J. Austral. Math. Soc. (Series A) 59 (1995), 287–303

THE LOCALLY CONVEX TOPOLOGY ON THE SPACE OF MEROMORPHIC FUNCTIONS

KARL-GOSWIN GROSSE-ERDMANN

(Received 21 May, 1993; revised 20 April, 1994)

Communicated by P. G. Dodds

Abstract

We give a positive answer to a question of Horst Tietz. A theorem of his that is related to the Mittag-Leffler theorem looks like a duality result under some locally convex topology on the space of meromorphic functions. Tietz has posed the problem of finding such a topology. It is shown that a topology introduced by Holdgrün in 1973 solves the problem. The main tool in the study of this topology is a projective description of it that is derived here. We also argue that Holdgrün's topology is the natural locally convex topology on the space of meromorphic functions.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 30D30, 46E10; secondary 12J99, 46E25.

1. A problem of Tietz

Let Ω be a domain in $\mathbb{C}$. One natural way of endowing the space $M(\Omega)$ of meromorphic functions in Ω with a topology is the following. One regards $M(\Omega)$ as a subspace of the space $C(\Omega, \widehat{\mathbb{C}})$ of all continuous functions on Ω with values in the extended complex plane $\widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ carries the chordal metric and $C(\Omega, \widehat{\mathbb{C}})$ is endowed with the topology of locally uniform convergence. With the inherited topology, τ_{chor} say, $M(\Omega)$ becomes a metric space; and it is complete if we add the function $f(z) \equiv \infty$ (cf. [13, VII.3]). However, this topology is not compatible with the linear structure of $M(\Omega)$. An interesting result of Cima and Schober [11] says that there is indeed no locally convex Hausdorff topology on $M(\Omega)$ that is comparable with τ_{chor}.

On the other hand a result of Horst Tietz [25] suggests that the space $M(\Omega)$ carries a natural linear (locally convex) topology. Recall that by the Mittag-Leffler theorem

© 1995 Australian Mathematical Society 0263-6115/95 $A2.00 + 0.00

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

every meromorphic function f in Ω can be written as

$$f(z) = g(z) + \sum_k \left(h^{\alpha_k}(z) - r_k(z)\right), \tag{1}$$

where the h^{α_k} are the principal parts of f at its poles α_k, and r_k and g are certain holomorphic functions in Ω. Now, Tietz characterised those meromorphic functions for which all r_k (and g) may be chosen 0 (for notation and definitions see below):

THEOREM 0. (Tietz [25, §5]) *Let $f \in M(\Omega)$ with poles $\alpha_1, \alpha_2, \dots$. Let (γ_n) be a sequence of Cauchy cycles in Ω such that* (int $\gamma_n)_n$ *is an exhaustion of Ω and such that $\alpha_1, \dots, \alpha_{k_n}$ lie inside γ_n and $\alpha_{k_{n+1}}, \alpha_{k_{n+2}}, \dots$ lie outside γ_n for $n \in \mathbb{N}$. Then*

(a) *The following assertions are equivalent:*

(i) *There is a function $g \in H(\Omega)$ with*

$$f(z) = g(z) + \lim_{n\to\infty} \sum_{k \le k_n} h^{\alpha_k}(z)$$

locally uniformly in $\Omega \setminus \{\alpha_1, \alpha_2, \dots\}$.

(ii) *For all $\varphi \in H_0(\widehat{\mathbb{C}} \setminus \Omega)$ the limits*

$$\lim_{n\to\infty} \int_{\gamma_n} f(\zeta)\varphi(\zeta)d\zeta$$

exist.

(b) *In* (i) *one has $g(z) \equiv 0$ if and only if all limits in* (ii) *vanish.*

Note that the integrals in (a)(ii) may only be defined for sufficiently large n depending on φ.

This theorem looks like a duality result for the space $M(\Omega)$ with respect to some locally convex topology. Tietz [26] has posed the problem of finding such a topology. Our primary aim in this paper is to study a certain locally convex topology on $M(\Omega)$ that may be considered natural. We will show in particular that it satisfies Tietz's demands.

This topology was introduced by Holdgrün [17] in 1973, but it has not yet been studied in any detail. After some preliminaries in Section 2 we recall its definition and obtain a useful projective description in Section 3. This allows us to derive its main properties in the following section. In Section 5 we determine the dual of $M(\Omega)$ and show that Holdgrün's topology does indeed solve Tietz's problem. In an appendix we give a survey of other topologies on $M(\Omega)$ that have been defined in the literature.

The investigations in this paper parallel those of Golovin [15, 16] for the space of holomorphic functions with (arbitrary) isolated singularities. The relevance of

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

Golovin's paper to the problem of topologising spaces of meromorphic functions does not seem to have been noticed before.

Notation and definitions. Let Ω be a domain in $\widehat{\mathbb{C}}$. The principal part of a meromorphic function in Ω at a point $\alpha \in \Omega$ is denoted by h^α. The linear mappings $f \mapsto a_j^\alpha(f)$ $(\alpha \in \Omega,\, j \in \mathbb{N})$ on $M(\Omega)$ are defined by $h^\alpha(z) = \sum_{j=1}^{\infty} a_j^\alpha(f)/(z-\alpha)^j$. Here and throughout we interpret $1/(z-\alpha)^j$ as z^j if $\alpha = \infty$.

An exhaustion (D_n) of a domain Ω is a sequence of relatively compact subsets D_n of Ω with $\Omega = \bigcup_{n=1}^{\infty} D_n$ and $\overline{D_n} \subset (D_{n+1})^\circ$ for $n \in \mathbb{N}$.

Let γ be a cycle ([24, 10.34]). A point $\alpha \in \mathbb{C}$ is said to lie inside γ if $\operatorname{ind}_\gamma(\alpha) = 1$ and outside γ if $\operatorname{ind}_\gamma(\alpha) = 0$, and we set $\operatorname{int}\gamma = \{\alpha \in \mathbb{C} : \operatorname{ind}_\gamma(\alpha) = 1\}$ (note that this differs from [22]). We call γ a *Cauchy cycle* (for a set D) in Ω if $\mathbb{C} \setminus \Omega$ lies outside γ (and D lies inside γ). For every compact subset K of any domain Ω in $\mathbb{C}$ there exists a Cauchy cycle for K in Ω ([24, proof of 13.5]). The name derives from the validity of Cauchy's theorem for such cycles ([24, 13.5]).

For further terminology from complex analysis we refer to [13, 22, 23, 24]. The terminology from functional analysis follows [18, 19]; for special areas see [6, 29, 30].

2. Preliminaries

1. Let Ω be a domain in $\widehat{\mathbb{C}}$. Then we denote by $H\mathscr{R}(\Omega)$ the space of all functions f on Ω that can be written as a sum of a holomorphic function and a rational function with poles in Ω, that is, with

$$f(z) = g(z) + \sum_{k=1}^{m}\sum_{j=1}^{n_k} \frac{a_j^{\alpha_k}}{(z-\alpha_k)^j} = g(z) + \sum_{\alpha\in\Omega}\sum_{j=1}^{\infty} \frac{a_j^{\alpha}}{(z-\alpha)^j}$$

in Ω, where $g \in H(\Omega)$ and $a_j^\alpha = a_j^\alpha(f) \in \mathbb{C}$ $(\alpha \in \Omega,\, j \in \mathbb{N})$ are uniquely determined. Thus we obtain an algebraic isomorphism

$$H\mathscr{R}(\Omega) \cong H(\Omega) \times \mathbb{C}^{(\Omega\times\mathbb{N})}, \quad f \mapsto \left(g, (a_j^\alpha)_{\alpha\in\Omega, j\in\mathbb{N}}\right).$$

We endow $H\mathscr{R}(\Omega)$ with the (locally convex) product topology of the topology of locally uniform convergence in $H(\Omega)$ with the locally convex direct sum topology of $\mathbb{C}^{(\Omega\times\mathbb{N})} = \bigoplus_{\Omega\times\mathbb{N}} \mathbb{C}$. Note that $H\mathscr{R}(\Omega)$ is simply the space of all meromorphic functions on Ω with (only) finitely many poles. In particular, $H\mathscr{R}(\widehat{\mathbb{C}}) = M(\widehat{\mathbb{C}}) = \mathbb{C}(z)$ is the space of all rational functions and carries its strongest locally convex topology (cf. [18, p. 111]).

