear resolution, and thus $|\tau| - |\sigma| = k - \ell$. Since

$$\begin{aligned}\sigma = \deg e(\Omega, T) &= \deg \prod_{p_i \in \Omega} y_i \prod_{p_i \notin \Omega} x_i \prod_{p_i \in \Omega \cap T} x_i, \\ \tau = \deg e(\Theta, S) &= \deg \prod_{p_i \in \Theta} y_i \prod_{p_i \notin \Theta} x_i \prod_{p_i \in \Theta \cap S} x_i\end{aligned}$$

and $\tau \succ \sigma$, it follows that

$$\begin{cases} \Omega \subset \Theta, \\ (V(\mathfrak{d}_G) \setminus \Omega) \cup (\Omega \cap T) \subset (V(\mathfrak{d}_G) \setminus \Theta) \cup (\Theta \cap S). \end{cases}$$

By $\Omega \subset \Theta$, we have $(V(\mathfrak{d}_G) \setminus \Theta) \cap \Omega = \emptyset$. Hence $\Omega \cap T \subset \Theta \cap S$. Then $A(\Theta) \supset \Theta \cap S \supset \Omega \cap T = A(\Omega)$, where the last equality follows from the condition (ii). Since $A(\Theta), A(\Omega) \in \Delta_{\mathfrak{d}_G}$ and $A(\Omega)$ is a facet of $\Delta_{\mathfrak{d}_G}$ by (i), we have $A(\Omega) = A(\Theta)$. Then it also follows that $\Omega = \Theta$. Again by (ii), this contradicts to $\tau \succ \sigma$. □

Now we can prove Theorem 1.1 for a Cohen–Macaulay very well-covered graph by showing the following proposition and using the Alexander duality: $\operatorname{pd} S/I(G) = \operatorname{reg} J(G)$ given by Terai[18].

PROPOSITION 3.5. *Let G be a very well-covered graph. If $\beta_{r,\sigma}(J(G)) \neq 0$ is extremal, then there exists a pairwise 3-disjoint set $\mathcal{B} = \{B_1, \ldots, B_r\}$ of complete bipartite subgraph of G with $V(\mathcal{B}) = \sigma$.*

COROLLARY 3.6. *Let G be a Cohen–Macaulay very well-covered graph. Then*

$$\begin{aligned} \operatorname{pd} S/I(G) &= \operatorname{height} I(G) \\ &= \max\left\{ \#V(\mathcal{B}) - r : \begin{array}{l} \mathcal{B} = \{B_1, \ldots, B_r\} \textit{ is a pairwise 3-disjoint set} \\ \textit{of complete bipartite subgraphs of } G \end{array} \right\}. \end{aligned}$$

Since $J(G)$ is squarefree, we only need to consider the case where the degree σ is a $(0, 1)$-vector because otherwise $\beta_{i,\sigma}(J(G)) = 0$. Hence we sometimes identify the degree σ with a subset of $V(G)$.

Proof of Proposition 3.5. Assume that $\beta_{r,\sigma}(J(G)) \neq 0$ is extremal. Then there exists a free basis $e(\Omega, T)$ with $\#(\Omega \cap T) = r$ and $\deg e(\Omega, T) = \sigma$. Also by Corollary 3.4, $A(\Omega)$ is a facet of $\Delta_{\mathfrak{d}_G}$ and $\Omega \cap T = A(\Omega)$ holds. Note that

$$\sigma = \deg u_{A(\Omega),T} = \deg \prod_{p_i \in \Omega} y_i \prod_{p_i \notin \Omega} x_i \prod_{p_i \in A(\Omega)} x_i.$$

Set $A(\Omega) := \{p_{\ell_1}, \ldots, p_{\ell_r}\}$. Since $A(\Omega)$ is independent in $\mathfrak{d}_G$, it follows that $\{x_{\ell_1}, y_{\ell_1}\}, \ldots, \{x_{\ell_r}, y_{\ell_r}\}$ are pairwise 3-disjoint in G. We define V_1 to be the set of $z \in V(G)$ which divides $u_{A(\Omega),T}$ and one of $\{z, x_{\ell_1}\}, \{z, y_{\ell_1}\}$ is an edge of G. Next we define V_2 to be the set of $z \in V(G) \setminus V_1$ which divides $u_{A(\Omega),T}$ and one of $\{z, x_{\ell_2}\}, \{z, y_{\ell_2}\}$ is an edge of G. Similarly, we define V_k to be the set of $z \in V(G) \setminus (V_1 \cup \cdots \cup V_{k-1})$ which divides $u_{A(\Omega),T}$ and one of $\{z, x_{\ell_k}\}, \{z, y_{\ell_k}\}$ is an edge of G. As a result, we obtain $V_1, \ldots, V_r$. Note that $x_{\ell_k}, y_{\ell_k} \in V_k$.

