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BOUNDS FOR MOMENTS OF QUADRATIC DIRICHLET CHARACTER SUMS

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Abstract

We establish upper bounds for moments of smoothed quadratic Dirichlet character sums under the generalized Riemann hypothesis, confirming a conjecture of M. Jutila ['On sums of real characters', *Tr. Mat. Inst. Steklova* **132** (1973), 247–250].

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1. Introduction

In this paper, we are interested in estimating the moments of quadratic Dirichlet character sums given by

$$S_m(X,Y) := \sum_{\chi\in\mathcal{S}(X)} \Big|\sum_{n\le Y}\chi(n)\Big|^{2m},$$

where $\mathcal{S}(X)$ denotes the set of all nonprincipal quadratic Dirichlet characters of modulus at most X. Here, $m > 0$ is any real number.

The case $m = 1$ was first studied by Jutila [6] and the best known estimation is given by Armon [1, Theorem 2], who showed that

$$S_1(X,Y) \ll XY(\log X). \tag{1.1}$$

For general m, a conjecture of Jutila [7] asserts that for a positive integer m, there are constants $c_1(m)$, $c_2(m)$, with values depending on m only, such that

$$S_m(X,Y) \le c_1(m)XY^m(\log X)^{c_2(m)}.$$

In [12], Virtanen established a weaker version of this conjecture for the case $m = 2$ with the expression $(\log X)^{c_2(m)}$ replaced by X^{ε} for any $\varepsilon > 0$. Other related bounds can be found in [8, 11].

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It is the aim of this paper to confirm a smoothed version of the conjecture of Jutila under the assumption of the generalised Riemann hypothesis (GRH). More precisely, we consider a sum of the form

$$S_m(X,Y;W) := \sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big|\sum_n \Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^{2m}, \tag{1.2}$$

where W is any nonnegative, smooth function compactly supported on the set of positive real numbers and $(\frac{\cdot}{\cdot})$ denotes the Jacobi symbol. Here we point out (see [10]) that the character $(\frac{\cdot}{\cdot})$ is primitive modulo $8d$ for any positive, odd and square-free d. Our result gives estimations for $S_m(X,Y;W)$ in terms of the conjectured size.

THEOREM 1.1. *With the notation as above and truth of GRH, we have, for large X, Y, $\varepsilon > 0$ and any real $m \ge 1/2$,*

$$S_m(X,Y;W) \ll XY^m(\log X)^{m(2m+1)}. \tag{1.3}$$

Using Hölder's inequality and the estimation in (1.1), we can further improve the result in Theorem 1.1 as follows.

THEOREM 1.2. *With the notation as above and the truth of GRH, we have, for large X, Y, $\varepsilon > 0$ and any real $m \ge 0$,*

$$S_m(X,Y;W) \ll \begin{cases} XY^m(\log X)^m, & 0\le m<1,\\ XY^m(\log X)^{9m-8}, & 1\le m<2,\\ XY^m(\log X)^{m(2m+1)}, & m\ge 2.\end{cases} \tag{1.4}$$

Here the estimation for the case $0 \le m \le 1$ holds unconditionally.

Our proof of Theorem 1.1 is rather simple and makes use of sharp upper bounds on moments of quadratic Dirichlet L-functions. Note that this enables our result to be valid for all real $m \ge 1/2$ instead of just positive integers. Our approach here certainly can be applied to bound moments of various other character sums as well. We also point out that by applying the methods in [2] or [10] to evaluate the moment of $|L(s,\chi_{8d})|^2$ twisted by a Dirichlet character for $\mathrm{Re}(s) = 1/2$ and arguing as in [3], one may establish (1.3) for $1/2 \le m \le 1$ unconditionally.

2. Proof of Theorem 1.1

We apply the Mellin inversion to obtain

$$S_m(X,Y;W) \ll \sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big|\int_{(2)} L(s,\chi_{8d})Y^s\widehat{W}(s)\,ds\Big|^{2m}, \tag{2.1}$$

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where $\widehat{W}$ stands for the Mellin transform of W given by

$$\widehat{W}(s) = \int_0^\infty W(t)t^s \frac{dt}{t}.$$

Observe that integration by parts implies that for any integer $E \geq 0$,

$$\widehat{W}(s) \ll \frac{1}{(1+|s|)^E}. \tag{2.2}$$

Further observe by [5, Corollary 5.20] that under GRH, for $\mathrm{Re}(s) \geq 1/2$ and any $\varepsilon > 0$,

$$L(s, \chi_{8d}) \ll |ds|^{\varepsilon}. \tag{2.3}$$

We note here that a bound weaker than (2.3) would be sufficient and GRH is not indispensable here.