In a similar fashion, if K is a compact subset of $\widehat{\mathbb{C}}$, we define and topologise the space $H\mathscr{R}(K) \cong H(K) \times \mathbb{C}^{(K\times\mathbb{N})}$ of (germs of) meromorphic functions on K and,

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

in case K has no isolated points, the space $A\mathscr{R}(K) \cong A(K) \times \mathbb{C}^{(K\times\mathbb{N})}$. Here, $A(K)$ denotes the space of continuous functions in K that are holomorphic in K°, endowed with the topology of uniform convergence on K.

2. Let Ω be a domain in $\widehat{\mathbb{C}}$ and $\delta : \Omega \to \mathbb{N}_0$ a positive divisor ([23, 3.1.1]), that is, a function with the property that the set $P_\delta = \{\alpha \in \Omega : \delta(\alpha) \neq 0\}$ is discrete in Ω. Then let $M(\Omega;\delta)$ denote the space of all meromorphic functions in Ω with the property that all its poles lie in P_δ and the order of a pole $\alpha \in P_\delta$ is at most $\delta(\alpha)$, that is,

$$M(\Omega;\delta) = \{f \in M(\Omega) : o_\alpha(f) \geq -\delta(\alpha) \text{ for all } \alpha \in \Omega\}.$$

Then it follows from the integral formula for Laurent coefficients that this space is a closed subspace of $H(\Omega \setminus P_\delta)$. We endow $M(\Omega;\delta)$ with the inherited topology, making it a Fréchet space; see [5].

3. The Holdgrün topology and its projective description

Let Ω be a fixed domain in $\widehat{\mathbb{C}}$. One may look at the concept of meromorphy of a function on Ω in (at least) two different ways.

1. A meromorphic function is everywhere holomorphic apart from a discrete set where it may have poles. In this spirit we can write $M(\Omega) = \bigcup_{\delta\in\mathscr{D}_\Omega} M(\Omega;\delta)$, where $\mathscr{D}_\Omega$ denotes the space of all positive divisors on Ω and the spaces $M(\Omega;\delta)$ are topologised as in the previous section. The set $\mathscr{D}_\Omega$ carries a natural order defined by $\delta_1 \leq \delta_2$ if $\delta_1(\alpha) \leq \delta_2(\alpha)$ for all $\alpha \in \Omega$. Then the inclusion mappings $M(\Omega;\delta_1) \hookrightarrow M(\Omega;\delta_2)$ are continuous for $\delta_1 \leq \delta_2$. Thus the locally convex inductive limit $\operatorname{ind}_{\delta\in\mathscr{D}_\Omega} M(\Omega;\delta)$ with respect to these inclusion mappings is defined and gives the Holdgrün topology τ_{Hol} on $M(\Omega)$.

REMARK 1. The topology was actually defined by Holdgrün [17, p.44] in a much more general setting. Earlier, Burmann and Holdgrün [9] had given a corresponding definition for spaces of meromorphic functions with only finitely many poles. Constantinescu and Gheondea [12] also obtain a (weaker) topology in some space of meromorphic functions via a similar inductive limit.

Although we have here a strict inductive limit – the topology of $M(\Omega;\delta_1)$ coincides with the one inherited from $M(\Omega;\delta_2)$ for $\delta_1 \leq \delta_2$ –, it is an uncountable limit so that we cannot apply the very strong results for countable strict inductive limits (see §24, in particular 3.3, in [14]). Thus we would have to study the topology τ_{Hol} in more detail to see, for example, if it is complete (cf. [17, pp. 46f]). Instead, we will here give an alternative description of this topology, namely, as a projective limit. This is based on an alternative view of meromorphic functions. The idea behind the Mittag-Leffler

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

theorem is to regard them as the sum of a holomorphic function and its principal parts. While in general one has to introduce additional terms r_k to make the Mittag-Leffler expansion (1) convergent, we always have the following.

2. Locally, a meromorphic function is the sum of a holomorphic function and its principal parts. This means that for any domain $O \subset \Omega$ that is relatively compact in Ω every meromorphic function f is the sum of a holomorphic function in O and the principal parts at its (finitely many) poles in O, that is, f belongs to the space $H\mathscr{R}(O)$. Hence we have $M(\Omega) = \bigcap_{O\in\mathscr{C}_\Omega} H\mathscr{R}(O)$, where $\mathscr{C}_\Omega$ denotes the set of all relatively compact subdomains of Ω, ordered by set inclusion.

Thus, for $O, O_1, O_2 \in \mathscr{C}_\Omega$ with $O_1 \subset O_2$ we consider the restriction mappings

$$\rho_O : M(\Omega) \to H\mathscr{R}(O), \quad f \mapsto f|_O \text{ and } \rho_{O_1,O_2} : H\mathscr{R}(O_2) \to H\mathscr{R}(O_1), \quad f \mapsto f|_{O_1}.$$

These correspond to the mappings

$$T_O : M(\Omega) \to H(O) \times \mathbb{C}^{(O\times\mathbb{N})}, \quad f \mapsto \left(f|_O - \sum_{\alpha\in O} h^\alpha,\ (a_j^\alpha(f))_{\alpha\in O, j\in\mathbb{N}} \right)$$

and

$$T_{O_1,O_2} : \begin{cases} H(O_2) \times \mathbb{C}^{(O_2\times\mathbb{N})} & \to \qquad H(O_1) \times \mathbb{C}^{(O_1\times\mathbb{N})} \\ (g, a) & \mapsto \left(g|_{O_1} + \displaystyle\sum_{\alpha\in O_2\setminus O_1} \sum_{j=1}^{\infty} \frac{a_j^\alpha}{(\cdot - \alpha)^j},\ a|_{O_1\times\mathbb{N}} \right). \end{cases}$$

The mappings T_{O_1,O_2} are easily seen to be continuous. Hence the projective limit $\text{proj}_{O\in\mathscr{C}_\Omega} H\mathscr{R}(O)$ with respect to the restriction mappings ρ_{O_1,O_2} is defined and is algebraically isomorphic to $M(\Omega)$. This provides us with a new locally convex topology on $M(\Omega)$ that we call the Mittag-Leffler topology, denoted by τ_{ML}. It is the weakest topology such that all mappings ρ_O, $O \in \mathscr{C}_\Omega$, are continuous.

REMARK 2. (i) In the particular case of $\Omega = \widehat{\mathbb{C}}$, when $M(\widehat{\mathbb{C}}) = \mathbb{C}(z)$, we have $\Omega \in \mathscr{C}_\Omega$, so that $\text{proj}_{O\in\mathscr{C}_\Omega} H\mathscr{R}(O) = H\mathscr{R}(\widehat{\mathbb{C}})$. Hence on the space of rational functions τ_{ML} is the strongest locally convex topology (cf. Section 2.1). This topology was also considered by Williamson [31, 4].