We prove the following 2 claims which derive the proposition:

Claim 1. $\sigma = V_1 \sqcup \cdots \sqcup V_r$.

Claim 2. G_{V_k} contains a complete bipartite graph as a spanning subgraph.

Proof of Claim 1. It is clear that $V_\ell \cap V_k \neq \emptyset$ if $k \neq \ell$. Also

$$\{x_{\ell_1}, \dots, x_{\ell_r}, y_{\ell_1}, \dots, y_{\ell_r}\} \subset V_1 \cup \cdots \cup V_r \subset \sigma.$$

We set $\sigma_0 := \sigma \setminus \{x_{\ell_1}, \dots, x_{\ell_r}, y_{\ell_1}, \dots, y_{\ell_r}\}$. We prove $\sigma_0 \subset V_1 \cup \cdots \cup V_r$. Then Claim 1 follows.

If $y_\ell \in \sigma_0$, then $p_\ell \in \Omega$. Therefore, $p_\ell \succ p_{\ell_k}$ for some $p_{\ell_k} \in A(\Omega)$. This implies $\{x_{\ell_k}, y_\ell\} \in E(G)$. Therefore, $y_\ell \in V_1 \cup \cdots \cup V_k$.

If $x_\ell \in \sigma_0$, then $p_\ell \notin \Omega$. Note that $p_\ell \notin A(\Omega)$. Since $A(\Omega)$ is a facet of $\Delta_{\mathfrak{d}_G}$ and $A(\Omega) \subsetneq A(\Omega) \cup \{p_\ell\}$, it follows that $A(\Omega) \cup \{p_\ell\} \notin \Delta_{\mathfrak{d}_G}$, in other words, $A(\Omega) \cup \{p_\ell\}$ is not independent in $\mathfrak{d}_G$. Since $p_\ell \notin \Omega$, it follows that $p_\ell p_{\ell_k} \in E_d(\mathfrak{d}_G)$ for some k or $\{p_\ell, p_{\ell_k}\} \in E_u(\mathfrak{d}_G)$ for some k. In the former case, we have $\{x_\ell, y_{\ell_k}\} \in E(G)$, and the latter case, we have $\{x_\ell, x_{\ell_k}\} \in E(G)$. Thus $x_\ell \in V_1 \cup \cdots \cup V_k$.

Proof of Claim 2. For each k, we set

$$V_{1k} := \{z \in V_k : \{z, x_{\ell_k}\} \in E(G)\},$$
$$V_{2k} := \{x_\ell \in V_k : \{x_\ell, y_{\ell_k}\} \in E(G)\}.$$

Note that $V_{1k} \cap V_{2k} = \emptyset$ by the condition (iv) of Theorem 2.1. Also $V_{1k} \cup V_{2k} = V_k$ holds. Take $z \in V_{1k}$ and $x_\ell \in V_{2k}$. Since $\{z, x_{\ell_k}\}, \{x_\ell, y_{\ell_k}\} \in E(G)$, we have $\{z, x_\ell\} \in E(G)$ by the condition (iii) of Theorem 2.1. $\square$

§4. Proof of the main theorem

In this section we prove Theorem 1.1. We use arguments similar to that of [9, Theorem 7.1].

