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CENTRAL RELATIONS ON LATTICES

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Abstract

A maximal tolerance of a lattice L without infinite chains is either a congruence or a central relation. A finite lattice L is order-polynomially complete if and only if L is simple and has no central relation.

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A binary relation ρ is called compatible on a lattice L if for $(a, b) \in \rho$ and $(c, d) \in \rho$ we have $(a \wedge c, b \wedge d) \in \rho$ and $(a \vee c, b \vee d) \in \rho$. A compatible binary relation ρ on L is a tolerance if ρ is reflexive and symmetric. Of course every congruence relation is also a tolerance. First results on tolerances besides congruence relations were already derived in the papers of Hashimoto [5], Grätzer and E. T. Schmidt [7]. Chajda gives in [3] an overview of the theory. In this paper we study central relations which are tolerances having the property that there is a set Z, $\varnothing \subsetneq Z \subsetneq L$, such that $(z, a) \in \rho$ for every $a \in L$ if and only if $z \in Z$. Z is called the center of ρ. The central relations play an important role in the theory of clones [8]. The aim of this paper is to show that every tolerance of a lattice L without infinite chains which is not the all relation is either contained in a congruence or a central relation.

This result can be applied to characterize order-polynomially complete lattices. A lattice L is called order-polynomially complete if every order-preserving function f: $L^n \to L$ is a polynomial function (= algebraic function). Lattices which correspond to finite projective geometries as well as the finite partition lattices have this combinatorial property. We show that a finite lattice is order-polynomially complete if and only if L is simple and has no central relation. This extends a result of Kindermann. Furthermore it is easy to check whether a finite lattice has a central relation because one has only to study single elements as a center for such a relation. This advantage can be used for deriving a new proof for a theorem of Wille [12]. In the following we call a binary relation ρ non trivial if ρ is neither the identity nor the all relation.

PROPOSITION 1. *Let ρ be a nontrivial tolerance of a complete lattice L and assume that ρ is a complete sublattice of L^2. Define $a = \sup\{x \in L \mid (0, x) \in \rho\}$ and $b = \inf\{x \in L \mid (x, 1) \in \rho\}$. If $b \leq a$, then ρ is a central relation.*

PROOF. Consider $Z = \{x \mid b \leq x \leq a\}$. Because of $(0, z) = (0, z) \wedge (z, z)$ we have $(0, z) \in \rho$ and similarly $(z, 1) \in \rho$. For every element $c \in L$ we have $(z, 1) \wedge (c, c) = (z \wedge c, c) \in \rho$ and $(z, 0) \in \rho$ hence $((z \wedge c) \vee z, c \vee 0) = (z, c) \in \rho$. Z is a proper subset of L otherwise ρ would be trivial.

PROPOSITION 2. *Let ρ be an intransitive tolerance of the lattice L. Then there are $a, b, d \in L$ such that $(a, d) \in \rho$, $(d, d) \in \rho$, $a < d < b$ but $(a, b) \notin \rho$.*

PROOF. As ρ is an intransitive tolerance we have $c, e, d \in L$ such that $(c, d) \in \rho$, $(d, e) \in \rho$ but $(c, e) \notin \rho$. If $d = 0$ we have $(c, 0) \vee (0, e) = (c, e) \in \rho$. If $d = 1$ we have $(c, 1) \wedge (1, e) = (c, e) \in \rho$. Therefore we have $0 < d < 1$. We put $a = c \wedge e \wedge d$ and $b = c \vee e \vee d$. Then we have $(a, d) \in \rho$ and $(d, b) \in \rho$. If $(a, b) \in \rho$ then we have $(a, e) \in \rho$ and $(c, a) \in \rho$ hence $(c, e) \in \rho$. Therefore $(a, b) \notin \rho$.

LEMMA 3. *Let L have no infinite chains and let ρ be an intransitive tolerance of L. If ρ is not a central relation then there is a non trivial tolerance η of L such that $\rho \underset{\neq}{\subset} \eta$.*

PROOF. As ρ is intransitive there is a triple (a, d, b) such that $a < d < b$ and $(a, d) \in \rho$, $(d, b) \in \rho$ but $(a, b) \notin \rho$. We put $\eta = \langle \rho \cup \{(a, b), (b, a)\} \rangle$ the tolerance generated by ρ and (a, b), (b, a). Since $\rho' = \rho \cup \{(a, b), (b, a)\}$ is reflexive and symmetric $\eta = \langle \rho' \rangle$ is clearly the sublattice of L^2 generated by ρ'. Assume η is trivial, then we have $\eta = L^2$ and $(0, 1) \in \eta$. Then there exists a term function φ of L^2 with $\varphi((c_1, e_1), \ldots, (c_n, e_n), (a, b), (b, a)) = (0, 1)$ where $\rho \supseteq \{(c_1, e_1), \ldots, (c_n, e_n)\}$ [5] page 46. Since φ is an isotonic function, we have $\varphi((c_1, g_1), \ldots, (c_n, g_n), (a, b), (b, b)) = (0, 1)$ where $g_i = e_i \vee c_i$, $i = 1, \ldots, n$. We have that $(c_i, g_i) \in \rho$ and $c_i \leq g_i$. We split $\varphi = (\psi, \psi)$ in two term functions and we have $\psi(c_1, \ldots, c_n, a, b) = 0$, $\psi(g_1, \ldots, g_n, b, b) = 1$. Furthermore we put $F((x, y)) = \varphi((c_1, g_1), \ldots, (c_n, g_n), (x, y), (b, b))$, $f(x) = \psi(c_1, \ldots, c_n, x, b)$ and

$g(x) = \psi(g_1, \ldots, g_n, x, b)$. We have $f(x) \leqslant g(x)$ for every $x \in L$. Consider $d_* = \inf\{x \mid (x, d) \in \rho,\ x \in L\}$ and $d^* = \sup\{x \mid (x, d) \in \rho,\ x \in L\}$. Then we have $F(d_*, d) = (0, s) \in \rho$ since $(d_*, d) \in \rho$ and $(a, d) \in \rho$ implies $d_* \leqslant a$ and similarly $F(d, d^*) = (t, 1) \in \rho$ for $t \in L$. We have $f(d) = t \leqslant g(d) = s$. Hence we have $(0, s) \in \rho$ and $(s, s) \vee (t, 1) = (s, 1) \in \rho$. s is an element of a center of ρ. Contradiction.

From this lemma it follows

THEOREM 4. *A maximal tolerance of a lattice L with no infinite chains is either a congruence relation or a central relation.*

This theorem can be applied to the following result of M. Kindermann in [7]. A finite lattice is order-polynomially complete if and only if L has only trivial tolerances. Therefore we have

THEOREM 5. *A finite lattice L is order-polynomially complete if and only if L is simple and has no central relation.*

This result is connected to the theory of clones because it states the following: Let $(A; \leqslant)$ be a lattice ordered finite set. The maximal subclones containing the functions $\wedge$ and $\vee$ of the clone of all order-preserving functions of A are either preserving a non trivial equivalence relation or a central relation.

Furthermore from the above theorem and from [10] we have the following

THEOREM 6. *A simple modular lattice L of finite length is a projective geometry if and only if L has no central relation.*

M. Szymańska has proved in [11] that a lattice L of finite length where 1 is the join of atoms has no central relation. Together with Theorem 5 this gives a new proof of Satz 5 in Wille [12].

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