(ii) Instead of restricting the elements $f \in M(\Omega)$ to subdomains O we may also restrict them to compact subsets K. This leads to the representations

$$M(\Omega) = \bigcap_{K\in\mathscr{K}_\Omega} H\mathscr{R}(K) \quad \text{and} \quad M(\Omega) = \bigcap_{K\in\mathscr{K}'_\Omega} A\mathscr{R}(K),$$

where $\mathscr{K}_\Omega$ ($\mathscr{K}'_\Omega$) denotes the set of compact subsets of Ω (without isolated points), and as above we may introduce topologies on $M(\Omega)$ as projective limits with respect

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

to the corresponding restriction maps. It follows easily from the general theory of projective limits that in fact we have equivalent projective systems, that is, that the new topologies coincide with τ_{ML} (see [14, p. 38]).

(iii) Bobillo Guerrero [7] and Cima and Schober [11] have defined topologies in a similar fashion. However, they topologise $A\mathscr{R}(K) \cong A(K) \times \mathbb{C}^{(K\times\mathbb{N})}$ by giving $\mathbb{C}^{(K\times\mathbb{N})}$ weaker norm topologies than its direct sum topology. The resulting metrisable topologies on $M(\Omega)$ are not complete and strictly weaker than τ_{ML} (see Remark 5(ii)(a) and Theorem 3(b) below).

We will now determine a convenient set of seminorms that defines τ_{ML}. Let $K \subset \Omega$ be compact and $b = (b_j^\alpha)_{\alpha\in K, j\in\mathbb{N}} \in \mathbb{C}^{K\times\mathbb{N}}$. For $f \in M(\Omega)$ we put

$$\|f\|_{K,b} = \sup_{z\in K}\Big|f(z) - \sum_{\alpha\in K} h^\alpha(z)\Big| + \sum_{\alpha\in K}\sum_{j=1}^{\infty} \big|a_j^\alpha(f)\, b_j^\alpha\big|.$$

Then $\|\cdot\|_{K,b}$ is well-defined and a seminorm on $M(\Omega)$.

THEOREM 1. *The topology* τ_{ML} *is generated by the directed system of seminorms* $\|\cdot\|_{K,b}$ ($K \subset \Omega$ *compact,* $b \in \mathbb{C}^{K\times\mathbb{N}}$).

PROOF. By Remark 2(ii), τ_{ML} is the projective topology on $M(\Omega)$ with respect to the mappings

$$T_K : M(\Omega) \to A(K) \times \mathbb{C}^{(K\times\mathbb{N})}, \quad f \mapsto \left(f|_K - \sum_{\alpha\in K} h^\alpha, (a_j^\alpha(f))_{\alpha\in K, j\in\mathbb{N}}\right).$$

Since the topology of $A(K)$ is induced by $\|g\| = \sup_{z\in K}|g(z)|$, that of $\mathbb{C}^{(K\times\mathbb{N})}$ by $\|a\|_b = \sum_{\alpha\in K}\sum_{j=1}^{\infty}|a_j^\alpha b_j^\alpha|$ for $b \in \mathbb{C}^{K\times\mathbb{N}}$ (cf. [19, §30.2]), the claim follows ([14, p. 36]).

Our next aim is to show that the Holdgrün topology and the Mittag-Leffler topology on $M(\Omega)$ coincide.

We need some preparation. For any non-empty subset D of Ω let $\mathscr{R}(\Omega; D)$ denote the space of all rational functions of the form $f(z) = \sum_{\alpha\in D}\sum_{j=1}^{\infty} a_j^\alpha/(z-\alpha)^j$, considered as a subspace of $M(\Omega)$. It is algebraically isomorphic to $\mathbb{C}^{(D\times\mathbb{N})}$.

LEMMA 1. *Let* D *be a non-empty and relatively compact subset of* Ω. *On* $\mathscr{R}(\Omega; D)$ *the topology* τ_{ML} *coincides with the one induced by* $\mathbb{C}^{(D\times\mathbb{N})}$.

PROOF. This follows since for every relatively compact subdomain O of Ω containing the (compact) closure $\overline{D}$ of D the topology on $\mathscr{R}(\Omega; D)$ induced by $H\mathscr{R}(O)$ coincides with the one induced by $\mathbb{C}^{(D\times\mathbb{N})}$.

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

Note that this result is not true for arbitrary subsets D of Ω: While $1/(\cdot - n) \to 0$ in $M(\mathbb{C})$ as $n \to \infty$, the corresponding elements do not converge in $\mathbb{C}^{(\mathbb{N}\times\mathbb{N})}$.

LEMMA 2. (a) *Every τ_{ML}-bounded subset of $M(\Omega)$ is contained and bounded in a space $M(\Omega; \delta)$ for some $\delta \in \mathscr{D}_\Omega$.*

(b) *The space $(M(\Omega), \tau_{\mathrm{ML}})$ is bornological.*

PROOF. (a) Let $B \subset M(\Omega)$ be τ_{ML}-bounded. For every relatively compact subdomain O of Ω, $T_O(B)$ is a bounded subset of $H(O) \times \mathbb{C}^{(O\times\mathbb{N})}$, so that for almost all $\alpha \in O$ and $j \in \mathbb{N}$ we have $a_j^\alpha(f) = 0$ for all $f \in B$. This implies that there is a positive divisor $\delta : \Omega \to \mathbb{N}_0$ such that $a_j^\alpha(f) = 0$ for all $\alpha \in \Omega$ and $j \in \mathbb{N}$ with $j > \delta(\alpha)$ and all $f \in B$. Hence B is contained in $M(\Omega; \delta)$. Now let K be a compact subset of $\Omega \setminus P_\delta$, where $P_\delta = \{\alpha \in \Omega : \delta(\alpha) \neq 0\}$. Since no element $f \in B$ has a pole in K, Theorem 1 and the boundedness of B imply that $\sup_{f\in B} \sup_{z\in K} |f(z)| < \infty$. Since $M(\Omega; \delta)$ carries the topology of $H(\Omega \setminus P_\delta)$, we see that B is bounded in this space.

(b) Let U be an absolutely convex subset of $M(\Omega)$ that absorbs every τ_{ML}-bounded subset of $M(\Omega)$. We firstly show:

(+) There exists a relatively compact subdomain O of Ω, a compact subset K of O and some $\varepsilon > 0$ such that, if $f \in M(\Omega)$ has no pole in O and satisfies $\sup_{z\in K} |f(z)| < \varepsilon$, then $f \in U$.

Assume that (+) does not hold. Choose an exhaustion (O_k) of Ω by relatively compact subdomains. Then there are functions $f_k \in M(\Omega)$ without poles in O_{k+1} such that $\sup_{z\in\overline{O_k}} |f_k(z)| < 1/k$ but $f_k \notin U$ for $k \in \mathbb{N}$. Consider the set $B = \{kf_k : k \in \mathbb{N}\}$, and let O_0 be a relatively compact subdomain of Ω. Since there is some $k_0 \in \mathbb{N}$ with $O_0 \subset O_k$ for $k \geq k_0$, it follows that $T_{O_0}(B)$ is a bounded subset of $H(O_0) \times \mathbb{C}^{(O_0\times\mathbb{N})}$. Hence B is a bounded subset of $M(\Omega)$, but clearly not absorbed by U.