The bounds in (2.3) and (2.2) allow us to shift the line of integration in (2.1) to $\mathrm{Re}(s) = 1/2$ to obtain

$$S_m(X, Y; W) \ll \sum_{\substack{d \leq X \\ (d,2)=1}} \mu^2(d) \bigg| \int_{(1/2)} L(s, \chi_{8d}) Y^s \widehat{W}(s)\, ds \bigg|^{2m}. \tag{2.4}$$

Applying (2.2) and Hölder's inequality (note that this requires the condition that $m \geq 1/2$) yields

$$\begin{aligned}
&\sum_{\substack{d \leq X \\ (d,2)=1}} \mu^2(d) \bigg| \int_{(1/2)} L(s, \chi_{8d}) Y^s \widehat{W}(s)\, ds \bigg|^{2m} \\
&\quad \ll \sum_{\substack{d \leq X \\ (d,2)=1}} \mu^2(d) \bigg(\int_{(1/2)} |\widehat{W}(s)|^{m/(2m-1)}\, |ds| \bigg)^{2m-1} \int_{(1/2)} |L(s, \chi_{8d}) Y^s|^{2m} |\widehat{W}(s)|^m\, |ds| \\
&\quad \ll Y^m \int_{(1/2)} \sum_{\substack{d \leq X \\ (d,2)=1}} \mu^2(d) |L(s, \chi_{8d})|^{2m} \cdot |\widehat{W}(s)|^m\, |ds|.
\end{aligned} \tag{2.5}$$

Now, by (2.2) and (2.3) again, we may truncate the integral in (2.5) to $|\mathrm{Im}(s)| \leq X^{\varepsilon}$ for any $\varepsilon > 0$ with a negligible error. Thus, we see from (2.4) and (2.5) that

$$S_m(X, Y; W) \ll Y^m \int_{|\mathrm{Im}(s)| \leq X^{\varepsilon}} \sum_{\substack{d \leq X \\ (d,2)=1}} \mu^2(d) |L(s, \chi_{8d})|^{2m} \cdot |\widehat{W}(s)|^m\, |ds|. \tag{2.6}$$

We modify the proof of [4, Theorem 2] and the proof of [9, Theorem 2.4] (particularly [9, (8.1)]) to see that under GRH, we have for $|t| \leq X^{\varepsilon}$ and all $m \geq 0$,

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$$\sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)|L(s,\chi_{8d})|^{2m} \ll_m X(\log X)^{m(2m+1)}. \tag{2.7}$$

Note that by applying the argument of Harper [4], one can remove the ε-power on the logarithm in the above-mentioned results in [9]. Now upon inserting (2.7) into (2.6), we immediately obtain the desired result given in (1.3). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We first note that the estimation given in (1.1) is still valid with $S_1(X,Y)$ being replaced by $S_1(X,Y;W)$ for any compactly supported W by going through the proof of [1, Theorem 2]. We apply Hölder's inequality to see from this and (1.2) that for $0\le m<1$,

$$\begin{aligned}S_m(X,Y;W) &= \sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big(1\cdot\Big|\sum_n\Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^{2m}\Big)\\ &\le \Big(\sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big)^{1-m}\Big(\sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big|\sum_n\Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^2\Big)^m \ll XY^m(\log X)^m.\end{aligned}$$

This gives the estimation for the case $0\le m<1$ in (1.4).

Similarly, if $1\le m<2$, we deduce from (1.1) and (1.3) that for any $p\ge 1$,

$$\begin{aligned}S_m(X,Y;W) &= \sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big(\Big|\sum_n\Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^{2/p}\cdot\Big|\sum_n\Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^{2m-2/p}\Big)\\ &\le \Big(\sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big|\sum_n\Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^2\Big)^{1/p}\\ &\quad\times\Big(\sum_{\substack{d\le X\\(d,2)=1}} \mu^2(d)\Big|\sum_n\Big(\frac{8d}{n}\Big)W\Big(\frac{n}{Y}\Big)\Big|^{(2m-2/p)/(1-1/p)}\Big)^{1-1/p}\\ &\ll XY^m(\log X)^{1/p+(m-1/p)((2m-2/p)(1-1/p)^{-1}+1)}.\end{aligned}$$

We optimize the exponent of $\log X$ by setting $1/p=2-m$ to obtain the desired estimation given in (1.4) for the case $1\le m<2$. Note that when $0\le m\le 1$, our estimations above are valid unconditionally since (1.1) holds unconditionally. As the case $m\ge 2$ in (1.4) is just that given in (1.3), this completes the proof of Theorem 1.2.

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