First, we recall some notation. Let G be a very well-covered graph on $V = \{x_1, \dots, x_h, y_1, \dots, y_h\}$ with the properties (i), (ii), (iii), (iv) of Theorem 2.1. Then we obtain the transitively closed semidirected graph $\mathfrak{d}_G$ on the vertex set $\{p_1, \dots, p_h\}$. Let $\mathcal{Z}_1, \dots, \mathcal{Z}_t$ be all strongly connected components of $\mathfrak{d}_G$. Then we obtain the acyclic reduction $\widehat{\mathfrak{d}}_G$ of $\mathfrak{d}_G$, whose vertex set is $\{q_1, \dots, q_t\}$. The semidirected graph $\widehat{\mathfrak{d}}_G$ corresponds to the acyclic reduction $\widehat{G}$ of G. Note that $\widehat{G}$ is a Cohen–Macaulay very well-covered graph and we may assume that the vertex set of $\widehat{G}$ is $\{u_1, \dots, u_t, v_1, \dots, v_t\}$

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

satisfying (i), (ii), (iii), (iv) of Theorem 2.1 and (v) of Theorem 2.2 (though we need to replace notation x_i, y_j, z_k by u_i, v_j, w_k, respectively). Also we set $\zeta_a = \#\mathcal{Z}_a$. Moreover, for $\sigma = \prod_a u_a^{s_a} \prod_b v_b^{r_b}$, we set $\sigma^\zeta = \prod_a u_a^{s_a \zeta_a} \prod_b v_b^{r_b \zeta_b}$.

The following result is an extension of the result for unmixed bipartite graphs by Kummini [10, Proposition 3.2] to very well-covered graphs.

PROPOSITION 4.1. *Let G be a very well-covered graph. Using the above notation we have*

$$\operatorname{pd} S/I(G) = \max\{|\sigma^\zeta| - r : \beta_{r,\sigma}(J(\widehat{G})) \neq 0\}.$$

One can prove this proposition by a similar argument due to Kummini [10]. We omit the proof.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We use the same notation as above. Let G be a very well-covered graph on $V = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ with the properties (i), (ii), (iii), (iv) of Theorem 2.1.

Take $\beta_{r,\sigma}(J(\widehat{G})) \neq 0$ which gives $\operatorname{pd} S/I(G)$. We may assume that $\beta_{r,\sigma}(J(\widehat{G}))$ is extremal. Indeed, suppose that $\beta_{r,\sigma}(J(\widehat{G}))$ is not extremal. Then there exist $s \geqslant r$ and $\tau \supsetneq \sigma$ with $|\tau| - |\sigma| \geqslant s - r$ satisfying $\beta_{s,\tau}(J(\widehat{G})) \neq 0$. In this case,

$$\begin{aligned}(|\tau^\zeta| - s) - (|\sigma^\zeta| - r) &= \sum_{u_a \in \tau \setminus \sigma} \zeta_a + \sum_{v_b \in \tau \setminus \sigma} \zeta_b - (s - r) \\ &\geqslant |\tau| - |\sigma| - (s - r) \geqslant 0.\end{aligned}$$

Therefore, we can replace $\beta_{r,\sigma}(J(\widehat{G}))$ by $\beta_{s,\tau}(J(\widehat{G}))$.

Since $\widehat{G}$ is Cohen–Macaulay, we can take a pairwise 3-disjoint set $\widehat{\mathcal{B}} = \{\widehat{B}_1, \ldots, \widehat{B}_r\}$ of complete bipartite subgraphs of $\widehat{G}$ with $V(\widehat{\mathcal{B}}) = \sigma$ as constructed in Proposition 3.5. We can assume that $\{u_{a_k}, v_{a_k}\} \in E(\widehat{B}_k)$ and $\{u_{a_1}, v_{a_1}\}, \ldots, \{u_{a_r}, v_{a_r}\}$ are pairwise 3-disjoint in $\widehat{G}$. Moreover, we can assume that $\widehat{B}_k$ is the complete bipartite subgraph of $\widehat{G}$ on $\widehat{V}_k := V(\widehat{B}_k)$ which is decomposed as $\widehat{V}_{1k} \sqcup \widehat{V}_{2k}$, where

$$\begin{aligned}\widehat{V}_{1k} &= \{w_b \in \widehat{V}_k : \{w_b, u_{a_k}\} \in E(\widehat{G})\}, \\ \widehat{V}_{2k} &= \{u_b \in \widehat{V}_k : \{u_b, v_{a_k}\} \in E(\widehat{G})\}.\end{aligned}$$

Set

$$V_{1k} := \left(\bigcup_{u_b \in \widehat{V}_{1k}} \{x_\ell : p_\ell \in \mathcal{Z}_b\} \right) \cup \left(\bigcup_{v_b \in \widehat{V}_{1k}} \{y_\ell : p_\ell \in \mathcal{Z}_b\} \right),$$