It follows that (+) holds with some $K \subset O \subset \Omega$ and $\varepsilon > 0$. By Lemma 1 the space $\mathscr{R}_O := \mathscr{R}(\Omega; O)$, considered as a subspace of $M(\Omega)$, is bornological. Now, $U \cap \mathscr{R}_O$ is an absolutely convex subset of $\mathscr{R}_O$ and absorbs every bounded subset of $\mathscr{R}_O$. Hence $U \cap \mathscr{R}_O$ is a 0-neighbourhood in $\mathscr{R}_O \cong \mathbb{C}^{(O\times\mathbb{N})}$, so that the set

$$V = \left\{ f \in M(\Omega) : \sup_{z\in K} \left| f(z) - \sum_{\alpha\in O} h^\alpha(z) \right| < \varepsilon \text{ and } \sum_{\alpha\in O} h^\alpha \in U \cap \mathscr{R}_O \right\}$$

is a 0-neighbourhood in $M(\Omega)$. For every $f \in V$ we have by (+)

$$f = \left(f - \sum_{\alpha\in O} h^\alpha \right) + \sum_{\alpha\in O} h^\alpha \in U + (U \cap \mathscr{R}_O) \subset 2U.$$

Thus we have $\frac{1}{2}V \subset U$, so that U is a 0-neighbourhood of $M(\Omega)$.

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

THEOREM 2. *In* $M(\Omega)$ *we have* $\tau_{\mathrm{ML}} = \tau_{\mathrm{Hol}}$.

PROOF. For $\tau_{\mathrm{Hol}} \geq \tau_{\mathrm{ML}}$ it suffices to show that the inclusion mappings $M(\Omega; \delta) \hookrightarrow H\mathscr{R}(O)$ are continuous for every $\delta \in \mathscr{D}_\Omega$ and $O \in \mathscr{C}_\Omega$. Let $P_\delta = \{\alpha \in \Omega : \delta(\alpha) \neq 0\}$ and let $f_k \to 0$ in $M(\Omega; \delta)$. Then we can write

$$f_k(z) = g_k(z) + \sum_{\alpha \in P_\delta \cap O} \sum_{j=1}^{\delta(\alpha)} \frac{a_j^\alpha(f_k)}{(z-\alpha)^j}$$

with $g_k \in H(O)$, and since $f_k(z) \to 0$ locally uniformly in $\Omega \setminus P_\delta$, the integral formula for Laurent coefficients implies that $a_j^\alpha(f_k) \to 0$ for all $\alpha \in P_\delta \cap O$, $j \leq \delta(\alpha)$, hence $\left(a_j^\alpha(f_k)\right)_{\alpha \in O, j \in \mathbb{N}} \to 0$ in $\mathbb{C}^{(O \times \mathbb{N})}$. As a consequence we have $g_k(z) \to 0$ locally uniformly in $O \setminus P_\delta$, hence in O by the maximum principle, that is, $g_k \to 0$ in $H(O)$. This shows that $f_k \to 0$ in $H\mathscr{R}(O)$.

Conversely, by Lemma 2(a) every τ_{ML}-bounded subset of $M(\Omega)$ is contained and bounded in some space $M(\Omega; \delta)$, $\delta \in \mathscr{D}_\Omega$, hence in $(M(\Omega), \tau_{\mathrm{Hol}})$. Since by Lemma 2(b) the space $(M(\Omega), \tau_{\mathrm{ML}})$ is bornological, this implies that $\tau_{\mathrm{Hol}} \leq \tau_{\mathrm{ML}}$. Altogether we have the equality of the two topologies.

Thus we see that there are two distinct but completely natural ways of defining a locally convex topology in the space $M(\Omega)$, and the two resulting topologies τ_{Hol} and τ_{ML} coincide. Since, in addition, we shall see in the next section that this topology is complete, we may consider it as the natural locally convex topology on $M(\Omega)$. Moreover $(M(\Omega), \tau_{\mathrm{ML}})$ is a projective limit of relatively simple spaces so that one may call τ_{ML} the projective description of Holdgrün's topology.

We also note the following.

REMARK 3. It is not difficult to see that $H\mathscr{R}(O) = \mathrm{ind}_{\delta \in \mathscr{D}_\Omega} M(O; \delta|_O)$ for any $O \in \mathscr{C}_\Omega$, the locally convex inductive limit being taken with respect to the inclusion mappings, while $M(\Omega; \delta) = \mathrm{proj}_{O \in \mathscr{C}_\Omega} M(O; \delta|_O)$ for any $\delta \in \mathscr{D}_\Omega$, the projective limit being taken with respect to the restriction mappings. Thus Theorem 2 can also be read as a result on interchanging limits (compare [16, p. 22]):

$$\mathrm{proj}_{O \in \mathscr{C}_\Omega} \mathrm{ind}_{\delta \in \mathscr{D}_\Omega} M(O; \delta|_O) = \mathrm{ind}_{\delta \in \mathscr{D}_\Omega} \mathrm{proj}_{O \in \mathscr{C}_\Omega} M(O; \delta|_O).$$

4. Properties of the locally convex topology on $M(\Omega)$

Let Ω be a domain in $\widehat{\mathbb{C}}$. We regard $M(\Omega)$ endowed with the topology $\tau_{\mathrm{ML}} = \tau_{\mathrm{Hol}}$.

THEOREM 3. (a) *$M(\Omega)$ is a complete Hausdorff locally convex space. It is ultrabornological, hence also barrelled and bornological. It is a Montel space, hence*

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

also reflexive and weakly sequentially complete with the Schur property (that is, every weakly convergent sequence converges (strongly)).

(b) *$M(\Omega)$ is not metrisable or separable, nor is it nuclear or a Schwartz space.*

PROOF. (a) The space $M(\Omega)$ is a complete Hausdorff space as a projective limit of such spaces. It is ultrabornological as a locally convex inductive limit of Fréchet spaces. It is a Montel space since it is barrelled and a projective limit of Montel spaces. The remainder follows from general results ([18]).

(b) By Lemma 1, $M(\Omega)$ contains a subspace isomorphic to $\mathbb{C}^{(A)}$ for some uncountable set A. Since this space is not metrisable, nuclear or a Schwartz space, the same follows for $M(\Omega)$ (see [18, p. 202]). To see that it is not separable, let (f_n) be any sequence in $M(\Omega)$. Let $\alpha \in \Omega$ be a point that is not a pole of any f_n, and let f be the function $f(z) = 1/(z - \alpha)$. Then, with the seminorms of Theorem 1, we have $\|f_n - f\|_{\{\alpha\},b} \geq |b_1^\alpha|$ for all $n \in \mathbb{N}$ and $b \in \mathbb{C}^{\{\alpha\}\times\mathbb{N}}$. Hence the f_n cannot form a dense set in $M(\Omega)$.

We had noted earlier that the inductive limit τ_{Hol} is a strict inductive limit. It is well known that countable strict inductive limits have strong properties. We will show that some of them are shared by the present uncountable inductive limit.

THEOREM 4. (a) *A subset B of $M(\Omega)$ is bounded if and only if it is contained and bounded in some step $M(\Omega; \delta)$, that is, the inductive limit $\mathrm{ind}_{\delta\in\mathscr{D}_\Omega} M(\Omega; \delta)$ is regular.*

(b) *Each space $M(\Omega; \delta)$ $(\delta \in \mathscr{D}_\Omega)$ is a closed subspace of $M(\Omega)$, and its topology coincides with the one inherited from $M(\Omega)$. In particular, $M(\Omega)$ induces on $H(\Omega)$ the topology of locally uniform convergence.*

(c) *A sequence (f_n) converges in $M(\Omega)$ if and only if it is contained and convergent in some step $M(\Omega; \delta)$.*

PROOF. (a) follows from Lemma 2(a) (with Theorem 2).

(b) The topology of $M(\Omega; \delta)$ is stronger than the one inherited from $M(\Omega)$. Conversely, for every compact subset K of $\Omega \setminus \{\alpha \in \Omega : \delta(\alpha) \neq 0\}$ we have, with the seminorms of Theorem 1, $\sup_{z\in K} |f(z)| = \|f\|_{K,b}$ for $f \in M(\Omega; \delta)$, where b is any fixed element in $\mathbb{C}^{K\times\mathbb{N}}$. This shows that the topology of $M(\Omega; \delta)$ is also weaker than the one inherited from $M(\Omega)$. As a complete space, $M(\Omega; \delta)$ now becomes a closed subspace of $M(\Omega)$. The last claim in (b) follows since $H(\Omega) = M(\Omega; \delta)$ for $\delta(\alpha) \equiv 0$.

(c) is now a consequence of (a) and (b).

REMARK 4. Golovin [15, 16] has defined a locally convex topology on the space of all functions that are holomorphic in Ω with possible exception of (arbitrary) isolated

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

singularities, hence a space that contains $M(\Omega)$. Our treatment here is analogous to his. By its definition, Golovin's topology, restricted to $M(\Omega)$, is weaker than $\tau_{\mathrm{Hol}} = \tau_{\mathrm{ML}}$. In fact it is strictly weaker: Fix some $\alpha \in \Omega$ and consider the functions $f_n \in M(\Omega)$, given by $f_n(z) = 1/((n^n(z-\alpha)^n)$ for $n \in \mathbb{N}$. From [16, Proposition 4] it follows that $f_n \to 0$ in Golovin's topology while we have that $f_n \not\to 0$ in $M(\Omega)$ by Assertion (c) of the last theorem.

Algebraically, the space $M(\Omega)$ has a rich structure. Apart from being a vector space it is also a field under the usual definition of the product of two meromorphic functions. In a different language, $M(\Omega)$ is a (commutative) division algebra. The locally convex topology of $M(\Omega)$ is well-behaved with respect to the linear structure of the space, but less so with respect to its multiplicative structure, as we see now.

THEOREM 5. *The operation of multiplication of two functions is separately continuous but not jointly continuous on* $M(\Omega)$. *Also, inversion is not continuous on* $M(\Omega)^\times = M(\Omega) \setminus \{0\}$.

PROOF. We consider the map $\cdot : M(\Omega) \times M(\Omega) \to M(\Omega)$, $(f, g) \mapsto fg$. Let us fix $g \in M(\Omega)$. Then to every positive divisor δ on Ω there is a positive divisor $\widetilde{\delta}$ so that the mapping $M(\Omega; \delta) \to M(\Omega; \widetilde{\delta})$, $f \mapsto fg$ is well-defined, and it is clearly continuous. Hence the linear mapping $M(\Omega) \to M(\Omega)$, $f \mapsto fg$ is continuous. By symmetry, the product $\cdot$ is separately continuous.

Now let us assume that the product is also jointly continuous. Let $K \subset \Omega$ be an uncountable compact set, and let $e = (1)_{\alpha \in K}$. Then there is a compact subset $\widetilde{K} \supset K$ of Ω, some $b \in \mathbb{C}^{\widetilde{K} \times \mathbb{N}}$ and some $c > 0$ such that

$$\|fg\|_{K,e} \le c \|f\|_{\widetilde{K},b} \|g\|_{\widetilde{K},b} \tag{2}$$

for all $f, g \in M(\Omega)$ (see Theorem 1). For $\alpha, \beta \in K$ define $f_{\alpha,\beta}(z) = 1/(z-\alpha) + 1/(z-\beta)$. Then, setting $f = g = f_{\alpha,\beta}$ in (2) with $\alpha \neq \beta$ gives

$$2 + \frac{4}{|\alpha - \beta|} \le c\ (|b_1^\alpha| + |b_1^\beta|)^2. \tag{3}$$

Now, since K is uncountable, there is an infinite subset K_0 of K with $\sup_{\alpha \in K_0} |b_1^\alpha| < \infty$. Since K_0 has an accumulation point, we obtain a contradiction to (3).

Finally, consider the functions f_n defined by $f_n(z) = z + 1/n$. Although (f_n) is convergent in $M(\Omega)^\times$, the sequence $(1/f_n)$ does not converge in $M(\Omega)$ by Theorem 4(c).

REMARKS 5. (i) The result generalises [31, Theorem 2] where the non-continuity of multiplication is proved for $M(\widehat{\mathbb{C}}) = \mathbb{C}(z)$.

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

(ii) The theorem says that $M(\Omega)$ is a topological algebra in a weak but not in a strong sense (i.e., multiplication is separately but not jointly continuous); and it is not a topological field. This may seem undesirable. But in fact there are limitations on how well a topology on $M(\Omega)$ can behave.

(a) *There is no completely metrisable vector space topology on $M(\Omega)$ in which, for some $\alpha \in \Omega$, every functional $f \mapsto a_j^\alpha(f)$ $(j \in \mathbb{N})$ is continuous.*

(b) *There is no Hausdorff locally convex topology on $M(\Omega)$ that is a field topology; in other words, there is no Hausdorff locally convex algebra topology (even in the weak sense) on $M(\Omega)$ in which inversion is continuous.*

For (a) note that $M(\Omega) = \bigcup_{j=1}^\infty \{f : a_j^\alpha(f) = 0\}$. Hence by the Baire category theorem some set $\{f : a_j^\alpha(f) = 0\}$ would have to have non-empty interior. This forces it to coincide with $M(\Omega)$, which is absurd. Finally, (b) is a consequence of Arens's extension of the Gelfand-Mazur theorem [1].

5. The dual of $M(\Omega)$ and Tietz's problem

For $\Omega = \widehat{\mathbb{C}}$ we have $M(\Omega) = \mathbb{C}(z) \cong \mathbb{C}^{(\widehat{\mathbb{C}} \times \mathbb{N})}$, and its dual is $\mathbb{C}^{\widehat{\mathbb{C}} \times \mathbb{N}}$. Thus we may assume that $\Omega \neq \widehat{\mathbb{C}}$, or, without loss of generality, that Ω is a domain in $\mathbb{C}$. We will use the projective description of the locally convex topology on $M(\Omega)$ to determine its dual $M(\Omega)'_b$ under the strong topology.

As usual, let $H_0(\widehat{\mathbb{C}} \setminus K)$ denote the space of functions that are holomorphic outside the compact set K and vanish at ∞. Then, for compact subsets K_1, K_2 of Ω with $K_1 \subset K_2$ we define the mappings

$$S_{K_2,K_1} : H_0(\widehat{\mathbb{C}} \setminus K_1) \times \mathbb{C}^{K_1 \times \mathbb{N}} \to H_0(\widehat{\mathbb{C}} \setminus K_2) \times \mathbb{C}^{K_2 \times \mathbb{N}}, \quad (\varphi, b) \mapsto (\varphi|_{\widehat{\mathbb{C}} \setminus K_2}, c),$$

where

$$c_j^\alpha = \begin{cases} b_j^\alpha & \text{for} \quad \alpha \in K_1,\, j \in \mathbb{N}, \\ -\dfrac{1}{(j-1)!}\varphi^{(j-1)}(\alpha) & \text{for} \quad \alpha \in K_2 \setminus K_1,\, j \in \mathbb{N}. \end{cases}$$

That $\left(H_0(\widehat{\mathbb{C}} \setminus K) \times \mathbb{C}^{K \times \mathbb{N}}, S_{K_2,K_1}\right)^{\mathscr{K}_\Omega}$ is an inductive system is part of the next result.