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

$$V_{2k} := \bigcup_{u_b \in \widehat{V}_{2k}} \{x_\ell : p_\ell \in \mathcal{Z}_b\}.$$

Since $\widehat{V}_{1k} \cap \widehat{V}_{2k} = \emptyset$, we have $V_{1k} \cap V_{2k} = \emptyset$. Note that for each b, at least one of u_b and v_b is not contained in $\widehat{V}_{1k}$. Let B_k be the complete bipartite graph with the bipartition $V_{1k} \sqcup V_{2k}$. Set $\mathcal{B} := \{B_1, \ldots, B_r\}$. It is clear that $V(B_k) \cap V(B_\ell) = \emptyset$ for $k \neq \ell$ and $\#V(\mathcal{B}) = |\sigma^\zeta|$. We prove the following 2 claims which derive the theorem.

Claim 1. B_k is a subgraph of G.

Claim 2. $\mathcal{B}$ is pairwise 3-disjoint in G.

Proof of Claim 1. Take $\{z_\ell, x_m\} \in E(B_k)$, where $z_\ell \in V_{1k}$ and $x_m \in V_{2k}$. Then there exist $w_a \in \widehat{V}_{1k}$ and $u_b \in \widehat{V}_{2k}$ such that $p_\ell \in \mathcal{Z}_a$ and $p_m \in \mathcal{Z}_b$.

We first assume $a = b$. Since $w_a \in \widehat{V}_{1k}$, $u_a \in \widehat{V}_{2k}$, and $\widehat{V}_{1k} \cap \widehat{V}_{2k} = \emptyset$, it follows that $w_a = v_a$ and thus $z_\ell = y_\ell$. On the other hand, $p_\ell, p_m \in \mathcal{Z}_a$. In particular, $p_m p_\ell \in E_d(\mathfrak{d}_G)$. This means $\{x_m, y_\ell\} \in E(G)$ as desired.

We next assume $a \neq b$. If $z_\ell = x_\ell$, then $w_a = u_a$ and we have $\{q_a, q_b\} \in E_u(\widehat{\mathfrak{d}}_G)$ since $\{u_a, u_b\} \in E(\widehat{B}_k) \subset E(\widehat{G})$. Therefore, $\{p_\ell, p_m\} \in E_u(\mathfrak{d}_G)$. This means that $\{z_\ell, x_m\} = \{x_\ell, x_m\} \in E(G)$. If $z_\ell = y_\ell$, then $w_a = v_a$ and we have $q_b q_a \in E_d(\widehat{\mathfrak{d}}_G)$ since $\{v_a, u_b\} \in E(\widehat{B}_k) \subset E(\widehat{G})$. Therefore, $p_m p_\ell \in E_d(\mathfrak{d}_G)$. This means that $\{z_\ell, x_m\} = \{y_\ell, x_m\} \in E(G)$.

Proof of Claim 2. Recall that $\{u_{a_1}, v_{a_1}\}, \ldots, \{u_{a_r}, v_{a_r}\}$ are pairwise 3-disjoint in $\widehat{G}$. Let $p_{\ell_s} \in \mathcal{Z}_{a_s}$. Then $\{x_{\ell_1}, y_{\ell_1}\}, \ldots, \{x_{\ell_r}, y_{\ell_r}\}$ are pairwise 3-disjoint in G. $\square$

§5. Projective dimension of symbolic powers of the edge ideal of a very well-covered graph

In this section we show that the projective dimension of symbolic powers of the edge ideal of a very well-covered graph is nondecreasing.

We recall the definition of a symbolic power of an ideal.

Let I be a radical ideal of a polynomial ring S. Let $\mathrm{Min}_S(S/I) = \{P_1, \ldots, P_r\}$ be the set of the minimal prime ideals of I, and put $W = S \setminus \bigcup_{i=1}^r P_i$. Given an integer $\ell \geqslant 1$, the ℓth *symbolic power* of I is defined to be the ideal

$$I^{(\ell)} = I^\ell S_W \cap S = \bigcap_{i=1}^{r} P_i^\ell S_{P_i} \cap S.$$

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

In particular, if I is a squarefree monomial ideal of S, then one has

$$I^{(\ell)} = P_1^{\ell} \cap \cdots \cap P_r^{\ell}.$$

We use the following lemma:

LEMMA 5.1. [14, Corollary 1.3] *Let I be a monomial ideal in a polynomial ring S. Take a monomial m such that $m \notin I$. Then*

$$\operatorname{depth} S/(I : m) \geqslant \operatorname{depth} S/I.$$

Now we state the main result in this section.