THEOREM 6. *We have $M(\Omega)'_b \cong \operatorname{ind}_{K \in \mathscr{K}_\Omega} \left(H_0(\widehat{\mathbb{C}} \setminus K) \times \mathbb{C}^{K \times \mathbb{N}}\right)$, the locally convex inductive limit being taken with respect to the mappings S_{K_2,K_1} $(K_1 \subset K_2)$.*

PROOF. By Remark 2(ii) we have $M(\Omega) \cong \operatorname{proj}_{K \in \mathscr{K}_\Omega} \left(H(K) \times \mathbb{C}^{(K \times \mathbb{N})}\right)$, where the projective limit is taken with respect to the mappings T_{K_1,K_2} $(K_1 \subset K_2)$ and the isomorphism is induced by the mappings T_K; here, T_{K_1,K_2} and T_K are defined in

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

analogy to the mappings T_{O_1,O_2} and T_O in the discussion preceding Remark 2. This projective limit is reduced by Runge's theorem ([23, p. 253]) according to which every element in $H(K)$ can be approximated uniformly on some neighbourhood U of K by rational functions with poles outside U. Hence

$$M(\Omega)'_b \cong \text{ind}_{K\in\mathscr{K}_\Omega} \left(H_0(\widehat{\mathbb{C}} \setminus K) \times \mathbb{C}^{K\times\mathbb{N}}\right),$$

noting that all spaces involved are reflexive ([19, 22.7(9), 23.3(1) and §27.3]). And the locally convex inductive limit is taken with respect to the adjoints T'_{K_1,K_2}. It remains to show that $T'_{K_1,K_2} = S_{K_2,K_1}$ for $K_1 \subset K_2$. To see this let $\varphi \in H_0(\widehat{\mathbb{C}}\setminus K_1) \cong H(K_1)'$, $b \in \mathbb{C}^{K_1\times\mathbb{N}} \cong (\mathbb{C}^{(K_1\times\mathbb{N})})'$, $g \in H(K_2)$ and $a \in \mathbb{C}^{(K_2\times\mathbb{N})}$. Then we have

$$\begin{aligned}\left\langle T'_{K_1,K_2}(\varphi, b), (g, a)\right\rangle &= \left\langle (\varphi, b), T_{K_1,K_2}(g, a)\right\rangle \\ &= \left\langle (\varphi, b), \left(g|_{K_1} + \sum_{\alpha\in K_2\setminus K_1} \sum_{j=1}^{\infty} \frac{a_j^\alpha}{(\cdot - \alpha)^j},\ a|_{K_1\times\mathbb{N}}\right)\right\rangle \\ &= \langle \varphi|_{\widehat{\mathbb{C}}\setminus K_2}, g\rangle + \sum_{\alpha\in K_2\setminus K_1} \sum_{j=1}^{\infty} \left\langle \varphi, \frac{1}{(\cdot - \alpha)^j}\right\rangle a_j^\alpha + \sum_{\alpha\in K_1} \sum_{j=1}^{\infty} b_j^\alpha a_j^\alpha \\ &= \left\langle S_{K_2,K_1}(\varphi, b), (g, a)\right\rangle,\end{aligned}$$

where we have used that

$$\left\langle \varphi, \frac{1}{(\cdot - \alpha)^j}\right\rangle = -\frac{1}{(j-1)!}\varphi^{(j-1)}(\alpha)$$

for $\alpha \in K_2 \setminus K_1$ (cf. [19, p. 373]). Hence we have $T'_{K_1,K_2} = S_{K_2,K_1}$.

From this result we obtain a concrete representation of $M(\Omega)'$.

Define $N(\Omega)$ as the following subspace of $H_0(\widehat{\mathbb{C}} \setminus \Omega) \times \mathbb{C}^{\Omega\times\mathbb{N}}$: An element (φ, b) of this product belongs to $N(\Omega)$ if there is a compact subset K of Ω with

$$(*) \qquad \varphi \in H_0(\widehat{\mathbb{C}} \setminus K) \text{ and } b_j^\alpha = -\frac{1}{(j-1)!}\varphi^{(j-1)}(\alpha) \text{ for } \alpha \in \Omega \setminus K,\ j \in \mathbb{N}.$$

For any $(\varphi, b) \in N(\Omega)$ and $f \in M(\Omega)$ we define

$$(4) \qquad u_{(\varphi,b)}(f) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\varphi(\zeta)d\zeta + \sum_{\alpha\in K} \sum_{j=1}^{\infty} a_j^\alpha(f) b_j^\alpha,$$

where K is any compact set in Ω satisfying $(*)$ and

$(**)$ γ is a Cauchy cycle for K in Ω such that every pole of f in $\Omega\setminus K$ lies outside γ.

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

LEMMA 3. *The functional $u_{(\varphi,b)}$ is well-defined.*

PROOF. We need to show that the definition does not depend on K and γ. The independence of γ follows from Cauchy's theorem ([24, 10.35]). Let K_ν be compact subsets of Ω satisfying $(*)$ and γ_ν cycles satisfying $(**)$ with respect to K_ν, $\nu = 1, 2$. We may assume that $K_1 \subset K_2$. Using $(*)$ we obtain by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{\varphi(\zeta)}{(\zeta-\alpha)^j}d\zeta - \frac{1}{2\pi i}\int_{\gamma_2}\frac{\varphi(\zeta)}{(\zeta-\alpha)^j}d\zeta = -\frac{1}{(j-1)!}\varphi^{(j-1)}(\alpha) = b_j^\alpha$$

for any pole α of f in $K_2 \setminus K_1$. This implies that

$$\begin{aligned}\frac{1}{2\pi i}&\int_{\gamma_1} f(\zeta)\varphi(\zeta)d\zeta + \sum_{\alpha\in K_1}\sum_{j=1}^{\infty} a_j^\alpha(f)b_j^\alpha \\ &= \frac{1}{2\pi i}\int_{\gamma_2} f(\zeta)\varphi(\zeta)d\zeta + \sum_{\alpha\in K_2\setminus K_1}\sum_{j=1}^{\infty} a_j^\alpha(f)b_j^\alpha + \sum_{\alpha\in K_1}\sum_{j=1}^{\infty} a_j^\alpha(f)b_j^\alpha \\ &= \frac{1}{2\pi i}\int_{\gamma_2} f(\zeta)\varphi(\zeta)d\zeta + \sum_{\alpha\in K_2}\sum_{j=1}^{\infty} a_j^\alpha(f)b_j^\alpha.\end{aligned}$$

THEOREM 7. *We have $M(\Omega)' \cong N(\Omega)$ in the following sense:*

(i) *A functional u belongs to $M(\Omega)'$ if and only if $u = u_{(\varphi,b)}$ for some element $(\varphi, b) \in N(\Omega)$.*

(ii) *The element $(\varphi, b) \in N(\Omega)$ in (i) is uniquely determined by u, and we have*

$$\varphi(z) = -u\left(\frac{1}{\cdot - z}\right) \text{ for } z \neq \infty \text{ in a neighbourhood of } \widehat{\mathbb{C}} \setminus \Omega \text{ and}$$

$$b_j^\alpha = u\left(\frac{1}{(\cdot-\alpha)^j}\right) \text{ for } \alpha \in \Omega \text{ and } j \in \mathbb{N}.$$

PROOF. (i) Firstly we note the following. Given $(\varphi, b) \in N(\Omega)$ we have for $f \in M(\Omega)$

$$u_{(\varphi,b)}(f) = \frac{1}{2\pi i}\int_\gamma \left(f(\zeta) - \sum_{\alpha\in K} h^\alpha(\zeta)\right)\varphi(\zeta)d\zeta + \sum_{\alpha\in K}\sum_{j=1}^{\infty} a_j^\alpha(f)b_j^\alpha, \tag{5}$$

where K is any compact subset of Ω satisfying $(*)$ and γ is any cycle satisfying $(**)$. For, if $\alpha \in K$, $j \in \mathbb{N}$, and γ_R denotes the positively oriented circle of radius $R > 0$, then

$$\int_\gamma \frac{1}{(\zeta-\alpha)^j}\varphi(\zeta)d\zeta = \int_{\gamma_R}\frac{\varphi(\zeta)}{(\zeta-\alpha)^j}d\zeta \to 0$$

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

as $R \to \infty$ since $\lim_{\zeta\to\infty} \varphi(\zeta) = 0$, so that in fact

$$\int_\gamma \frac{1}{(\zeta-\alpha)^j}\varphi(\zeta)d\zeta = 0, \tag{6}$$

which implies (5).