THEOREM 5.2. *Let G be a very well-covered graph or a graph with a leaf. Then for $i \geqslant 2$,*

$$\operatorname{pd} S/I(G)^{(i)} \geqslant \operatorname{pd} S/I(G)^{(i-1)}.$$

Proof. Let P be a minimal prime ideal of $I(G)$. We first assume that G is a very well-covered graph whose vertices are labeled as in Theorem 2.1. Then P is of the form

$$P = (z_1, z_2, \ldots, z_h)$$

where $z_i = x_i$ or $z_i = y_i$ for $i = 1, 2, \ldots, h$ by [2, Corollary 2.2].

Next we assume that G has a leaf x_1. And we assume that $\{x_1, y_1\} \in E(G)$. Then just either one of x_1 and y_1 is contained in P.

In either case we show that

$$P^i : x_1y_1 = P^{i-1}.$$

Take a minimal monomial generator m of P^i. Then m is not divided by x_1y_1 and $x_1y_1 \in P$. Hence $(m) : x_1y_1 \subset P^{i-1}$. Hence $P^i : x_1y_1 \subset P^{i-1}$. Conversely, take a minimal monomial generator m of P^{i-1}. Since we have $x_1 \in P$ or $y_1 \in P$, we have $mx_1y_1 \in P^i$. Then we have $P^{i-1} \subset P^i : x_1y_1$.

Now we have

$$\begin{aligned} I(G)^{(i)} : x_1y_1 &= \left(\bigcap_{P \in \operatorname{Min}_S S/I(G)} P^i \right) : x_1y_1 \\ &= \bigcap_{P \in \operatorname{Min}_S S/I(G)} (P^i : x_1y_1) \end{aligned}$$

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press

$$= \bigcap_{P \in \mathrm{Min}_S S/I(G)} P^{i-1}$$
$$= I(G)^{(i-1)}.$$

By Lemma 5.1 we have

$$\begin{aligned} \mathrm{pd}\, S/I(G)^{(i)} &\geqslant \mathrm{pd}\, S/(I(G)^{(i)} : x_1 y_1) \\ &= \mathrm{pd}\, S/I(G)^{(i-1)}. \end{aligned} \qquad \square$$

As an application we show that the projective dimension of the ordinary powers of the edge ideal is also nondecreasing for certain bipartite graphs.

COROLLARY 5.3. *Let G be an unmixed or sequentially Cohen–Macaulay bipartite graph. Then for $i \geqslant 2$,*

$$\mathrm{pd}\, S/I(G)^i \geqslant \mathrm{pd}\, S/I(G)^{i-1}.$$

The above corollary follows from the next facts:

LEMMA 5.4. [15, Lemma 3.10], [16, Lemma 5.8, Theorem 5.9] *Let $I(G)$ be the edge ideal of a graph G. Let $t \geqslant 2$ be an integer. Then $I(G)^{(t)} = I(G)^t$ holds if and only if G contains no odd cycles of length $2s-1$ for any $2 \leqslant s \leqslant t$.*

LEMMA 5.5. [20, Lemma 2.8] *Let G be a sequentially Cohen–Macaulay bipartite graph. Then G has a leaf.*

REMARK 5.6. Using the corresponding result to Lemma 5.1 for Stanley depth sdepth instead of depth in [1], we can prove the following nonincreasing property of Stanley depth, similarly:

(1) sdepth $S/I(G)^{(i)} \leqslant$ sdepth $S/I(G)^{(i-1)}$ for a very well-covered graph G with $i \geqslant 2$.
(2) sdepth $S/I(G)^i \leqslant$ sdepth $S/I(G)^{i-1}$ for an unmixed or sequentially Cohen–Macaulay bipartite graph G with $i \geqslant 2$.