Now, by (the proof of) Theorem 6 a functional u belongs to $M(\Omega)'$ if and only if there is a compact subset K of Ω and elements $\varphi \in H_0(\widehat{\mathbb{C}} \setminus K)$ and $b \in \mathbb{C}^{K\times\mathbb{N}}$ such that $u = S_K(\varphi, b)$, where $S_K = T_K'$ is the adjoint of the mapping

$$T_K : M(\Omega) \to H(K) \times \mathbb{C}^{(K\times\mathbb{N})}, \quad f \mapsto \left(f - \sum_{\alpha\in K} h^\alpha, (a_j^\alpha(f))_{\alpha\in K, j\in\mathbb{N}}\right)$$

(see Remark 2(i) and the discussion preceding Remark 2); in other words,

$$\begin{aligned} u(f) &= \left\langle T_K'(\varphi, b), f\right\rangle = \left\langle(\varphi, b), T_K(f)\right\rangle \\ &= \frac{1}{2\pi i}\int_\gamma \left(f(\zeta) - \sum_{\alpha\in K} h^\alpha(\zeta)\right)\varphi(\zeta)d\zeta + \sum_{\alpha\in K}\sum_{j=1}^\infty a_j^\alpha(f) b_j^\alpha \end{aligned} \tag{7}$$

for $f \in M(\Omega)$, where γ is any cycle satisfying (**). Thus, if we define $b_j^\alpha = -\varphi^{(j-1)}(\alpha)/(j-1)!$ for $\alpha \in \Omega \setminus K$ and $j \in \mathbb{N}$, we see, comparing (5) and (7), that $u \in M(\Omega)'$ if and only if $u = u_{(\varphi,b)}$ for some $(\varphi, b) \in N(\Omega)$.

(ii) Let $u \in M(\Omega)'$ be given by some $(\varphi, b) \in N(\Omega)$ via $u = u_{(\varphi,b)}$. Define $f_j^\alpha \in M(\Omega)$ by $f_j^\alpha(\zeta) = 1/(\zeta-\alpha)^j$ for $\alpha \in \Omega$, $j \in \mathbb{N}$. Suppose that φ is holomorphic in $\widehat{\mathbb{C}} \setminus K$, and let $z \in \mathbb{C} \setminus K$. Let γ be a Cauchy cycle for K in $\Omega \setminus \{z\}$ and γ' a Cauchy cycle for $K \cup \{z\}$ in Ω. Then, by (4) and (6), we have

$$u(f_1^z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\left(\int_\gamma - \int_{\gamma'}\right)\frac{\varphi(\zeta)}{\zeta - z}d\zeta = -\varphi(z).$$

If $\alpha \in \Omega$ and $j \in \mathbb{N}$, we obtain by (5) applied to $K \cup \{\alpha\}$: $u(f_j^\alpha) = b_j^\alpha$. This proves (ii).

We are now in a position to give a new proof of Tietz's theorem that is based on the duality theory for the space $M(\Omega)$. This then solves Tietz's problem.

PROOF OF THEOREM 0. Let $f \in M(\Omega)$ and $P = \{\alpha_1, \alpha_2, \dots\}$ its set of poles. Let (γ_n) be a sequence of Cauchy cycles in Ω with the stated properties.

(a) Condition (i) in the theorem is equivalent to

(i$_1$) $\left(f - \sum_{k\le k_n} h^{\alpha_k}\right)_n$ converges in the space $M(\Omega; N)$,

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

where $N \in \mathscr{D}_\Omega$ is defined by $N(\alpha) = -o_\alpha(f)$ if $\alpha \in P$ and $N(\alpha) = 0$ otherwise; note that the limit function is automatically holomorphic in all of Ω. By Theorem 4(c) this is equivalent to

(i_2) $\left(f - \sum_{k \le k_n} h^{\alpha_k}\right)_n$ converges in $M(\Omega)$.

Since by Theorem 3(a) the space $M(\Omega)$ is weakly sequentially complete and has the Schur property, this in turn is equivalent to

(i_3) $\left(u\left(f - \sum_{k \le k_n} h^{\alpha_k}\right)\right)_n$ converges for every $u \in M(\Omega)'$.

Now represent $u \in M(\Omega)'$ as $u = u_{(\varphi,b)}$ with $(\varphi, b) \in N(\Omega)$. Let K be a compact subset of Ω satisfying $(*)$. Then γ_n satisfies condition $(**)$ for the function $f - \sum_{k \le k_n} h^{\alpha_k}$ if n is sufficiently large. Hence we have for these n

$$u\left(f - \sum_{k \le k_n} h^{\alpha_k}\right) = \frac{1}{2\pi i}\int_{\gamma_n}\left(f(\zeta) - \sum_{k \le k_n} h^{\alpha_k}(\zeta)\right)\varphi(\zeta)d\zeta$$
$$= \frac{1}{2\pi i}\int_{\gamma_n} f(\zeta)\varphi(\zeta)d\zeta$$

(see (6)). Since, on the other hand, every $\varphi \in H_0(\widehat{\mathbb{C}} \setminus \Omega)$ appears in some pair $(\varphi, b) \in N(\Omega)$, we see that ($i_3$) is equivalent to condition (ii) in the theorem.

(b) The reasoning in (a) shows that if (i) (or (ii)) holds, then $\lim_{n\to\infty}(2\pi i)^{-1}\int_{\gamma_n} f(\zeta)\varphi(\zeta)d\zeta = u_{(\varphi,b)}(g)$ where $g = \lim_{n\to\infty}\left(f - \sum_{k \le k_n} h^{\alpha_k}\right)$. Hence (b) follows.

Appendix: Literature on the problem of topologising spaces of meromorphic functions

We give here a survey of topologies on $M(\Omega)$ that have been considered in the literature.

1. The most natural topology in the space of meromorphic functions is the topology τ_{chor} of locally uniform convergence with respect to the chordal metric in the range space $\widehat{\mathbb{C}}$; see the introduction. This topology was introduced by Ostrowski [21] to capture Montel's notion of a normal family and is widely used in complex analysis. However, it does not reflect the linear structure of $M(\Omega)$. For details see, for example, [13, VII.1–3].

2. Topologies on $M(\Omega)$ as extensions of the topology (of locally uniform convergence) on $H(\Omega)$; typically, these topologies have the property that every functional $f \mapsto a_j^\alpha(f)$ ($\alpha \in \Omega$, $j \in \mathbb{N}$) is continuous. We have already mentioned [7, 9, 11, 12, 17] in Remarks 1 and 2(iii). Topologies on subspaces of $M(\Omega)$ with prescribed poles are given in [5, 10, 27, 32] (see also [2, 3, 4]). Some of the topologies discussed in these papers are non-linear. We believe that $\tau_{\text{Hol}} = \tau_{\text{ML}}$ is the natural locally convex topology on $M(\Omega)$.

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

3. Topologies on $M(\Omega)$ with an emphasis on compatibility with its algebraic structure – in particular field topologies. A common procedure in field theory is to define a field topology via a given absolute value on the field, where the absolute value may come from a valuation, see [29, 30]. Thus the natural – and essentially only ([23, p. 96]) – valuations $v(f) = o_\alpha(f)$ with $\alpha \in \Omega$ define a metric field topology on $M(\Omega)$ ([30, pp. 9f]) which is not a vector space topology. On $\mathbb{C}(z)$ this topology was introduced by Kürschák [20] as early as 1913.

Field topologies on $\mathbb{C}(z)$ that are also vector space topologies were defined by Williamson [31, Section 2] (see also [28, IX.3] and [6, Example 4.9–1]) and Boehme [8]. They are metrisable, non-complete and, necessarily, non-locally convex (see Remark 5(ii)(b)). It is not difficult to see that Williamson's idea can be extended to define a metrisable vector space and field topology on any space $M(\Omega)$. Williamson [31, Section 3] also defined a metrisable locally convex algebra topology on $\mathbb{C}(z)$; it is non-complete and inversion is necessarily discontinuous (see Remark 5(ii)(b)). Another interesting topology on $\mathbb{C}(z)$ was defined by Waelbroeck [28, IX.4]: It is its strongest vector space and field topology; it is non-metrisable but sequentially complete. We do not know if it is complete.

We finish with two problems that seem to be open:

(1) Is there a complete Hausdorff vector space and field topology on $M(\Omega)$? Can such a topology even be metrisable?

(2) Is there a complete Hausdorff locally convex algebra topology (in the strong sense; see Remark 5(ii)) on $M(\Omega)$? Can such a topology even be metrisable?

The questions are already of interest for $M(\widehat{\mathbb{C}}) = \mathbb{C}(z)$. In this case, Waelbroeck's topology on $\mathbb{C}(z)$ may be a solution to the first question in (1).

References

[1] R. Arens, 'Linear topological division algebras', *Bull. Amer. Math. Soc.* **53** (1947), 623–630.

[2] E. G. Barsukov, 'Complete systems in the space of analytic functions with isolated singularities of unique character', in: *Mathematical analysis and its applications, vol. VI* (Rostov State University, Rostov-on-Don, 1974) pp. 228–237 (in Russian).

[3] ———, 'Maximal ideals in an algebra of meromorphic functions', in: *Theory of functions. Differential equations and their applications, No. 1* (Kalmyk State University, Elista, 1976) pp. 23–34 (in Russian).

[4] ———, 'On abstract meromorphic algebras', in: *Mathematical analysis and its applications* (Rostov State University, Rostov-on-Don, 1981) pp. 7–10 (in Russian).

[5] E. G. Barsukov and M. G. Khaplanov, 'A space of meromorphic functions', *Mat. Zametki* **17** (1975), 589–598 (in Russian). English translation: *Math. Notes* **17** (1975), 350–355.

[6] E. Beckenstein, L. Narici and C. Suffel, *Topological algebras* (North-Holland, Amsterdam, 1977).

[7] P. Bobillo Guerrero, 'Topologies on $M(A)$', in: *Proceedings of the eleventh annual conference of Spanish mathematicians (Murcia, 1970)* (Universidad Complutense de Madrid, Madrid, 1973) pp. 22–30 (in Spanish).

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press

[8] T. K. Boehme, 'On the limits of the Gelfand-Mazur theorem', in: *Proceedings of the conference on convergence spaces* (Reno, 1976) (University of Nevada, Reno, 1976) pp. 1–4.
[9] H.-W. Burmann and H. S. Holdgrün, 'Ein vollständiger induktiver Limes aus gewissen meromorphen Funktionen', *Math. Z.* **102** (1967), 89–109.
[10] J. A. Cima and J. A. Pfaltzgraff, 'The Hornich topology for meromorphic functions in the disk', *J. Reine Angew. Math.* **235** (1969), 207–220.
[11] J. Cima and G. Schober, 'On spaces of meromorphic functions', *Rocky Mountain J. Math.* **9** (1979), 527–532.
[12] T. Constantinescu and A. Gheondea, 'Algebraic aspects of the functional calculus for meromorphic functions', *Rev. Roumaine Math. Pures Appl.* **27** (1982), 949–956.
[13] J. B. Conway, *Functions of one complex variable*, 2nd edition (Springer, New York, 1978).
[14] K. Floret and J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume* (Springer, Berlin, 1968).
[15] V. D. Golovin, 'Duality in spaces of holomorphic functions with singularities', *Dokl. Akad. Nauk SSSR* **168** (1966), 9–12 (in Russian). English translation: *Soviet Math. Dokl.* **7** (1966), 571–574.
[16] ———, 'On some spaces of holomorphic functions with isolated singularities', *Mat. Sb.* **73(115)** (1967), 21–41 (in Russian). English translation: *Math. USSR - Sb.* **2** (1967), 17–33.
[17] H. S. Holdgrün, 'Fastautomorphe Funktionen auf komplexen Räumen', *Math. Ann.* **203** (1973), 35–64.
[18] H. Jarchow, *Locally convex spaces* (Teubner, Stuttgart, 1981).
[19] G. Köthe, *Topological vector spaces, I* (Springer, Berlin, 1969).
[20] J. Kürschák, 'Über Limesbildung und allgemeine Körpertheorie', *J. Reine Angew. Math.* **142** (1913), 211–253.
[21] A. Ostrowski, 'Über Folgen analytischer Funktionen und einige Verschärfungen des Picardschen Satzes', *Math. Z.* **24** (1926), 215–258.
[22] R. Remmert, *Theory of complex functions* (Springer, New York, 1991).
[23] ———, *Funktionentheorie, II* (Springer, Berlin, 1991).
[24] W. Rudin, *Real and complex analysis*, 3rd edition (McGraw-Hill, New York, 1987).
[25] H. Tietz, 'Zur Klassifizierung meromorpher Funktionen auf Riemannschen Flächen', *Math. Ann.* **142** (1961), 441–449.
[26] ———, Private communication.
[27] F. S. Vakher and V. V. Ryndina, 'Algebras of meromorphic functions', in: *Mathematical analysis and its applications* (Rostov State University, Rostov-on-Don, 1981) pp. 23–33 (in Russian).
[28] L. Waelbroeck, *Topological vector spaces and algebras* (Springer, Berlin, 1971).
[29] S. Warner, *Topological fields* (North-Holland, Amsterdam, 1989).
[30] W. Więsław, *Topological fields* (Marcel Dekker, New York, 1988).
[31] J. H. Williamson, 'On topologising the field $C(t)$', *Proc. Amer. Math. Soc.* **5** (1954), 729–734.
[32] V. P. Zakharyuta and A. I. Matvienko, 'On bases in spaces of meromorphic functions with given poles', in: *Mathematical analysis and its applications* (Rostov State University, Rostov-on-Don, 1981) pp. 55–60 (in Russian).

Fachbereich Mathematik
Fernuniversität Hagen
Postfach 940
D-58084 Hagen
Germany

https://doi.org/10.1017/S1446788700037198 Published online by Cambridge University Press