Acknowledgments. Kimura was partially supported by JSPS Grant-in-Aid for Young Scientists (B) 24740008/15K17507. Terai was partially supported by JSPS Grant-in-Aid (C) 26400049 and thanks the American Institute of Mathematics for giving the chance to participate in SQuaRE "Ordinary powers and symbolic powers." Yassemi was partially supported by a grant from University of Tehran. The authors thank the referee for many comments.

References

[1] M. Cimpoeaş, *Several inequalities regarding Stanley depth*, Romanian J. Math. Comput. Sci. **2** (2012), 28–40.

[2] M. Crupi, G. Rinaldo and N. Terai, *Cohen–Macaulay edge ideals whose height is half of the number of vertices*, Nagoya Math. J. **201** (2011), 116–130.

[3] A. Dochtermann and A. Engström, *Algebraic properties of edge ideals via combinatorial topology*, Electron. J. Combin. **16** (2009), Special volume in honor of Anders Björner, Research Paper 2, 24pp.

[4] I. Gitler and C. E. Valencia, *Bounds for invariants of edge-rings*, Comm. Algebra **33** (2005), 1603–1616.

[5] H. T. Hà and A. Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. **27** (2008), 215–245.

[6] J. Herzog and T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality*, J. Algebraic Combin. **22** (2005), 289–302.

[7] M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory Ser. A **113** (2006), 435–454.

[8] K. Kimura, "*Non-vanishingness of Betti numbers of edge ideals*," in *Harmony of Gröbner Bases and the Modern Industrial Society*, World Scientific Publishing, Hackensack, NJ, 2012, 153–168.

[9] K. Kimura, *Non-vanishing of Betti numbers of edge ideals and complete bipartite subgraphs*, Comm. Algebra **44** (2016), 710–730.

[10] M. Kummini, *Regularity, depth and arithmetic rank of bipartite edge ideals*, J. Algebraic Combin. **30** (2009), 429–445.

[11] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai and S. Yassemi, *Vertex decomposability and regularity of very well-covered graphs*, J. Pure Appl. Algebra **215** (2011), 2473–2480.

[12] F. Mohammadi and S. Moradi, *Resolution of unmixed bipartite graphs*, Bull. Korean Math. Soc. **52** (2015), 977–986.

[13] S. Morey and R. H. Villarreal, "*Edge ideals: algebraic and combinatorial properties*," in *Progress in Commutative Algebra 1*, de Gruyter, Berlin, 2012, 85–126.

[14] A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra **38** (2010), 773–784.

[15] G. Rinaldo, N. Terai and K. Yoshida, *Cohen–Macaulayness for symbolic power ideals of edge ideals*, J. Algebra **347** (2011), 1–22.

[16] A. Simis, W. V. Vasconcelos and R. H. Villarreal, *On the ideal theory of graphs*, J. Algebra **167** (1994), 389–416.

[17] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications*, Springer, Berlin/Heidelberg/New York, 1986.

[18] N. Terai, *Alexander duality theorem and Stanley–Reisner rings*, Sūrikaisekikenkyūsho Kōkyūroku **1078** (1999), 174–184; Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998).

[19] A. Van Tuyl, *Sequentially Cohen–Macaulay bipartite graphs: vertex decomposability and regularity*, Arch. Math. (Basel) **93** (2009), 451–459.

[20] A. Van Tuyl and R. H. Villarreal, *Shellable graphs and sequentially Cohen–Macaulay bipartite graphs*, J. Combin. Theory Ser. A **115** (2008), 799–814.

[21] R. Woodroofe, *Matchings, coverings, and Castelnuovo–Mumford regularity*, J. Commut. Algebra **6** (2014), 287–304.

[22] X. Zheng, *Resolutions of facet ideals*, Comm. Algebra **32** (2004), 2301–2324.

Kyouko Kimura
Department of Mathematics
Faculty of Science
Shizuoka University
836 Ohya
Suruga-ku
Shizuoka 422-8529
Japan

kimura.kyoko.a@shizuoka.ac.jp

Naoki Terai
Department of Mathematics
Faculty of Culture and Education
Saga University
Saga 840-8502
Japan

terai@cc.saga-u.ac.jp

Siamak Yassemi
School of Mathematics
Statistics and Computer Sciences
College of Science
University of Tehran
P.O. Box 14155-6455
Tehran
Iran

yassemi@ut.ac.ir

https://doi.org/10.1017/nmj.2017.7 Published online by Cambridge University Press